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CONJUGATE ANALOGY FOR SPACE STRUCTURES

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INTRODUCTION

For elastic, straight, single-span beams or cantilevers, the small deflections and rotations may be calculated using the well-known conjugate beam method<sup>2,3,4,5,6,7,8,9</sup> which is expressed in the so-called Mohr's theorems. Its origin dates to the paper by O. Mohr,<sup>10</sup> and the concept of conjugate beam was also introduced by H. F. B. Müller-Breslau.<sup>11</sup>

In principle, this method is based on the formal analogy between the geometry of small deformations and equilibrium of internal forces. In view of a simple calculation allowed by this method, further developments yielded suc-

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<sup>2</sup> Timoshenko, S. P., and MacCullough, G. H., "Elements of Strength of Materials," 3rd Edition, McGraw-Hill Book Co., Inc., New York, N. Y., 1949, Chapter VI.

<sup>3</sup> Timoshenko, S. P., and Young, D. H., "Theory of Structures," McGraw-Hill Book Co., Inc., New York, N. Y., 1945.

<sup>4</sup> Timoshenko, S. P., "History of the Strength of Materials," McGraw-Hill Book Co., Inc., New York, N. Y., 1953.

<sup>5</sup> Schleichner, F., "Sul theorema de Mohr," *Costruzioni in cemento armato*, Rendiconti e pubblicazioni del corso di perfezionamento del Politecnico di Milano, Lib. ed. Politt. Tamburini, Milano, No. 11, 1960, p. 140.

<sup>6</sup> Dašek, V., "Mechanics of Framed Structures," State Publishing House for Tech. Literature, Prague, Czechoslovakia, 1959 (in Czech).

<sup>7</sup> Westergaard, H. M., "Deflection of Beams by the Conjugate Beam Method," *Journal of the Western Society of Engineers*, Vol. 26, No. 11, 1921, p. 369.

<sup>8</sup> Bazant, Z., and Bazant, Z. P., "Analysis of Framed Structures," *Applied Mechanics Surveys*, edited by ASME, Applied Mechanics Reviews, San Antonio, Tex., 1966.

<sup>9</sup> Bazant, Z., "Structural Mechanics," 5th Edition, Vol. II, Czechoslovakian Academy of Science, Prague, Czechoslovakia, 1954 (in Czech).

<sup>10</sup> Mohr, O., "Beitrag zur Theorie der Holz- und Eisenkonstruktionen," *Zeitschrift des Architekten und Ingenieuren Vereines zu Hannover*, Hannover, Germany, 1868, p. 19.

<sup>11</sup> Müller, Breslau, H. F. B., "Beitrag zur Theorie des Fachwerks," *Zeitschrift des Architekten und Ingenieuren Vereines zu Hannover*, Germany, 1868, p. 10.

cessive generalizations. They are represented, for instance, by the moment-area method,<sup>12,13,14</sup> the method of elastic weights (fictitious loads), or the string polygon method,<sup>15</sup> the method of fictitious equilibrium conditions,<sup>16</sup> and the conjugate frame method,<sup>17</sup> which has been also extended for certain space frames.<sup>18</sup>

The complete force-deformation conjugate analogy for structures consisting of one-dimensional members (bars or beams), however, has not, to the writer's knowledge, been published. This is the purpose of the present paper in which the conjugate analogy will be generalized for bars, beams, and structures of arbitrary shape (including curved bars) and with arbitrary supports, connections, and joints; for tension and shear; for inelastic material; for thin-walled bars with rigid cross sections that are allowed to warp longitudinally; and for the cases of elastic (nonrigid) restraints, supports and connections, and plastic hinges.

At the same time, a brief, intuitive, and simple way of derivation, using integral vector equations, will be presented, and some new possibilities for practical use will be shown. The conjugate correspondence between all kinematically possible types of supports and connections will be studied in detail.

#### ASSUMPTIONS AND SCOPE

The analysis will be concerned only with structures, consisting of members that can be mathematically treated as one-dimensional. This definition includes bars and beams of arbitrary shape (e. g., curved beams), frames, arches, lattices, girder grillages, etc. The idealization, understood in the term one-dimensional members, consists of the fact that the members may be replaced by their axes and cross sections corresponding to the points of the axis—the cross sections being considered absolutely rigid (in their planes). These members may be subjected to bending, torsion, shear, and tension or compression, and their deformation is fully described by three rotation components resulting in vector,  $\bar{\phi}$ , and three displacement components resulting in vector,  $\bar{u}$ , at each cross section. This idealization requires that the dimension of the cross sections be sufficiently small with respect to the length of bar and also with respect to the curvature radius of the axis of the bar (except for apexes at broken axis bars). According to this definition, the analysis will

<sup>12</sup> Matheson, J. A. L., Murray, N. W., and Livesley, R. K., "Hyperstatic Structures," John Wiley & Sons, Inc., New York, N. Y., 1955.

<sup>13</sup> Parcel, J. I., and Moorman, R. B. B., "Analysis of Statically Indeterminate Structures," John Wiley & Sons, Inc., New York, N. Y., 1955.

<sup>14</sup> Robertson, R. G., et al., "Portal Frame Analysis by Moment Area Method," *Structural Engineer*, London, Vol. 34, No. 5, 1956, pp. 173-178.

<sup>15</sup> Tuma, J. J., and Oden, J. T., "String Polygon Analysis of Frames with Straight Members," *Journal of the Structural Division*, ASCE, Vol. 87, No. ST7, Proc. Paper 2956, October, 1961, pp. 63-89.

<sup>16</sup> Havelka, K., Harvančík, J., and Trokan, J., "Framed Bridge and Factory Building Structures Solved by the Method of Fictitious Equilibrium Conditions," *Slovakian Academy of Science, Bratislava, Czechoslovakia, 1958* (in Slovak).

<sup>17</sup> Lee, S. L., "The Conjugate Frame Method and its Application in the Elastic and Plastic Theories of Structures," *Journal of the Franklin Institute*, Philadelphia, Pa., Vol. 266, No. 3, 1958, p. 207.

<sup>18</sup> Kiusalaas, J., Lee, S. L., and Makino, M., "Analysis of Continuous Space Frames by Conjugate Frame Analogy," *Structural Engineer*, London, Vol. 26, No. 6, 1963, p. 203.

be concerned not only with bars, rods, and beams of massive cross section, but also to thin-walled bars (straight and curved) with a rigid cross section subjected to warping, with open as well as closed profile.

The type of stress-strain law (elastic, nonlinear elastic, viscoelastic, plastic, etc.) is not decisive. Only small deflections and rotations are considered subsequently.

#### ANALOGY OF SMALL DEFORMATION GEOMETRY WITH EQUILIBRIUM CONDITIONS

The axial vector of relative rotation of two adjacent cross sections at a small distance,  $ds$ , along the bar axis equals  $\bar{k} ds$ , in which  $s$  = the length of the bar axis. This defines the change of curvature,  $\bar{k}$ , of the bar, corresponding to its deformation (with respect to its initial curvature before deformation). The direction of the axial vector is always clockwise. The components of  $\bar{k}$  in the local cartesian system,  $x, y, z$ , with the  $z$ -axis tangent to the axis of the bar (in the sense of increasing  $s$ ), are the bending curvatures,  $k_x$  and  $k_y$ , and the torsional curvature or specific angle of twist,  $k_z$ . The relative displacement of two adjacent sections in the direction of the bar axis, or perpendicular to it, will be designated  $\bar{\epsilon} ds$  or  $\bar{\gamma} ds$ . This defines the specific elongation (extension)  $\bar{\epsilon}$ , the bar in its axis and the shear angle,  $\bar{\gamma}$ , with components,  $\gamma_x$  and  $\gamma_y$ .

Starting from the given rotation,  $\bar{\phi}_0$ , and displacement,  $\bar{u}_0$ , in a certain bar cross section with coordinate,  $s_0$ , and summing the contributions of all elements,  $ds$ , of the bar, the rotations and displacements at an arbitrary cross section,  $s$ , of the bar may be expressed as

$$\bar{\phi}(s) = \bar{\phi}_0 + \int_{s_0}^s \bar{k}(s') ds' + \sum_i \bar{\psi}(i) \dots \dots \dots (1)$$

$$\text{and } \bar{u}(s) = \bar{u}_0 + \int_{s_0}^s \left\{ \bar{k}(s') \times [\bar{r}(s) - \bar{r}(s')] \right\} ds'$$

$$+ \int_{s_0}^s \left[ \bar{\epsilon}(s') + \bar{\gamma}(s') \right] ds' + \sum_i \left\{ \bar{\psi}(i) \times [r(s) - r(s(i))] \right\} + \sum_i \bar{v}(i) \dots (2)$$

in which  $\times$  = the vector product (the system of three vectors  $\bar{k}$ ,  $\bar{r}(s)$ , and  $-\bar{r}(s')$  and their product form a right-hand system). The radius vector of the points of the bar axis with respect to the given origin is designated by  $\bar{r} = \bar{r}(s)$ ; vectors,  $\bar{\psi}(i)$ , denote the relative rotation in elastic hinges or end constraints or in plastic hinges; vectors,  $\bar{v}(i)$ , represent the relative displacement in an elastic (or plastic) connection or end support; and  $s'$  represents the integration variable in contrast to the upper limit,  $s$ , of the integral.

Consider the conjugate bar, the equilibrium of which should be analogous to the deformation of the given bar. All quantities concerning the conjugate bar are denoted by an asterisk, \*.

The equilibrium conditions of an unsupported part of the bar between the sections  $s_0$  and  $s$  require that

$$\bar{S}^*(s) = \bar{S}_0^* - \int_{s_0}^s \bar{p}^*(s') ds' - \sum_i \bar{P}^*_{(i)} \dots \dots \dots (3)$$

and 
$$\bar{M}^*(s) = \bar{M}_0^* - \int_{s_0}^s \left\{ \bar{p}^*(s') \times [\bar{r}(s) - \bar{r}(s')] \right\} ds' - \int_{s_0}^s \bar{m}^*(s') ds' - \sum_i \left\{ \bar{P}^*_{(i)} \times [\bar{r}(s) - \bar{r}(s_{(i)})] \right\} - \sum_i \bar{C}^*_{(i)} \dots \dots \dots (4)$$

in which  $\bar{S}^*$  and  $\bar{M}^*$  = the resultant force and the resultant moment in the cross section, respectively. The components of  $\bar{M}^*$  in the local cartesian

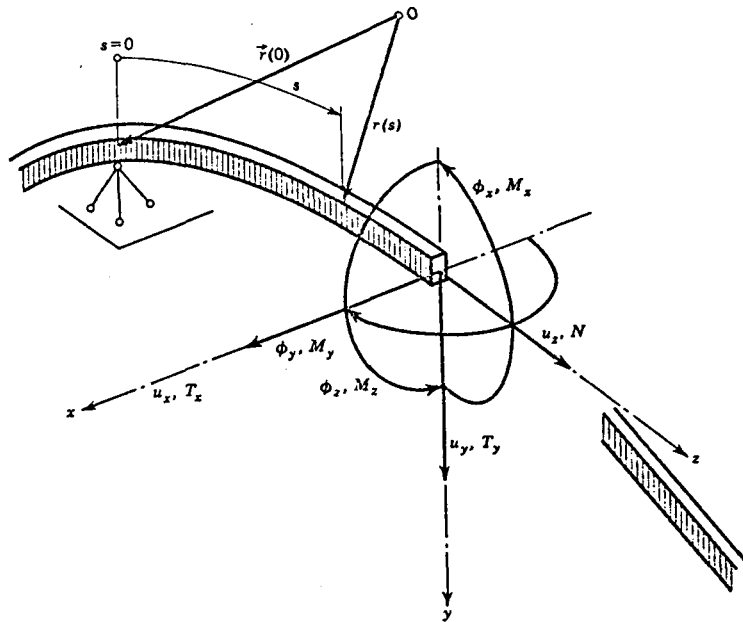


FIG. 1.—LOCAL COORDINATE SYSTEM OF BAR CROSS SECTION, INTERNAL FORCES, ROTATIONS, AND DISPLACEMENTS

system x, y, z are the bending moments  $M_x$  and  $M_y$  about the x-y axes, and the torsional moment (torque)  $M_z$  about the tangent, z; the internal force,  $\bar{S}^*$ , is the resultant of the shear force,  $\bar{T}^*$ , perpendicular to the bar axis and with components  $S_x^* = T_x^*$ ,  $S_y^* = T_y^*$ , and of the normal force,  $\bar{N}$ , in the direction of the bar axis,  $\bar{S}^* = \bar{T}^* + \bar{N}$ ; the vectors  $\bar{S}_0^*$  and  $\bar{M}_0^*$  = the force and moment in section,  $s_0$ , respectively;  $\bar{p}^*$  and  $\bar{m}^*$  = the specific load vector and the specific moment vector, respectively; and vectors  $\bar{P}^*$  and  $\bar{C}^*$  = the concentrated loads and the concentrated loading couples (moments).

For internal forces  $\bar{S}$  and  $\bar{M}$  in the given bar, the same relationships as Eqs. 3 and 4 apply. The positive sense of the internal forces will be determined by the way that the vectors of internal forces acting on the part of bar behind the section (e. g., in the sense of decreasing s) coincide with the direction of the x-y-z axes (Fig. 1).

For straight and circular bars, sometimes another sign convention is used, and some terms in Eqs. 1 to 6 have another sign.

Comparing Eqs. 1 and 2, and 3 and 4, their formal analogy is obvious. In this analogy, the quantities

$$\bar{p}^*, \bar{m}^*, \bar{S}^*, \bar{M}^*, \bar{P}^*, \text{ and } \bar{C}^* \dots \dots \dots (5a)$$

correspond to the quantities

$$-\bar{k}, -(\bar{\epsilon} + \bar{\gamma}), \bar{\phi}, \bar{u}, -\bar{\psi} \text{ and } -\bar{v} \dots \dots \dots (5b)$$

This is the simple basis of the conjugate analogy between the deformations and the internal forces of one-dimensional members.

Eqs. 1 and 2 represent the integral expressions for the deformations, Eq. 2 determines the so-called "deflection line." Eqs. 3 and 4 give the integral expressions for the internal forces. If values  $\phi$  and  $\bar{u}$  or values  $\bar{M}^*$  and  $\bar{S}^*$  are given, these equations, on the contrary, represent the integral equations for  $\bar{k}$ ,  $\bar{\gamma}$ , and  $\bar{\epsilon}$  or  $\bar{p}$  and  $\bar{m}$ .

The analytical expression for  $\bar{r} = \bar{r}(s)$  (i.e., for the bar axis) being known, the integral equations (Eqs. 1 and 2) may be replaced by equivalent differential equations. This is done by differentiating successively Eqs. 1 and 2 or 3 and 4 at which in the differentiated equations, the same integrals as in Eqs. 1 and 2 and 3 and 4 reproduce, and can be then eliminated from the system of original equations and their derivatives, to a certain order.<sup>19</sup>

These differential equations of deformation, as well as the differential equations of equilibrium, can be derived directly, considering the deformation or the equilibrium of an element, ds, of the member. This is the usual procedure. For differential equations of equilibrium, this is the simplest method, but for differential deformation equations, this method is complicated (for curved bars) because of the nonlinearity of deformation geometry and correct neglecting of nonlinear terms (linearization).

Therefore, one possible way of making use of the conjugate analogy presents itself. Along with the integral equations, Eqs. 1, 2 and Eqs. 3, 4, the differential equations of deformation and of equilibrium must also be conjugated. Hence, by interchanging, in the differential equilibrium equations, the quantities according to Eq. 5, the differential geometric equations for small deformation are directly obtained.

*Example 1.*—To demonstrate this, consider a circular curved bar, the radius of the bar axis being R, and the x-axis of the sections being directed to the center of the circle ( $|\bar{r}| = R = \text{const.}$ ). Neglecting the influence of  $\bar{\gamma}$  (in which case the equations for  $\bar{\gamma}_x$  and  $\bar{\gamma}_y$  are not independent and are thus surplus), Eqs. 1 and 2 take in their components, the form

<sup>19</sup> Bažant, Z. P., "Nonuniform Torsion of Thin-Walled Bars With Variable Section, Publications-Memoires-Abhandlungen, Internatl. Assn. for Bridges and Structural Engrg., Zurich, 1965.

$$\phi_z(s) = \phi_{z_0} + \int_{s_0}^s k_z \cos \frac{s-s'}{R} ds' + \int_{s_0}^s k_x \sin \frac{s-s'}{R} ds' \dots (6a)$$

$$u_x(s) = u_{x_0} - R \int_{s_0}^s k_y \sin \frac{s-s'}{R} ds' - \int_{s_0}^s \epsilon \sin \frac{s-s'}{R} ds' \dots (6b)$$

$$u_y(s) = u_{y_0} + R \int_{s_0}^s k_x \sin \frac{s-s'}{R} ds' - R \int_{s_0}^s k_z \left(1 - \cos \frac{s-s'}{R}\right) ds' \dots (6c)$$

$$\text{and } u_z(s) = u_{z_0} - R \int_{s_0}^s k_y \left(1 - \cos \frac{s-s'}{R}\right) ds' + \int_{s_0}^s \epsilon \cos \frac{s-s'}{R} ds' \dots (6d)$$

The integral equations of equilibrium are conjugated with them according to Eq. 5 and need not to be introduced. Writing the first and second derivatives of these equations with respect to  $s$  and eliminating the integrals from them, the following differential geometric equations for small deformations are obtained: or

$$k_x = \frac{d^2 u_y}{ds^2} + \frac{\phi_z}{R} \dots (7a)$$

$$k_y = \frac{-d^2 u_x}{ds^2} - \frac{1}{R} \frac{du_z}{ds} \dots (7b)$$

and  $k_z = \frac{d\phi_z}{ds} - \frac{1}{R} \frac{du_y}{ds} \dots (7c)$

and  $\epsilon = \frac{du_z}{ds} - \frac{u_x}{R} \dots (7d)$

Eqs. 7 can be more easily obtained (by interchanging the quantities, according to Eq. 5) from the following well-known differential equilibrium equations:

$$p_x = -\frac{d^2 M_y}{ds^2} - \frac{N}{R} \dots (8a)$$

$$p_y = \frac{d^2 M_x}{ds^2} + \frac{1}{R} \frac{dM_z}{ds} \dots (8b)$$

$$p_z = -\frac{dN}{ds} + \frac{1}{R} \frac{dM_y}{ds} \dots (8c)$$

$$m_z = -\frac{dM_x}{ds} + \frac{M_y}{R} \dots$$

which result directly from the equilibrium conditions of the element  $ds$  of the bar.<sup>20</sup>

To make this analogy complete, it is necessary to examine the conjugate boundary conditions, i.e., to determine the support or connection (joint) conditions of the bar or structure that will correspond in the same way to the given support or connection conditions. These corresponding supports and connections will be called conjugate supports and conjugate connections. The bar, beam, or structure with conjugate supports and connections will be called a conjugate bar, conjugate beam, or conjugate structure.

DETERMINATION OF CONJUGATE SUPPORTS AND CONNECTIONS

The action of support or connection on bars with rigid sections may be mathematically described by kinematic or static conditions, both conditions being equivalent.

An unsupported rigid section of a bar has six degrees of kinematic freedom that may be represented by three noncoplanar rotation components and three noncoplanar displacement components. The same is true for relative rotations and displacements of two different bar sections.

Any support or connection may be kinematically determined by the unit vectors of rotation and displacement, in the sense of which the rotation or the displacement is prevented (i.e., not allowed). These unit vectors are  $\bar{o}_p$  and  $\bar{a}_r$ , respectively. Possible values of their indices,  $p$ , and  $r$ , are, 1, 2, or 3. Simultaneously, the unit vectors  $\bar{o}_p$  and  $\bar{a}_r$  represent statically these components of reaction moment and reaction force, respectively, which can be transmitted by the support or connection under consideration.

Conversely, any support or connection may also be kinematically represented by the unit vectors  $\bar{o}_q$  and  $\bar{a}_s$  of free rotation or displacement which is allowed (i.e., not prevented) by them.

Consider now a support at an intermediate cross section of the bar [Figs. 2(a) and 2(c)] that can transmit the components of moment and force which are given by the unit vectors  $\bar{o}_p$  and  $\bar{a}_r$ . From continuity of the bar in the support section

$$\bar{\phi}_0 = \bar{\phi}'_0 \text{ and } \bar{u}_0 = \bar{u}'_0 \dots (9)$$

in which  $\bar{\phi}_0$  and  $\bar{u}_0$  correspond to the nearest section at one side and  $\bar{\phi}'_0$  and  $\bar{u}'_0$  correspond to the nearest section at the other side of the support. The kinematic conditions of this support are

$$\bar{\phi}_0 \cdot \bar{o}_p = 0 \text{ and } \bar{u}_0 \cdot \bar{a}_r = 0 \dots (10)$$

in which the dots indicate the scalar vector products.

The equivalent equilibrium conditions are

$$\bar{M}_0 \cdot \bar{o}_q = \bar{M}'_0 \cdot \bar{o}'_q \text{ and } \bar{S}_0 \cdot \bar{a}_s = \bar{S}'_0 \cdot \bar{a}'_s \dots (11)$$

in which  $\bar{o}_q$  and  $\bar{a}_s$  = the unit vectors of free rotations and displacements, respectively ( $q \neq p$ ;  $p, q = 1, 2, 3$ ;  $s \neq r$ ; and  $s, r = 1, 2, 3$ ).

In place of an intermediate support of the actual bar, consider a connection of two parts of the conjugate bar such that its unit vectors  $\bar{o}_s$  and  $\bar{a}_q$  of prevented relative rotation or displacement, i. e., its components of reaction

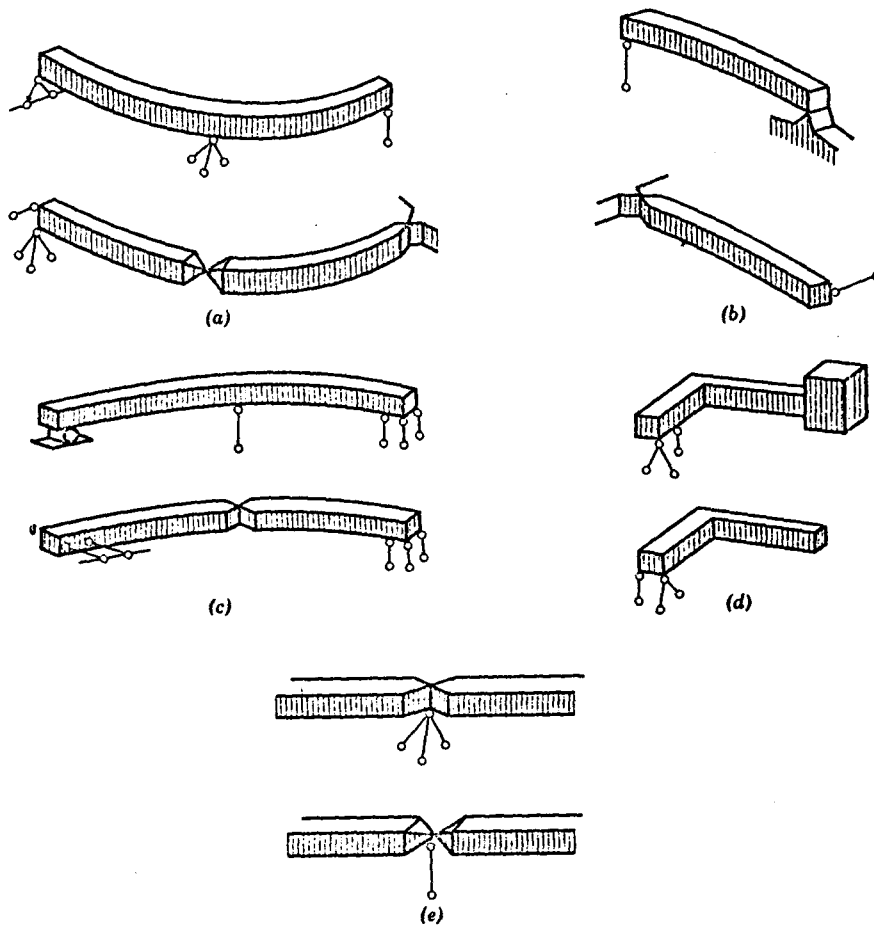


FIG. 2.—EXAMPLES OF CONJUGATED STRUCTURES

moments and forces, coincide with the unit vectors of displacement or rotation respectively, which are allowed by the intermediate support of the actual bar. Because the connected parts of the bar have no support in the section of connection

$$\bar{S}_0^* = \bar{S}_0^{*'} \text{ and } \bar{M}_0^* = \bar{M}_0^{*'} \dots \dots \dots (12)$$

The equilibrium conditions of this connection are

$$\bar{S}^* \cdot \bar{o}_p = 0 \text{ and } \bar{M}^* \cdot \bar{a}_r = 0 \dots \dots \dots (13)$$

in which  $\bar{o}_p$  and  $\bar{a}_r$  = the unit vectors of enabled free relative displacement and rotation, respectively, coinciding with the unit vectors of prevented rotation and displacement in the intermediate support of the actual bar considered previously. The equivalent kinematic conditions of this connection are

$$\bar{u}_0^* \cdot \bar{a}_q = \bar{u}_0^{*'} \cdot \bar{a}_q \text{ and } \bar{\phi}_0^* \cdot \bar{o}_s = \bar{\phi}_0^{*'} \cdot \bar{o}_s \dots \dots (14)$$

Comparing Eqs. 9 to 11 with Eqs. 12 to 14 (or both sides of Table 1), it is obvious that the connection being studied is conjugated to the intermediate sup-

TABLE 1.—CONJUGATE CORRESPONDENCE OF SUPPORTS AND CONNECTIONS

Intermediate support of a bar	Connections of two parts of a bar
Unit vectors of reaction moment and force $\bar{o}_p, \bar{a}_r$	Unit vectors of reaction moment and force $\bar{o}_s, \bar{a}_q$ $q \neq p; p, q = 1, 2, 3$ $s \neq r; r, s = 1, 2, 3$
kinematic conditions $\bar{\phi}_0 = \bar{\phi}_0'$ $\bar{u}_0 = \bar{u}_0'$ $\bar{\phi}_0, \bar{o}_p = 0$ $\bar{u}_0, \bar{a}_r = 0$	equilibrium conditions $\bar{S}_0^* = \bar{S}_0^{*'}$ $\bar{M}_0^* = \bar{M}_0^{*'}$ $\bar{S}_0^* \cdot \bar{o}_p = 0$ $\bar{M}_0^* \cdot \bar{a}_r = 0$
equivalent equilibrium conditions $\bar{M}_0 \cdot \bar{o}_q = \bar{M}_0' \cdot \bar{o}_q$ $\bar{S}_0 \cdot \bar{a}_s = \bar{S}_0' \cdot \bar{a}_s$	equivalent kinematic conditions $\bar{u}_0^* \cdot \bar{a}_q = \bar{u}_0^{*'} \cdot \bar{a}_q$ $\bar{\phi}_0^* \cdot \bar{o}_s = \bar{\phi}_0^{*'} \cdot \bar{o}_s$

port under consideration, because all their conditions mutually correspond in the same manner as Eqs. 1 and 2, and 3 and 4. It is also evident that the conjugate relation is reciprocal, i.e., that this intermediate support is conjugate to this connection.

An end support (support at the end of the bar, connecting it to a rigid foundation) may be treated as a special case (Figs. 2 and 3). Similar conditions are obtained in which

$$\bar{\phi}_0' = \bar{u}_0' = 0 \text{ and } \bar{S}_0^{*'} = \bar{M}_0^{*'} = 0 \dots \dots \dots (15)$$

Obviously, the end support has again an end support for its conjugate.

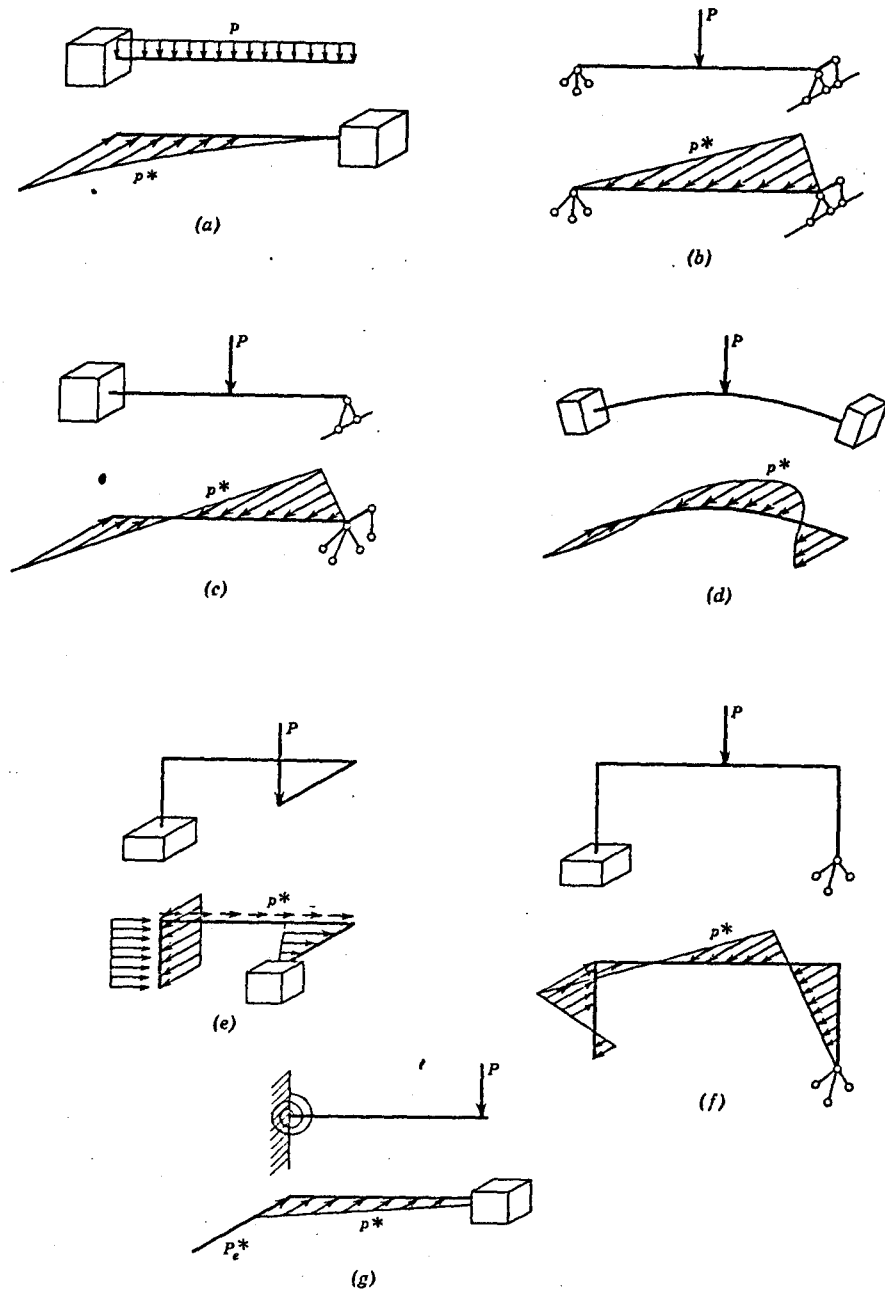


FIG. 3.—EXAMPLES OF SIMPLE CONJUGATED STRUCTURES

Furthermore, the case of a multiple joint in which three or more bars are meeting must be examined. [ Fig. 4(a) ]. If the connection of the ends of bars in a multiple joint is rigid [the restraint, Fig. 4(a) ], the equations

$$\bar{u}_0 = \bar{u}'_0 = \bar{u}''_0 \text{ and } \bar{\phi}_0 = \bar{\phi}'_0 = \bar{\phi}''_0 \dots \dots \dots (16a)$$

are valid, in which  $\bar{u}_0, \bar{u}'_0, \bar{u}''_0$ , etc., denote the displacements at the ends of individual bars. The corresponding equations for the conjugate joint would be

$$\bar{S}_0 = \bar{S}'_0 = \bar{S}''_0 \text{ and } \bar{M}_0 = \bar{M}'_0 = \bar{M}''_0 \dots \dots \dots (16b)$$

It is clear that such a connection of bars cannot be realized. For determining the deformations of the structure, however, it is not necessary to find the con-

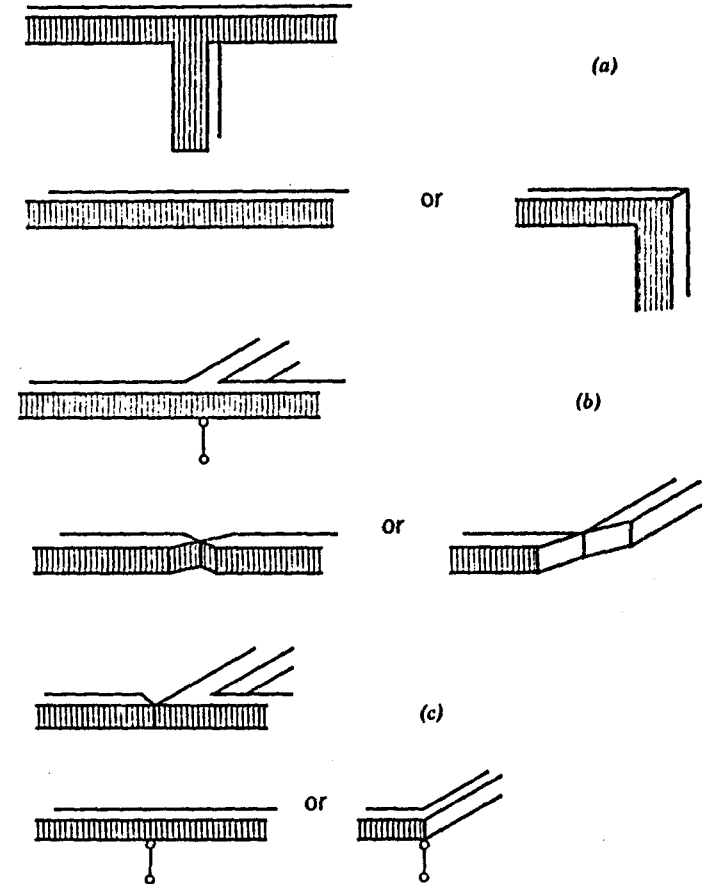


FIG. 4.—EXAMPLES OF CONJUGATE JOINTS, CONNECTIONS, AND SUPPORTS

jugate of the entire structure, and only one of each of the latter double equations will be sufficient, that is, either

$$\bar{S}_0 = \bar{S}'_0 \text{ and } \bar{M}_0 = \bar{M}'_0, \text{ or } \bar{S}_0 = \bar{S}''_0 \text{ and } \bar{M}_0 = \bar{M}''_0 \dots (16c)$$

Thus, the actual multiple joint must be primarily reduced to a connection of only two bars, i.e., to one continuous bar or the so-called "bar (or beam) circuit" that runs continuously through the joint section. This is done by cutting off the arbitrarily chosen redundant bars from the multiple joint. Then it is possible to determine the conjugates for the bar circuits obtained this way. For all bar circuits that can be cut off in various ways from the actual structure, the conjugate internal forces in the same section produced by conjugate loads (i.e., the deformations of actual structure) must be equal. This may serve also as a check of the internal force distribution in redundant structures.

In the latter case, the determination of the conjugate structure is not unique. The term "conjugated structure," as used herein, indicates the set of conjugates to all the bar circuits that may be cut off from the actual structure.

Finally, the combined case of support-connection may occur, in which the support is situated (theoretically) directly in the connection of bars [ Fig. 2(a) ]. In this case, the support may be treated as if lying in the nearest (infinitely near) adjacent section of the connection. Thus it follows that the conjugate may be determined as for the independent support and the connection existing separately.

If the support is situated directly in the multiple joint of three or more bars, the redundant bars must be primarily cut off, which can be done in an arbitrary combination, as mentioned previously [ Fig. 4(b) ].

When, at an intermediate bar section, the end of another bar is nonrigidly connected [ Fig. 4(c) ], the joint can be treated in the same way as the previously examined rigid joint of three beams with the nonrigid connection situated in the nearest adjacent section. If all three bars are nonrigidly connected in the joint, two of them can be arbitrarily chosen to form the continuous bar circuit—the third bar being regarded as a bar connected with the circuit.

**THEOREM OF CONJUGATE ANALOGY**

The preceding results lead to the following theorem:

The rotation vector,  $\bar{\phi}$ , and displacement vector,  $\bar{u}$ , at any section of the given bar, beam, or structure consisting of bars is equal to the internal force,  $\bar{S}^*$ , or to the internal moment,  $\bar{M}^*$ , produced in the same section of the conjugate bar or structure by the conjugate specific loads,  $\bar{p}^*$ , by the conjugate specific loading moments,  $\bar{m}^*$ , and by the conjugate concentrated forces,  $\bar{P}_{(i)}^*$ , and couples,  $\bar{C}_{(i)}^*$ . The loads,  $\bar{p}^*$ , are equal to the curvature change vectors,  $\bar{k}$ , at the actual structure; the loading moments,  $\bar{m}^*$ , are equal to the sums of the specific elongations,  $\bar{\epsilon}$ , and of the shear angles,  $\bar{\gamma}$ ; and the concentrated loads,  $\bar{P}_{(i)}^*$ , and couples,  $\bar{C}_{(i)}^*$ , are equal to the relative rotations,  $\bar{\psi}_{(i)}$ , and relative displacements,  $\bar{v}_{(i)}$ , in the nonrigid restraints, supports, connections, plastic hinges, etc.

The conjugate bar or structure is determined by replacing all supports and connections (joints) with their conjugates. The conjugate to an intermediate

support of a bar is a connection (joint) of two parts of the bar, and *vice versa*, whereas in the conjugate correspondence the conjugate to each axis of allowed (i.e., free, not prevented) rotation is a unit vector of reaction force (i.e., axis of prevented displacement), and the conjugate to each axis of allowed displacement is a unit vector of reaction moment (i.e., axis of prevented rotation), and *vice versa*. As a special case, it follows that an end support corresponds in the same manner to the given end support of bar.

Having used vector notation, it is not necessary to repeat the same statement for each component of  $\bar{\phi}$  or  $\bar{u}$  and  $\bar{S}^*$  or  $\bar{M}^*$  ( $T_x, T_y, N, M_x, M_y, M_z$  correspond to  $k_x, k_y, k_t, u_x, u_y, u_z$ ).

**ANALYSIS OF DIFFERENT TYPES OF CONJUGATED SUPPORTS AND CONNECTIONS**

According to the number of degrees of freedom—i.e., the number of unit vectors of free rotation and displacement—various kinematic types of supports and connections may be distinguished. Each type may be characterized by the number,  $n_\alpha$ , of prevented degrees of freedom in rotation (prevented rotation components) that coincides with the number of reaction moment components, and by the number,  $n_a$ , of prevented degrees of freedom in displacement (prevented displacement components), which is equal to the number of reaction force components.

The possible values of  $n_\alpha$  or  $n_a$  are 0, 1, 2, and 3, so that the number of all possible types of kinematic supports or connections is  $4 \times 4 = 16$ . Among these types, four types of supports exist, each of which has two different modifications; these are the cases for which  $n_\alpha$  and  $n_a$  have the values 1 or 2, because in this case one direction is characteristic for rotation and one is characteristic for displacement; so that it depends on whether these directions are (a) parallel or (b) concurrent (perpendicular).

The conjugate to a support or connection of type ( $n_\alpha, n_a$ ) is, according to the preceding theorem, a connection or support of the type

$$n_\alpha^* = 3 - n_a \text{ and } n_a^* = 3 - n_\alpha \dots (17)$$

with the total number of prevented degrees of freedom equal to  $6 - n_\alpha - n_a$ .

Four types of supports or connections exist, the conjugate of which are the connections or supports of the same type. Evidently, these are supports at which  $n_\alpha + n_a = 3$  (namely, the types  $n_\alpha = 0, n_a = 3$  and  $n_\alpha = 3, n_a = 0$  and further the types, each with two modifications,  $n_\alpha = 1, n_a = 2$  and  $n_\alpha = 2, n_a = 1$ ).

All possible types of supports and connections with their conjugates are presented in Table 2. The axes  $\rightarrow$  indicate the unit vectors of reaction forces and the axes  $\rightarrow\rightarrow$  indicate the unit vectors of reaction moments which can be transmitted by the support or connection. At the same time, an axonometric schematization with help of hinged bars is given. As shown, this schematization becomes rather complicated for certain space supports or connections.

An important case is, for instance, the fixed end (a restraint), ( $n_\alpha = n_a = 3$ ), which is conjugated with the free end ( $n_\alpha = n_a = 0$ ) [ Figs. 2(d), 3(a), 3(c), 3(d), 3(e), 3(f), and 3(g) ]. The hinged bar support ( $n_\alpha = 0, n_a = 1$ ) is conjugated with uniaxial hinge (articulation), ( $n_\alpha = 2, n_a = 3$ ),

TABLE 2.—ALL KINEMATIC TYPES OF SPACE SUPPORTS OR CONNECTIONS WITH THEIR CONJUGATES

	Support or connection			Connection or support			
	$n_o$ $n_a$	Reaction scheme	Hinged bar scheme	$\bar{n}_o$ $\bar{n}_a$	Reaction scheme	Hinged bar scheme	
1	0 0			3 3			16
2	0 1			2 3			15
3	0 2			1 3			14
4	0 3			ditto	ditto		
5	1 0			3 2			13
6a	1 1			2 2			12a
6b							
7a	1 2			ditto	ditto		12b
7b							
8	2 0			3 1			11
9a	2 1			ditto	ditto		11
9b							
10	3 0			ditto	ditto		

[ Figs. 2(a), 2(b), and 2(c) ]. The point support ( $n_o = 0, n_a = 3$ ) is conjugated with the spherehinge (Cardan joint, triaxial hinge), ( $n_o = 0, n_a = 3$ ), [ Figs. 2(a), 3(a), 3(b), and 3(f), etc. ].

RELATION OF DEFORMATIONS AND INTERNAL FORCES

For structural analysis, it is necessary to introduce the stress-strain law and the corresponding relationships between internal forces  $\bar{S}, \bar{M}$  and deformations,  $\bar{k}, \bar{\epsilon}, \bar{\gamma}$  under certain deformation hypotheses.

*Massive Elastic Bars.*—For bending and tension of massive elastic bars, assuming that the cross sections after deformation remain plane and perpendicular to the axis of bar,

$$k_x = \frac{M_x}{E I_x} \dots \dots \dots (18)$$

$$k_y = \frac{M_y}{E I_y} \dots \dots \dots (19)$$

and

$$\epsilon = \frac{N}{E F} \dots \dots \dots (20)$$

in which  $E$  = the elastic modulus,  $F$  = the area of section, and  $I_x, I_y$  = the principal centroidal moments of inertia of the cross section. For Saint-Venant's pure (simple) torsion in which the influence of normal strain is neglected,

$$k_z = \frac{M_z}{G I_k} \dots \dots \dots (21)$$

in which  $G$  = the shear modulus and  $I_k$  = the moment of inertia in pure torsion, according to Saint-Venant. These equations also apply approximately for closed (box) thin-walled profiles, in which  $I_k$  is given by Bredt's formulas.

If the effect of shear forces cannot be neglected, i.e., if the section after deformation is not perpendicular to the bar axis, the relationships

$$\gamma_x = \frac{T_x}{G F_x} \text{ and } \gamma_y = \frac{T_y}{G F_y} \dots \dots \dots (22)$$

in which  $F_x$  or  $F_y$  = the reduced area of section, must be considered.

*Curved Elastic Thin-Walled Bars With Rigid Cross Section.*—For thin-walled bars, it is necessary to consider the effect of normal strain in torsion, i.e., the so-called bending-torsion (or nonuniform torsion, warping torsion) at which rotation is accompanied by the longitudinal warping of cross sections. For the torsion of open as well as of closed profiles, Umankly's assumption may be adopted that the warping (longitudinal displacements) has the same distribution in the section as for pure torsion. This is identical, at open profiles, with Wagner's assumption of zero shear strain in the middle surface of walls. Then, for open profiles of constant cross section, 20 when ( $\nu = 0$ ), instead of Eq. 21,



$$M_z = G I_k k_z - E I_\omega \frac{d^2 k_z}{ds^2} \dots \dots \dots (23)$$

For bars of variable open cross section that are symmetrical with respect to the y-axis, a similar normal strain distribution, as in the case of constant section, may be assumed. Then, instead of independent Eqs. 19 and 21, the following system of simultaneous equations for open profiles is<sup>19</sup>

$$M_z = - \frac{d}{ds} \left( E I_{\omega x} k_y + E I_\omega \frac{dk_z}{ds} \right) + G I_k k_z \dots \dots \dots (24a)$$

and 
$$M_y = E I_y k_y + E I_{\omega x} \frac{dk_z}{ds} \dots \dots \dots (24b)$$

For bars of one-cell or two-cell closed section (box section) symmetrical with respect to the y-axis, under the same assumptions,<sup>19</sup>

$$M_z = \frac{1}{\nu} \frac{d}{ds} \left( -E I_{\omega x} k_y + E I_\omega \frac{d\kappa}{ds} \right) - G I_k \kappa \dots \dots \dots (25a)$$

and 
$$M_y = E I_y k_y - E I_{\omega x} \frac{d\kappa}{ds} \dots \dots \dots (25b)$$

in which 
$$\kappa = - \frac{k_z}{\nu} + \frac{M_z}{G I_p} \text{ and } \nu = 1 - \frac{I_k}{I_p} \dots \dots \dots (26)$$

in which  $I_\omega$ ,  $I_{\omega x}$ , and  $I_p$  = certain constants (moments of inertia for warping of the cross sections. Note that by substituting Eqs. 7 and 8, the differential equations for the direct solution of circular thin-walled bars are obtained.

For thin-walled bars, the conjugate analogy is generally valid, as for massive bars. The boundary conditions for displacements and rotations are, of course not sufficient at thin-walled bars, and the conditions for the specific angle of twist,  $k_z$  (or for the value of  $\kappa$  representing a warping section parameter, at closed profiles) also apply. However, this fact does not restrict the validity of conjugate analogy. If  $k_z$  is prescribed in a certain section, the only consequence is that the load,  $p_z$ , on the conjugate bar is also prescribed. The boundary conditions for bi-moments (warping moments) affect only the solution of Eqs 23 through 26 and not the determination of the conjugate structure. However, the hinged bar schemes in Table 2 are not satisfactory in this case, because the sections may warp, i.e., do not remain plane. The conjugated bars, of course, need not be regarded as thin-walled bars, because at conjugate bars only the equilibrium equations (Eqs. 3 and 4) come into play, which is the same for massive bars.

*Elastic Constraints, Supports, and Connections.*—In this case, it is necessary to introduce also the relationship between the relative rotations,  $\bar{\psi}_{(i)}$ , and moments,  $\bar{M}_{(i)}$ , or relative displacements,  $\bar{u}_{(i)}$ , and forces,  $\bar{S}_{(i)}$ , occurring in them (e.g.,  $\bar{\psi}_{x(i)} = c M_{x(i)}$ , etc.).

*Inelastic Material.*—The conjugate analogy has been derived without introducing the stress-strain law and is therefore independent of it and holds true for linearly elastic as well as for nonlinear and inelastic bars.

For a nonlinearly elastic material, Eqs. 18 to 25 may be kept valid with the only difference being that the moduli E and G are not constant but can be regarded as values that depend on internal forces or deformations. The same holds for partly plasticized sections. For viscoelastic material, the constants E and G (or C) have to be replaced by the corresponding time-differential operators for creep and their initial conditions, or by the corresponding Volterra's time-integral operators, so that Eqs. 18 to 25 eventually become differential or integral equations. In the range of completely plastic sections (plastic hinges), instead of Eqs. 18 to 25 the conditions of plasticity for internal forces apply (e. g.,  $M = M_{ultimate}$ ), while the deformations are indeterminate.

ANALYSIS OF CONJUGATED STRUCTURES

Consider a n-times statically indeterminate beam or structure in which connections (joints) of only two bars occur (e.g., continuous beams, simple frames). Because the number of degrees of freedom in rotation or displacement prevented by the external supports enlarges the static indeterminacy and, conversely, the number of degrees of freedom in rotation or displacement permitted by the internal connections reduces the static indeterminacy, it may be stated that the conjugate structure must be n-times kinematically indeterminate (i.e., of form). This follows from the fact that the number of degrees prevented by the supports and the number of degrees of freedom permitted by connections, mutually correspond [ Figs. 2(c), 2(d), 3(c), 3(d), and 3(f) ]. As a special case, from this statement the result is that a statically determinate structure has for its conjugate a statically determinate structure [ Figs. 2(a), 2(b), 3(a), 3(b), and 3(g) ].

For the structure with multiple joints, in which more than two bars are meeting, the latter statement makes no sense on the whole, but it can be applied for each bar circuit cut off from the actual structure. By cutting off the redundant beams at multiple joints, the static indeterminacy of the actual structure decreases eventually to a structure indeterminate of form; and the static (or kinematic) indeterminacy of the separate conjugate structure by the same number increases, or decreases, respectively.

If the deformation of a redundant structure is to be computed, while its internal forces distribution is known (and checked), first the given structure should be reduced to a primary statically determinate system, because the redundant deformation conditions may be disregarded, necessarily having been fulfilled. This is usually done for continuous beams, the deformations of which are calculated on a primary system of simply supported one-span beams [ Figs. 2(b) and 3(b) ].

Note that for a structure made of thinwalled bars with sections that may warp, the statical indeterminacy, as used herein, concerns only the number of components of internal forces,  $\bar{M}$  and  $\bar{S}$ , which are not determined by equilibrium conditions, and does not concern the analysis of deformation and stress

distribution in the sections (which itself is, "infinite times," statically indeterminate).

### STRESS ANALYSIS OF REDUNDANT STRUCTURES

The redundant compatibility (deformation) conditions of the actual statically indeterminate structure have for their conjugate counterparts the equilibrium conditions of the conjugate kinematically (of form) indeterminate structure. This property may serve as a check for the correctness of the internal force distribution, if it has been computed previously by another method. It may also be used to compute the internal force distribution directly, i.e., the statically indeterminate quantities. Then the equations of the force (flexibility) method are obtained as equilibrium conditions of the conjugate structure. For a structure with multiple joints (Fig. 4), of course, the condition of equal values of internal forces in the same sections in all possible conjugate structures corresponding to various bar circuits has to be used.

The conjugate analogy may also be used only partly and with the help of it, only the flexibility coefficients of the primary (basic) statically determinate system (which has for its conjugate, a statically determinate structure) in the sense of chosen redundant quantities determined; while the equations of the force method are written as compatibility conditions of the given structure. This is the matter of a different approach only; mathematically it is the same.

In the first procedure, however, in equilibrium conditions of the conjugate kinematically indeterminate structure, the use of all rules of statics can be made, and the equations can often be obtained in a more simple form corresponding to a system of partly or entirely orthogonal redundant quantities, i.e., in the form of a selection of redundant quantities, for which nondiagonal flexibility coefficients are at least partly zero.

For the special case of a plane structure this method is identical with the well-known area-moment method as well as with its further improvements.<sup>6,16,17,18</sup> The obtained equations for the statically indeterminate quantities must be identical, of course, with the equations of the force method (flexibility method, method of least work) obtained by another method (e.g., from the principle of virtual work).

### EXAMPLES OF SOME CONJUGATE STRUCTURES

To demonstrate the results of this method, certain types of structures will now be examined. The conjugate for a horizontally curved beam (or a beam with broken axis) and loaded vertically is an arch beam loaded radially and tangentially (Fig. 5)—an arch loaded in its plane, in fact—and vice versa [Fig. 3(d)]. The vertical deflection of a horizontally curved elastic beam (Fig. 5) is equal to the bending moment at the conjugated arch beam loaded in its plane with  $M_x/EJ_x$  in the direction perpendicular to the beam axis and with  $M_z/GJ_k$  in the direction tangential to the beam axis. A reciprocal statement is that the vertical deflection of a vertically loaded plane arch [Fig. 3(d)] equals the bending moment about the vertical axis at the conjugate curved beam loaded transversely by  $M_x/EJ_x$ . For the special case of a straight beam, the well-known conjugate beam method is obtained [Figs. 3(a) and 3(b)]. Note that up to 1966, this method was presented in textbooks only in a plane; e.g., in Fig. 3(a), the

conjugate cantilever with loads,  $p^*$ , was rotated about its axis to a vertical plane. A plane nonvectorial formulation is possible only in certain simple cases and this is the reason why the general conjugate analogy method has been used until this time. The conjugate for a cantilever—straight, curved, or with broken axis—is a cantilever that is fixed at the free end of the given beam and free at the fixed end [Figs. 3(a) and 3(e)].

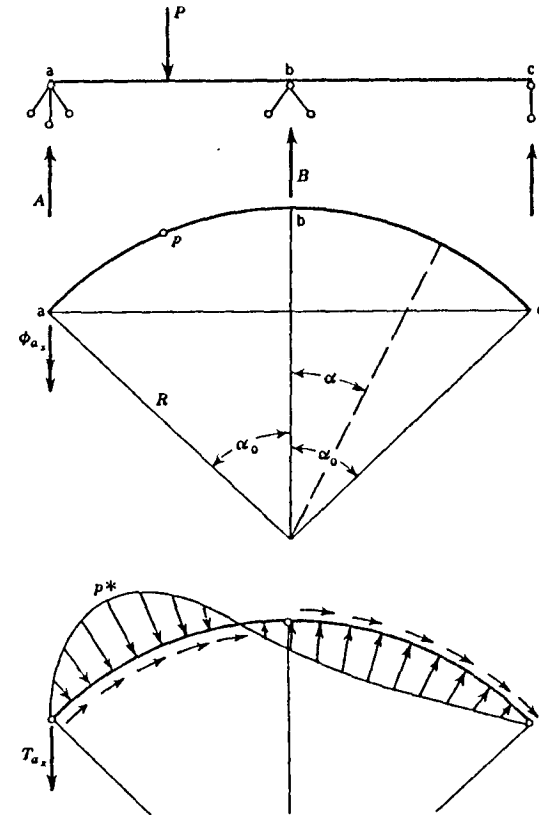


FIG. 5.—HORIZONTALLY CURVED BEAM SOLVED IN EXAMPLE 2 (PLAN AND FRONT VIEWS AND CONJUGATE LOADS)

The conjugate for a one-span beam, supported at one end by a uniaxial hinge (edge) and at the other end by a vertical hinged bar [Fig. 2(b)], is a one-span beam that is supported by a horizontal hinged bar and vertical uniaxial hinge, respectively.

The conjugate for a continuous beam—straight—horizontally curved, or with broken axis—supported by point supports, is the beam point—supported only at the end, with spherical hinges in place of the intermediate supports. The con-

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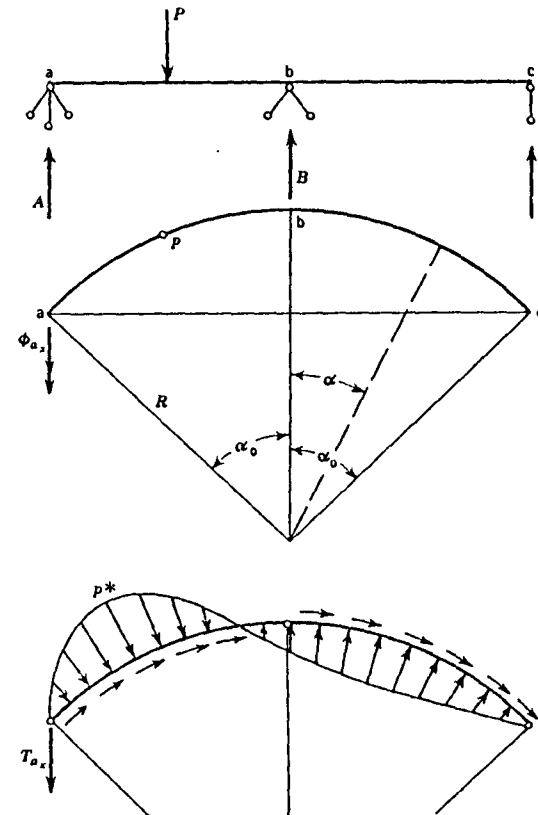


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jugate of its primary system consisting of one-span beams is represented again by a system of one-span beams, but with conjugate loads. If the actual continuous beam has intermediate spherical hinges (Gerber's girder), the conjugate beam then has point supports in their place. If this beam has only two spans, then the conjugate structure is a three-hinged arch, and vice versa.

The continuous beam, supported by horizontal uniaxial hinges, has for its conjugate a set of beams corresponding to separate spans that are connected together by horizontal hinged bars with the direction of original uniaxial hinges.

The beam or frame that is restrained at both ends has for its conjugate a beam or frame that is freely situated without any support in the space [Fig. 3(d)]. The conjugate of a horizontally curved two-hinged frame with horizontal uniaxial hinges at its feet is the same frame that is supported in the place of feet by one horizontal hinged bar in the direction of original uniaxial hinges.

*Example 2.*—Consider now a statically determinate circular horizontal continuous two-span beam, supported in three points (a, b, and c) and loaded vertically by a point load, P, in the middle of the span, ab (Fig. 5). Calculate the rotation  $\phi_{aX}$  at end, a, in a vertical plane given by the chord, ac.

From the equilibrium conditions of the beam it follows that the support reactions (according to Fig. 5) have the values

$$A = P \frac{\sin \frac{\alpha_0}{4} \sin \frac{3}{4} \alpha_0}{\sin \frac{\alpha_0}{2} \sin \alpha_0} \dots \dots \dots (27a)$$

$$C = - \frac{P}{2} \frac{1 - \cos \frac{\alpha_0}{2}}{\sin \frac{\alpha_0}{2} \sin \alpha_0} \dots \dots \dots (27b)$$

and  $B = P - A - C \dots \dots \dots (27c)$

The bending moments,  $M_x$ , in the vertical plane and the torsional moments,  $M_z$ , are

$$- \alpha_0 \leq \alpha \leq - \frac{\alpha_0}{2}, M_x = AR \sin (\alpha_0 + \alpha), M_z = AR [1 - \cos(\alpha_0 + \alpha)] \dots (28a)$$

$$- \frac{\alpha_0}{2} \leq \alpha \leq 0, M_x = AR \sin (\alpha_0 + \alpha) + PR \sin \alpha \dots \dots \dots (28b)$$

$$M_z = AR [1 - \cos (\alpha_0 + \alpha)] - PR \left[ 1 - \cos \left( \frac{\alpha_0}{2} + \alpha \right) \right] \dots \dots \dots (28c)$$

and  $0 \leq \alpha \leq \alpha_0, M_x = CR \sin (\alpha_0 - \alpha), M_z = - CR [1 - \cos(\alpha_0 - \alpha)] \dots (28d)$

The conjugate structure is a three-hinged arch loaded radially by  $M_x/EJ_x$  and tangentially by  $M_z/GJ_k$ . From the moment equilibrium condition of this arch. abc, to the axis in the plane of arch, perpendicular to the chord, ac, at the point c, there results

$$\phi_{aX} = T_{aX}^* = \frac{R}{2 \sin \alpha_0} \left\{ - \int_{-\alpha_0}^{+\alpha_0} \frac{M_x}{EJ_x} \sin (\alpha_0 - \alpha) d\alpha + \int_{-\alpha_0}^{+\alpha_0} \frac{M_z}{GJ_k} [1 - \cos (\alpha_0 - \alpha)] d\alpha \right\} \dots \dots \dots (29)$$

The simplicity of this calculation is evident. For example, a direct solution of influence lines as deflection lines for horizontally curved beams and frames was given in a paper by the writer,<sup>21</sup> in which, however, this analogy could speed up the solution.

For a large radius, R, and small central angle of individual spans, this beam with point support becomes inapplicable, because of the magnitude of the torsional moments and the instability for eccentric load in the limit case of a straight beam. It may also be noticed that for R tending to infinity, this solution does not tend to a solution of a straight continuous beam, but to different values. For instance,  $\lim A = 12/32 P$ , but for a straight continuous beam,  $A = 13/32 P$ .

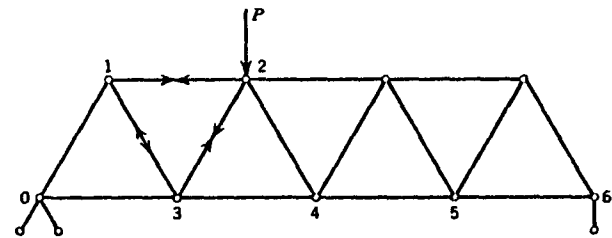


FIG. 6

The conjugate analogy is also valid for trusses (frameworks, lattices) made of bars with hinge joints and results in the well-known Mohr's method of fictitious loads used for determining their deflections with the help of funicular curves. Namely, in the truss in Fig. 6, the extension of chord member 12 causes a rotation in joint 3 to which a horizontal panel load in joint 3 is conjugated. Similarly, the extension of member 1 causes a relative vertical displacement between joints 2 and 3 to which a loading moment in the truss plane about the vertical axis or a couple of panel loads in joints 2 and 3, perpendicular to the truss plane, is conjugated. These conjugate loads, which are identical with Mohr's fictitious loads, have to be applied to beam 03456, which is

<sup>21</sup> Bažant, Z. P., "Influence Lines of Horizontally Curved Bridges," Slovakian Academy of Science, Bratislava, Czechoslovakia, Vol. XII, No. 1, 1964, p. 18 (in Czech, with English Summary).

conjugated with the bottom chord. (A strict procedure with cut off bar circuits is not advantageous in this instance.)

### CONCLUSIONS

Besides the principle of virtual work and the direct solution of pertinent differential equations between deformations and loads, the conjugate analogy of the geometry of small deformations with equilibrium conditions, is a further general method for determining the deformations of arbitrary bars, beams, and structures consisting of one-dimensional members. The deformations of the given structure are equal to the internal forces in the conjugate structure, i.e., a structure with conjugate supports and connections under the action of conjugate load.

*Practical Use.*—This analogy may serve for the computation of deformations, deflection lines, and influence lines for the determination of flexibility coefficients of the primary system in the force method and also in formulation of stability problems. It is in many cases simpler than other methods. Furthermore, it may also be used for determining the differential geometric equations. Besides simple plane beams, frames, and girder grillages, this analogy is especially useful for laterally loaded arches and frames, helicoidal beams, and for horizontally curved beams and frames that are rather important in highway bridge design.

If a general computer program for the calculation of the internal force distribution in a certain statically determinate structure is available, it may serve simultaneously for the computation of deformations of the conjugate structure as well as of all redundant structures, the primary system of which this conjugate structure represents.

### APPENDIX.—NOTATION

The following symbols are used in this paper:

- $\bar{a}_r, \bar{a}_s$  and  $\bar{o}_r, \bar{o}_s$  = components of reaction forces and reaction moments which can be transmitted by the support or connection = prevented directions of displacement and axes of rotation;
- $C(i)$  = concentrated loading couple (moment);
- $E, G$  = tensile and shear moduli;
- $F, F_x, F_y$  = area of cross section and reduced areas for determining the shear deformation;
- $I_x, I_y$  = principal centroidal moments of inertia of section;
- $I_k$  = moment of inertia in pure (simple) torsion;
- $I_p, I_\omega, I_{\omega x}$  = polar moment of inertia of section and moments of inertia due to warping;
- $\bar{k}, k_x, k_y, k_z$  = vector of change of curvature of axis of bar due to deformation and its components bending curvatures and specific angle of twist;

- $\bar{M}, M_x, M_y, M_z$  = internal moment vector in the section and its components, bending moments about the x, y axes and torsional moment (torque) about the tangent, z;
- $\bar{m}, m_x, m_y, m_z$  = axial vector of specific moment load and its components;
- $\bar{N}$  = vector of normal force;
- $n_a, n_\alpha$  = number of degrees of freedom in displacement or in rotation prevented by the support or connection;
- $\bar{P}(i)$  = concentrated (point) loads;
- $\bar{p}, p_x, p_y, p_z$  = vector of specific load and its components;
- $R$  = radius of curvature of bar (before loading);
- $\bar{r}$  = radius vector of the axis of bar,  $\bar{r} = \bar{r}(s)$ ;
- $\bar{S}$  = vector of internal force in the section,  $\bar{S} = \bar{T} + \bar{N}$ ;
- $s$  = length of the axis of bar;
- $\bar{T}, T_x, T_y$  = shear force and its components;
- $\bar{u}, u_x, u_y, u_z$  = vector of displacement and its components;
- $v(i)$  = relative displacement in elastic (nonrigid) connections or supports;
- $x, y, z$  = local cartesian coordinate system in the section of bar (right-handed);
- $z$  = tangent to the axis of bar;
- $\bar{\gamma}, \gamma_x, \gamma_y$  = shear angle and its components;
- $\bar{\epsilon}, \epsilon$  = specific elongation of bar in its axis;
- $\bar{\phi}, \phi_x, \phi_y, \phi_z$  = vector of rotation of the cross section and its components about the axes x, y, z;
- $\psi(i)$  = relative rotation in elastic (nonrigid) connections or restraints, plastic hinges, etc.;
- (\*) = quantities concerning the conjugate structure, i.e.,  $\bar{p}^*$ ,  $\bar{M}^*$ ,  $\bar{T}_x^*$ , etc.;
- $\cdot$  = scalar product of two vectors, e.g.,  $\bar{M}_O \cdot \bar{o}_r$ ; and
- $\times$  = vector product of two vectors, e.g.,  $\bar{k} \times \bar{r}$ .

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 CONJUGATE ANALOGY FOR SPACE STRUCTURES<sup>a</sup>


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 Closure
 

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ZDENĚK P. BAŽANT.<sup>26</sup>—The writer wishes to thank Mavis for his valuable addition to the survey of the literature, concerning the papers of Hardy Cross on virtual work, column analogy, and moment distribution. In these papers, Cross pointed out several aspects of the conjugate analogy and some of his ideas, e.g., column analogy, became widespread and significantly influence further development. For a more general and compact formulation, however, the vector (or matrix) formulation is unavoidable.

The writer does not claim that his survey of the literature is complete. Many additional sources may be found in the works cited, especially by Schleicher.<sup>5</sup> In this connection the writer also wishes to thank Laredo<sup>27</sup> for his note alluding to the book by Gheorghiu,<sup>28</sup> in which many special cases of analogy of frames and arches are treated (in a way similar to that of Dašek<sup>6</sup>); also, a vector static analogy for the differential equation of the deflection line of a plane bar without shears is quoted. We emphasize, however, that for general treatment, the integral rather than differential formulation is necessary, and the analogy of supports is the main problem. Many important cases of analogy were also introduced by Baron and Michalos.<sup>29</sup>

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 LIMIT DESIGN OF CONTINUOUS REINFORCED  
 CONCRETE CRANE GIRDERS<sup>b</sup>


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 Closure
 

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## 4849 CONJUGATE ANALOGY OF SPACE STRUCTURES

KEY WORDS: arches; bars; beams (structural); conjugate beam method; deflections; flexibility methods (structural); frames; influence lines; structural engineering

ABSTRACT: A theorem of conjugate analogy is derived which generalizes the well-known conjugate beam method for bars, beams, and structures consisting of one-dimensional members of arbitrary shape (including curved bars) and with arbitrary supports, connections, and joints (including nonrigid ones). It also applies to thin-walled bars, for tension and shear, and for inelastic materials. It develops from the analogy between the geometry of small deformations and the equilibrium condition which is presented in vectorial integral form. This analogy permits the determination of deformations, deflection lines, influence lines, flexibility coefficients, as well as differential relations between deformations according to the equilibrium conditions of the conjugate structure which is obtained by replacing all supports and connections with their conjugates and loaded by conjugate loads (corresponding to curvature changes, extensions and shear angles at deformation).

REFERENCE: Bazant, Zdeněk P., "Conjugate Analogy of Space Structures," *Journal of the Structural Division, ASCE*, Vol. 92, No. ST3, Proc. Paper 4849, June, 1966,

M. Z. COHN,<sup>9</sup> M. ASCE.—Gerstle's interest in the paper and his comments concerning Assumption 6 are appreciated.

<sup>a</sup> June, 1966, by Zdeněk P. Bažant (Proc. Paper 4849).

<sup>26</sup> Research Engr., Building Research Inst., Tech. Univ. of Prague, Czechoslovakia.

<sup>27</sup> Laredo, M., *Omnium Technique d' Etudes d' Habitation*, Paris, personal communications, December 9, 1966.

<sup>28</sup> Gheorghiu, A., "Conjugate Systems in Structural Analysis," Editura Technica, Bucharest, Rumania, 1957 (in Rumanian).

<sup>29</sup> Baron, F., and Michalos, J. P., "Laterally Loaded Plane Structures and Structures Curved in Space," *Proc. Ser. No. 51, ASCE*, Vol. 77, Jan., 1951.

<sup>b</sup> June, 1966, by M. Z. Cohn (Proc. Paper 4850).

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The method described in the paper implies that possible modes of collapse are restricted to individual spans, Figs. 1(b) and 1(c). While Assumption 6 has been adopted as a matter of convenience it is probably acceptable for the assumed modes of collapse. For collapse schemes involving more than one span, Assumption 6 becomes questionable.

From the theoretical point of view, Gerstle's remark is justified whenever mechanisms as in Fig. 9 (i.e., with two or more spans collapsing), can actually occur. However, it is doubtful that such mechanisms may become critical in practice, at least when the joint between the girder and the supporting corbel is monolithic or when the dead load is high enough to eliminate the negative moments in spans.

Indeed, in the first case a rigid rotation of the segment  $kj$  about support section 10 (Fig. 9) is prevented either by the torsional stiffness of the supporting corbel, or by the longitudinal bending stiffness of the column on line 10. It is more likely that girders built monolithically with their corbels and supporting columns will collapse in the modes sketched in Fig. 1 rather than in that of Fig. 9.

On the other hand, Eqs. 2 and 6 and Figs. 5 to 7 indicate that to have negative plastic hinges in spans,  $a_i w L^2 + b_i^{(-)} W L < 0$ . Only for those sections for which this condition is satisfied can a negative plastic hinge occur. It is thus apparent that for some cases (large spans, small lifting loads) the positive dead load moment may be in excess of the negative live load moment, which will make it impossible for a mechanism of the type in Fig. 9 to form.

Validity of Assumption 6 is thus dependent on a realistic assessment of possible collapse modes and corresponding limit criteria for the typical crane girders studied in the paper. It is believed that the investigation of actual mechanism failures of a sufficient number of small-scale models reproducing the main parameters in crane girders design could offer a sound basis for checking the assumptions adopted. It is hoped that a better knowledge of actual failure modes and of corresponding limit criteria will contribute to developing further the method described in the paper.

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## BEHAVIOR OF REINFORCED CONCRETE COLUMNS WITH SIDESWAY<sup>a</sup>

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### Closure

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EDWARD O. PFRANG,<sup>b</sup> M. ASCE.—The writer wishes to thank Erdem for his valuable comments concerning the application of the results of this in-

<sup>a</sup> June, 1967, by Edward O. Pfrang, M. ASCE, and John A. Wilson, Jr., M. ASCE.

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