

# **STABILITY OF STRUCTURES**

**Elastic, Inelastic, Fracture,  
and Damage Theories**

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## Preface

It is our hope that this book will serve both as a textbook for graduate courses on stability of structures and a reference volume for engineers and scientists. We assume the student has a background in mathematics and mechanics only at the level of the B.S. degree in civil or mechanical engineering, though in the last four chapters we assume a more advanced background. We cover subjects relevant to civil, structural, mechanical, aerospace, and nuclear engineering, as well as materials science, although in the first half of the book we place somewhat more emphasis on the civil engineering applications than on others. We include many original derivations as well as some new research results not yet published in periodicals.

Our desire is to achieve understanding rather than just knowledge. We try to proceed in each problem from special to general, from simple to complex, treating each subject as concisely as we can and at the lowest possible level of mathematical apparatus we know, but not so low as to sacrifice efficiency of presentation. We include a large number (almost 700) of exercise problems. Solving many of them is, in our experience, essential for the student to master the subject.

In some curricula, the teaching of stability is fragmented into courses on structural mechanics, design of steel structures, design of concrete structures, structural dynamics, plates and shells, finite elements, plasticity, viscoelasticity, and continuum mechanics. Stability theory, however, stands at the heart of structural and continuum mechanics. Whoever understands it understands mechanics. The methods of stability analysis in various applications are similar, resting on the same principles. A fundamental understanding of these principles, which is not easy to acquire, is likely to be sacrificed when stability is taught by bits, in various courses. Therefore, in our opinion, it is preferable to teach stability in a single course, which should represent the core of the mechanics program in civil, mechanical, and aerospace engineering.

Existing textbooks of structural stability, except for touching on elastoplastic columns, deal almost exclusively with elastic stability. The modern stability problems of fracture and damage, as well as the thermodynamic principles of stability of irreversible systems, have not been covered in textbooks. Even the catastrophe theory, as general as it purports to be, has been limited to systems that possess a potential, which implies elastic behavior. Reflecting recent research results, we depart from tradition, devoting about half of the book to nonelastic stability.

Various kinds of graduate courses can be fashioned from this book. The first-year quarter-length course for structural engineering students may, for example, consist of Sections 1.2–1.7, 2.1–2.4, 2.8, 3.1, 3.2, 3.5, 3.6, 4.2–4.6,

5.1–5.4, 6.1–6.3, 7.1–7.3, 7.5, 7.8, 8.1, 8.3, and 8.4, although about one-third of these sections can be covered in one quarter only partly. A semester-length course can cover them fully and may be expanded by Sections 1.8, 1.9, 2.7, 3.3, 4.5, 4.6, 5.5, 7.4, 7.8, 8.2, and 8.6. The first-year course for mechanical and aerospace engineers may, for example, be composed of Sections 1.1–1.5, 1.7, 1.9, 2.1–2.3, 3.1–3.7, 4.2–4.6, 5.1–5.4, 6.1–6.3, 7.1–7.3, 7.5, 7.8, 8.1–8.3, and 9.1–9.3, again with some sections covered only partly. A second-year sequel for structural engineering students, dealing with inelastic structural stability, can, for example, consist of Sections 8.1–8.6, 9.1–9.6, 10.1–10.4, 13.2–13.4, and 13.6, preceded as necessary by a review of some highlights from the first course. Another possible second-year sequel, suitable for students in theoretical and applied mechanics, is a course on material modeling and stability, which can be set up from Sections 11.1–11.7, 10.1–10.6, 13.1–13.4, 13.8–13.10, and 12.1–12.5 supplemented by a detailed explanation of a few of the constitutive models mentioned in Section 13.11. A course on Stability of Thin-Wall Structures (including plates and shells) can consist of a review of Sections 1.1–1.8 and detailed presentation of Chapters 6 and 7. A course on Inelastic Columns can be based on a review of Sections 1.1–1.8 and detailed presentation of Chapters 8 and 9. A course on Stability of Multidimensional Structures can be based on a review of Sections 1.1–1.9 and detailed presentation of Chapters 7 and 11. A course on Energy Approach to Structural Stability can be based on a review of Sections 1.1–1.8 and detailed presentation of Chapters 4, 5, and 10. A course on Buckling of Frames can be based on Chapters 1, 2, and 3. Chapter 3, along with Section 8.6, can serve as the basis for a large part of a course on Dynamic Stability.

The present book grew out of lecture notes for a course on stability of structures that Professor Bažant has been teaching at Northwestern University every year since 1969. An initial version of these notes was completed during Bažant's Guggenheim fellowship in 1978, spent partly at Stanford and Caltech. Most of the final version of the book was written during Professor Cedolin's visiting appointment at Northwestern between 1986 and 1988, when he enriched the text with his experience from teaching a course on structural analysis at Politecnico di Milano. Most of the last six chapters are based on Bažant's lecture notes for second-year graduate courses on inelastic structural stability, on material modeling principles, and on fracture of concrete, rock, and ceramics. Various drafts of the last chapters were finalized in connection with Bažant's stay as NATO Senior Guest Scientist at the Ecole Normale Supérieure, Cachan, France, and various sections of the book were initially presented by Bažant during specialized intensive courses and guest seminars at the Royal Institute of Technology (Cement och Betonginstitutet, CBI), Stockholm; Ecole des Ponts et Chaussées, Paris; Politecnico di Milano; University of Cape Town; University of Adelaide; University of Tokyo; and Swiss Federal Institute of Technology. Thanks go to Northwestern University and the Politecnico di Milano for providing environments conducive to scholarly pursuits. Professor Bažant had the good fortune to receive financial support from the U.S. National Science Foundation and the Air Force Office of Scientific Research, through grants to Northwestern University; this funding supported research on which the last six chapters are partly based. Professor Bažant wishes to express his thanks to his father, Zdeněk J. Bažant, Professor Emeritus of Foundation Engineering at the

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