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LONG-WAVE EXTENSIONAL BUCKLING OF LARGE REGULAR FRAMES

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Buckling of rectangular building frames is customarily analyzed under the assumption that the effect of axial extensions is negligible. The structure can then buckle only in modes whose wave-length is short, i.e., of the order column length. The negligence of axial extensions is certainly justified if the building is not too high as compared with its width. However, the limiting height is not known precisely. Thus, in view of the permanent trend to taller and slenderer buildings, the question of safety against overall, long-wave buckling modes, in which the columns undergo significant extensions or shortenings, the floors tilt and the whole building frame buckles as a single column, gains in importance.

Buckling modes with axial extensions have been studied for 1-bay frames modeling built-up columns (7,8,9,14,19,21), in which case the problem leads directly to one-dimensional difference equations. A computer program capable of handling incremental deflections of frames under axial forces (including plastic hinge formation) has been developed by Korn and Galambos (18) who also solved several examples of 1-bay and 2-bay multistory frames. A method of numerical analysis of buckling with axial extensions of very large frames was reported in Ref. 3, with an example of a 52-story 11-bay frame.

The numerical approach, however, does not suffice to reach general results and good understanding of the phenomenon. But it seems that no general analytical study has yet been devoted to the long-wave buckling modes of multistory multibay frames. Therefore, such a study has been chosen as the objective of this study. Attention will be restricted to regular planar unbraced rectangular frames with a rectangular boundary and uniform spacing of joints in the horizontal and vertical directions. Furthermore, to make field solutions feasible, a somewhat idealized problem will be solved, in which all horizontal members are identical to each other and carry equal axial forces, and the same is true of all vertical members. In addition, both the base and the top boundaries will be considered to be rigidly supported in the vertical directions. However, it will be shown that the solutions are in approximate correlation with the practical case of a free standing frame.

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The solution will be carried out exactly, using the methods of finite difference calculus (17), whose concise description may be found in the book by Wah and Calcote (22), along with applications to inextensible frames and a detailed literature survey. The finite difference calculus has already been applied successfully in a number of problems of stress and stability analysis of regular frames and built-up columns (7,8,9,10,11,12,14,21).

The problem which will be analyzed in the sequel is not directly applicable to frames with heavy diagonal bracing and further extension would be required to include this case. However, at present very rigid frames without diagonal bracing appear to be the most economic means of providing adequate lateral stiffness, even for the highest steel buildings presently constructed (Sears Tower in Chicago, World Trade Center in New York).

MATHEMATICAL FORMULATION OF THE PROBLEM

A planar rectangular frame is considered to be initially in equilibrium under axial forces P_y^0 in all columns and P_x^0 in all beams (although in practical cases usually $P_x^0 = 0$). Subsequently, the initial equilibrium is disturbed by infinitely small load increments so that the joints undergo small displacements u and v in the horizontal, x, and vertical y directions and small rotations ϕ (positive if counterclockwise); see Fig. 1. If all members of each direction have the same properties, the conditions of equilibrium of incremental horizontal forces, vertical forces and moments acting on interior joint r, s lead to the equations (cf. Ref. 3):

$$E_x'(u_{r+1,s} - 2u_{r,s} + u_{r-1,s}) + \frac{k_y s_y'}{L_y}(\phi_{r,s+1} - \phi_{r,s-1})$$

$$+ \frac{k_y s_y''}{L_y^2}(u_{r,s+1} - 2u_{r,s} + u_{r,s-1}) + f_{x_{r,s}} = 0 \qquad (1a)$$

$$E_y'(v_{r,s+1} - 2v_{r,s} + v_{r,s-1}) - \frac{k_x s_x'}{L_x}(\phi_{r+1,s} - \phi_{r-1,s})$$

$$+ \frac{k_x s_x''}{L_x^2}(v_{r+1,s} - 2v_{r,s} + v_{r-1,s}) + f_{y_{r,s}} = 0 \qquad (1b)$$

$$k_x s_x c_x(\phi_{r+1,s} - 2\phi_{r,s} + \phi_{r-1,s}) + 2k_x s_x'\phi_{r,s}$$

$$- \frac{k_x s_x'}{L_x}(v_{r+1,s} - v_{r-1,s}) + k_y s_y c_y(\phi_{r,s+1} - 2\phi_{r,s} + \phi_{r,s-1})$$

$$+ 2k_y s_y'\phi_{r,s} + \frac{k_y s_y'}{L_y}(u_{r,s+1} - u_{r,s-1}) - m_{r,s} = 0 \qquad (1c)$$

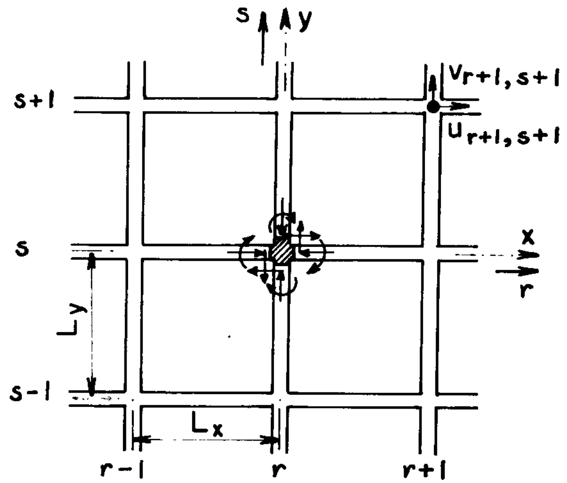

FIG. 1.—NUMBERING OF JOINTS AND FORCES ACTING ON JOINT

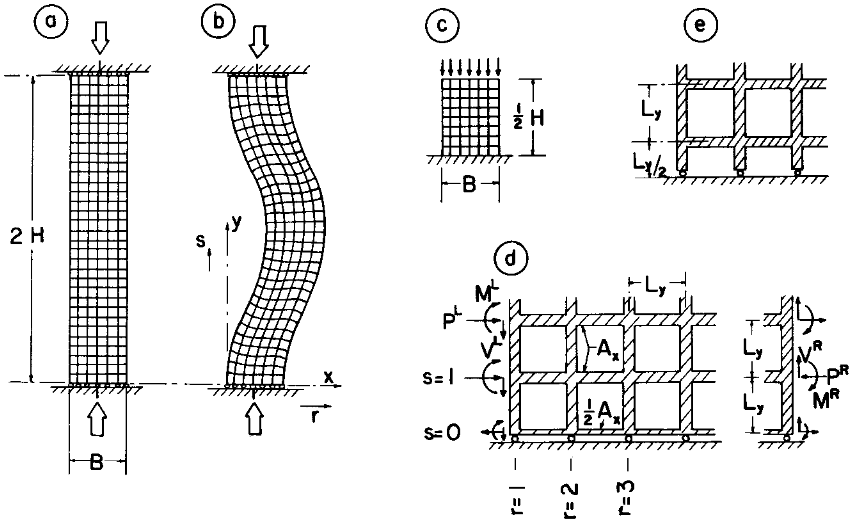

FIG. 2.—(*a*) GEOMETRY OF FRAME SOLVED; (*b*) LONG-WAVE BUCKLING MODE; (*c*) FREE STANDING FRAME OF APPROXIMATELY SAME INITIAL LOAD; (*d*) DETAIL OF BOUNDARIES OF FRAME SOLVED; (*e*) ALTERNATE BOUNDARY SUPPORT WHICH COULD BE ANALYZED SIMILARLY

in which subscripts x and y refer to members in the horizontal and vertical directions and subscripts r and s to the numbers of the joint from the left vertical side and from the base; L_x, L_y = length of horizontal and vertical members; $E'_x = EA_x/L_x$, $E'_y = EA_y/L_y$ in which A_x, A_y = cross-sectional areas; $k_x = EI_x/L_x$, $k_y = EI_y/L_y$ in which I_x, I_y = cross-sectional moments of inertia; f_x, f_y, m = small incremental forces and moment applied at the joint; and s_x, c_x or s_y, c_y = the well-known stability functions of P^0_x or P^0_y, which are expressed, in the case of constant cross section, as (1)

$$s_x = \frac{\alpha(\sin\alpha - \bar{\alpha}\cos\alpha)}{2 - 2\cos\alpha - \bar{\alpha}\sin\alpha};\quad c_x = \frac{\bar{\alpha} - \sin\alpha}{\sin\alpha - \bar{\alpha}\cos\alpha};\quad P^0 > 0 \qquad (2a)$$

$$s_x = \frac{\alpha(\bar{\alpha}\,\mathrm{ch}\,\alpha - \mathrm{sh}\,\alpha)}{2 - 2\,\mathrm{ch}\,\alpha + \bar{\alpha}\,\mathrm{sh}\,\alpha};\quad c_x = \frac{\mathrm{sh}\,\alpha - \bar{\alpha}}{\bar{\alpha}\,\mathrm{ch}\,\alpha - \mathrm{sh}\,\alpha};\quad P^0 < 0 \qquad (2b)$$

while the expressions for s_y, c_y are analagous; $\alpha = \sqrt{|P^0_x|L_x/(k_x\beta)}$; $\bar{\alpha} = \alpha\beta$; $\beta = 1 - P^0_x/G_xA_x$; P^0_x is negative for tension; and G_xA_x = the shear rigidity of the cross section. Eqs. 2*a* and 2*b* take into account the deflections of member due to shear forces. For slender members these may be neglected ($G_xA_x \rightarrow \infty$), as has been done in all numerical studies reported herein. For this case the s_x and c_x functions (15) have been first introduced by Goldberg (13). (For $P^0_x = 0$, $s_x = 4$, $c_x = 1/2$.) Eqs. 1*a*, 1*b*, and 1*c* also hold when the cross section is variable within each member, but the values of s_x and c_x must then be in general determined by numerical integration of the differential equation for bending. Finally

$$s'_x = s_x(1 + c_x);\quad s''_x = 2s'_x - \frac{P^0_x L_x}{k_x} \qquad (3)$$

Now consider incremental deformation modes which are sinusoidal in the vertical direction (Fig. 2)

$$\left.\begin{aligned} u_{r,s} &= U_c - U_r\cos\gamma s \\ v_{r,s} &= V_r\sin\gamma s \\ \phi_{r,s} &= R_r\sin\gamma s \end{aligned}\right\} \qquad (4)$$

in which U_r, V_r, R_r are discrete functions of only one integer subscript r; and U_c is a constant which can be determined from the manner of fixation of the frame against lateral sliding. [It would be an easy extension to consider arbitrary nonsinusoidal deformation. Expanding the displacements (as well as the incremental loads) in a finite Fourier series, one finds that, because of linearity of the problem, each sine component of the series may be analysed separately, as is done here, and the solutions superimposed. The lowest critical load can be shown to be dominated by a single sine component, the one of the longest wavelength.]: It can easily be verified that the deformation modes given by Eq. 4 satisfy the conditions

$$\left.\begin{array}{l} v = \phi = 0 \ \text{for any} \ r \\ u = 0 \ \text{for} \ r = c \end{array}\right\} \qquad (5)$$

and $s = 0$ ($y = 0$) or $s = 2n_y(y = 2H)$, in which $n_y = \pi/\gamma$ will be assumed to be an integer, denoting the number of floors contained within the half-wavelength H. In addition, Eq. 4 satisfy a certain condition on shear force V_y in vertical members. Generally, V_y is expressed (cf. Ref. 3) as

$$V_{y_{r,s}} = \frac{k_y s_y''}{L_y^2}(u_{r,s+1} + u_{r,s}) + \frac{k_y s_y'}{L_y}(\phi_{r,s} + \phi_{r,s+1}) \qquad (6a)$$

Considering the average of the shear forces in two consecutive members, $\overline{V}_{y_{r,s}} = (1/2)\left(V_{y_{r,s-1}} + V_{y_{r,s}}\right)$, it is possible to find from Eqs. 6*a* and 4 that

$$\overline{V}_y = 0 \ \text{for any} \ r; \ s = 0 \ \text{or} \ s = 2n_y \qquad (6b)$$

Physically, $\overline{V}_y$ represents the shear force per length L_x transmitted along a horizontal (longitudinal) section of horizontal members passed through their neutral axis.

Therefore, Eqs. 5 and 6*b* represent the boundary conditions of the base and top of a frame with a rectangular boundary and height $2H = 2n_yL_y$ as shown in Fig. 2. The base and top boundaries are supported against rigid half-planes producing the initial axial forces P_y^0 in vertical columns; they slide without friction while the central joint is fixed. The horizontal beams at the base and top extremities have half the cross-sectional area, $A_x/2$, as compared with all interior members A_x, as in Fig. 2(*d*). (In reality, the solution would be almost exact for any cross section areas of the horizontal boundary beams, because the horizontal incremental axial forces are negligible.) The moment of inertia of these beams may have any value because, according to Eq. 4, the boundary joints do not rotate. It should be noted that a frame supported at the boundary as shown in Fig. 1(*e*) could be solved exactly in a similar manner.

The joints at the left and right boundary of the frame will be considered as free and the incremental loads applied on them as specified. The boundary conditions are formulated most conveniently if the frame is imagined to be extended beyond the actual boundary, adding vertical rows of fictitious joints $r = 0$ and $r = m + 1$. This has the advantage that the boundary joints $r = 1$ and $r = m = n_x + 1$ can be treated as interior joints of the frame, for which Eqs. 1*a*, 1*b*, and 1*c* apply. The internal forces transmitted into the joints $r = 1$ and $r = m$ by the added fictitious horizontal beams extending beyond the actual boundary must then be made equal to the prescribed incremental loads applied at the boundary joints. Thus

$$E_x'(u_{0,s} - u_{1,s}) = P^L; \ E_x'(u_{m,s} - u_{m+1,s}) = P^R;$$

$$k_x s_x(c_x\phi_{0,s} + \phi_{1,s}) - \frac{k_x s_x'}{L_x}(v_{1,s} - v_{0,s}) = M^L;$$

$$k_x s_x(\phi_{m,s} + c_x\phi_{m+1,s}) - \frac{k_x s_x'}{L_x}(v_{m+1,s} - v_{m,s}) = M^R;$$

$$\frac{k_x s_x''}{L_x^2}(v_{1,s} - v_{0,s}) - \frac{k_x s_x'}{L_x}(\phi_{0,s} + \phi_{1,s}) = V^L;$$

$$\frac{k_x s_x''}{L_x^2}(v_{m+1,s} - v_{m,s}) - \frac{k_x s_x'}{L_x}(\phi_{m,s} + \phi_{m+1,s}) = V^R \qquad (7)$$

in which the left-hand sides are the expressions for the preceding internal forces (3) and P^L, V^L, M^L or P^R, V^R, M^R are the incremental horizontal force, vertical force and moment applied at the left or right boundary joint, considered positive in the same sense as the axial force, shear force and bending moment which replace them [Fig. 2(*d*)].

The problem to be solved, physically representing the sinusoidal buckling of the frame shown in Fig. 2, is thus mathematically formulated by three second-order partial finite difference equations, Eqs. 1, and boundary conditions Eq. 7 with $P^L = V^L = \ldots = M^R = 0$ and a prescribed class of solutions of form of Eq. 4. The smallest magnitude of the initial axial forces P_y^0 for which non-zero solutions exist is to be found (while P_x^0 is considered as fixed, in most practical cases as zero).

CONVERSION TO ORDINARY FINITE DIFFERENCE EQUATIONS

Deformation modes of form of Eq. 4 have been introduced because for zero loads at interior joints, $f_x = f_y = 0$, (as well as for sinusoidally distributed joint loads) they satisfy Eq. 1 identically with respect to discrete variables, leaving the following system of three simultaneous second-order ordinary finite difference equations:

$$-E_x'(U_{r+1} - 2U_r + U_{r-1}) + \frac{k_y s_y'}{L_y}(2\sin\gamma)R_r$$

$$-\frac{k_y s_y''}{L_y} 2(\cos\gamma - 1)U_r = 0 \qquad (8a)$$

$$E_y' \, 2(\cos\gamma - 1)V_r - \frac{k_x s_x'}{L_x}(R_{r+1} - R_{r-1})$$

$$+\frac{k_x s_x''}{L_x}(V_{r+1} - 2V_r + V_{r-1}) = 0 \qquad (8b)$$

$$k_x s_x c_x(R_{r+1} - 2R_r + R_{r-1}) + (2k_x s_x' + 2k_y s_y')R_r$$

$$+ k_y s_y c_y \, 2(\cos\gamma - 1)R_r + \frac{k_y s_y'}{L_y}(2\sin\gamma)U_r$$

$$-\frac{k_x s_x'}{L_x}(V_{r+1} - V_{r-1}) = 0 \qquad (8c)$$

For the analytical solution it is convenient to convert these equations to a system of six first-order equations. To this end, new discrete functions may be defined as

$$F_{1_r} = U_{r-1};\; F_{3_r} = V_{r-1};\; F_{5_r} = R_{r-1};$$

$$F_{2_r} = U_r;\; F_{4_r} = V_r;\; F_{6_r} = R_r \quad \text{.......................} \quad (9)$$

Next, U_{r+1} may be expressed explicitly from Eq. 8*a* and the system of Eqs. 8*b* and 8*c* may be solved for V_{r+1} and R_{r+1}. Finally, appending three finite difference equations $F_{1_{r+1}} = F_{2_r}$, $F_{3_{r+1}} = F_{4_r}$, $F_{5_{r+1}} = F_{6_r}$ which follow from Eq. 9:

$$\begin{Bmatrix} F_{1_{r+1}} \\ F_{2_{r+1}} \\ F_{3_{r+1}} \\ F_{4_{r+1}} \\ F_{5_{r+1}} \\ F_{6_{r+1}} \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & a_{22} & 0 & 0 & 0 & a_{26} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} \end{bmatrix} \begin{Bmatrix} F_{1_r} \\ F_{2_r} \\ F_{3_r} \\ F_{4_r} \\ F_{5_r} \\ F_{6_r} \end{Bmatrix} \quad \text{.......} \quad (10)$$

in which $a_{22} = 2\left[\dfrac{k_y s''_y}{L_y^2 E'_x}(1 - \cos\gamma) + 1\right]$; $a_{26} = \dfrac{2k_y s'_y}{L_y E'_x} \sin\gamma$;

$$a_{42} = \frac{2L_x s'_x k_y s'_y \sin\gamma}{L_y k_x (s'^2_x - s''_x s_x c_x)};\quad a_{43} = \frac{s'^2_x + s''_x s_x c_x}{s'^2_x - s''_x s_x c_x};$$

$$a_{44} = \frac{-2s''_x s_x c_x}{(s'^2_x - s''_x s_x c_x)}\left[\frac{E'_y L_x^2}{k_x s''_x}(1 - \cos\gamma) + 1\right];$$

$$a_{45} = \frac{2L_x s'_x s_x c_x}{(s'^2_x - s''_x s_x c_x)};\quad a_{46} = \frac{2L_x s'_x (k_x s_x + k_y s_y + k_y s_y c_y \cos\gamma)}{k_x (s'^2_x - s''_x s_x c_x)};$$

$$a_{62} = \frac{2k_y s'_y s''_x \sin\gamma}{L_y k_x (s'^2_x - s''_x s_x c_x)};\quad a_{63} = \frac{2s''_x s'_x}{L_x (s'^2_x - s''_x s_x c_x)};$$

$$a_{64} = -\frac{2s'_x}{L_x k_x}\left[\frac{E'_y L_x^2 (1 - \cos\gamma) + k_x s''_x}{(s'^2_x - s''_x s_x c_x)}\right];\quad a_{65} = \frac{s'^2_x + s''_x s_x c_x}{s'^2_x - s''_x s_x c_x};$$

$$a_{66} = \frac{2s''_x (k_x s_x + k_y s_y + k_y s_y c_y \cos\gamma)}{k_x (s'^2_x - s''_x s_x c_x)} \quad \text{.................} \quad (11)$$

Eq. 10 represents a system of six linear first-order finite difference equations in the canonical (standard) form. (Note that while the second-order Eqs. 8*a*, 8*b*, and 8*c* are symmetrical, these equations are not.)

The method of solution of the linear ordinary finite difference equations is

completely analogous to that for linear ordinary differential equations (17,22). Because Eq. 10 has constant coefficients, their general solutions may be assumed to be composed of functions of form

$$F_1(r) = K_1\rho^r,\quad F_2(r) = K_2\rho^r,\; \ldots,\; F_6(r) = K_6\rho^r \quad \text{...........} \quad (12)$$

in which ρ, K_1, . . ., K_6 are constants.

Substitution in Eq. 10 yields a system of linear homogeneous algebraic equations for K_1, K_2, . . ., K_6 whose determinant must vanish, i.e.

$$\begin{vmatrix} -\rho & 1 & 0 & 0 & 0 & 0 \\ -1 & (a_{22} - \rho) & 0 & 0 & 0 & a_{26} \\ 0 & 0 & -\rho & 1 & 0 & 0 \\ 0 & a_{42} & a_{43} & (a_{44} - \rho) & a_{45} & a_{46} \\ 0 & 0 & 0 & 0 & -\rho & 1 \\ 0 & a_{62} & a_{63} & a_{64} & a_{65} & (a_{66} - \rho) \end{vmatrix} = 0 \quad \text{....} \quad (13)$$

This is a characteristic equation which has six roots $\rho = \rho_1, \rho_2, \ldots, \rho_6$ which are in general complex (but which for most of the cases studied happened to be real). They will be assumed to be all distinct, i.e., no double roots among them. This has been found to occur in all of the practical cases computed. However, the program was set up so as to abandon the given case and solve a slightly different one if a double root or nearly double root was encountered. The case of multiple roots could, of course, be incorporated into the program, but the effort on the part of the programmer would not be worthwhile. The general solution of Eq. 10 is then expressed as

$$F_i(r) = C_1 K_i^1 \rho_1^r + C_2 K_i^2 \rho_2^r + \ldots + C_6 K_i^6 \rho_6^r \qquad i = 1, 2, \ldots, 6 \qquad (14)$$

in which C_1, C_2, . . ., C_6 are arbitrary constants to be determined from boundary conditions Eq. 7*a* on the sides and $(K_1^j, K_2^j, \ldots, K_6^j)$ is the *j*th eigenvector of the matrix Eq. 13, corresponding to root ρ_j. The eigenvalues, eigenvectors and constants C_j are in general complex. Noting that $F_1(r + 1) = F_2(r)$, $F_3(r + 1) = F_4(r)$, $F_5(r + 1) = F_6(r)$, it is seen that the eigenvectors must be such that

$$K_1^j \rho_j = K_2^j,\; K_3^j \rho_j = K_4^j,\; K_5^j \rho_j = K_6^j \quad \text{.....................} \quad (15)$$

BOUNDARY CONDITIONS AND COMPUTATION OF CRITICAL LOADS

In view of the symmetry of the frame considered, attention may be restricted to symmetric and antisymmetric deformation modes. The symmetric buckling mode (bulging), which involves significant extensions of horizontal beams, would obviously lead to much higher critical values of P_y^0 than the antisymmetric buckling mode shown in Fig. 2(*b*). Thus, only the antisymmetric buckling modes will be investigated in the sequel. Function V_r must then be antisymmetric with respect to the vertical axis of symmetry of the frame, while functions U_r and R_r must be symmetric.

Although the ultimate aim is to find P_y^0 for which deformation is possible at zero incremental loads at the left and right boundaries, it is helpful, as will be seen later, to consider in general nonzero horizontal incremental loads P^L and P^R. In accordance with the antisymmetry of the buckling modes sought, the incremental loads at the left and right boundaries will be assumed in the form

$$P^L = P^* \cos \gamma s; \; P^R = - P^* \cos \gamma s;$$

$$V^L = V^R = M^L = M^R = 0 \quad \text{(16)}$$

which satisfies the condition of antisymmetry and agrees with the assumption of sinusoidal distributions, as in Eq. 4. Substituting Eqs. 4 and 16 into the boundary conditions of Eq. 7, functions $\sin \gamma s$ and $\cos \gamma s$ can be eliminated, and the following conditions ensue:

$$E_x'(U_1 - U_0) = \frac{P^L}{\cos \gamma s} = P^* \quad \text{(17a)}$$

$$E_x'(U_{m+1} - U_m) = \frac{P^R}{\cos \gamma s} = - P^* \quad \text{(17b)}$$

$$k_x s_x(R_1 + c_x R_0) - \frac{k_x s_x'(V_1 - V_0)}{L_x} = \frac{M^L}{\sin \gamma s} = 0 \quad \text{(17c)}$$

$$k_x s_x(R_m + c_x R_{m+1}) - \frac{k_x s_x'(V_{m+1} - V_m)}{L_x} = \frac{M^R}{\sin \gamma s} = 0 \quad \text{(17d)}$$

$$\frac{k_x s_x''(V_1 - V_0)}{L_x^2} - \frac{k_x s_x'(R_1 + R_0)}{L_x} = \frac{V^L}{\sin \gamma s} = 0 \quad \text{(17e)}$$

$$\frac{k_x s_x''(V_{m+1} - V_m)}{L_x^2} - \frac{k_x s_x'(R_m + R_{m+1})}{L_x} = \frac{V^R}{\sin \gamma s} = 0 \quad \text{(17f)}$$

Note that these equations satisfy the symmetry and antisymmetry properties required.

When all incremental loads at the boundary joints are zero, $P^* = 0$, the boundary conditions (Eq. 17) yield a system of six linear homogeneous equations for $C_1, C_2, \ldots, C_6$. The critical values of P_y^0 are then characterized by the condition that the determinant of these equations must be zero for buckling to occur. The coefficients of the determinant, however, depend on P_y^0 in a very complicated manner; they are functions of the eigenvalues and eigenvectors of the matrix (Eq. 10) which do not possess explicit expressions and which, according to Eq. 11, depend nonlinearly on s_y, c_y, s_y', s_y''. These in turn are also nonlinear functions of P_y^0. Thus, although the present problem has the nature of an eigenvalue problem, it is much more complicated than its usual form.

For this reason a trial-and-error procedure must be adopted in order to determine the value of P_y^0 for which incremental deformation is possible with-

out any incremental loads. The joints lying on the vertical axis of symmetry (or adjacent to it) will be forced to undergo incremental horizontal displacements:

$$u_{c,s} = 1 - \cos \gamma s \text{ or } U_c = 1 \text{ for } c = \frac{1}{2}(m + 1), m \text{ odd};$$

$$\frac{1}{2}(u_{c,s} + u_{c+1,s}) = 1 - \cos \gamma s \text{ or } \frac{1}{2}(U_c + U_{c+1}) = 1$$

$$\text{for } c = \frac{m}{2}, m \text{ even} \quad \text{(18)}$$

in which m is the number of column lines.

The value of P^*, the amplitude of incremental horizontal loads P^L and P^R, which need to be applied to enforce these displacements, is a continuous and smooth function of P_y^0. It must vanish if and only if P_y^0 assumes the critical value. This value may thus be found by trying various values of P_y^0 in a procedure which is analogous to the numerical solution of nonlinear algebraic equations by the method regula falsi.

The condition (Eq. 18) now replaces the boundary condition of Eq. 17*a*. Instead of Eq. 17*b*, the condition of symmetry

$$U_1 - U_0 = U_m - U_{m+1} \quad \text{(19)}$$

has to be imposed. Substituting Eq. 14 into Eqs. 17*c* to 17*f*, 18 and 19 leads to the following system of linear nonhomogeneous algebraic equations for $C_1, \ldots, C_6$:

$$\left.\begin{aligned} \sum_{j=1}^{6} b_{ij} C_j &= 0 \text{ for } i = 1, \ldots, 5 \\ &= 1 \text{ for } i = 6 \end{aligned}\right\} \quad \text{(20)}$$

$$\left.\begin{aligned}
\text{in which } b_{1j} &= \left[c_x K_5^j + K_6^j - (K_4^j - K_3^j)\frac{s_x'}{s_x L_x}\right]\rho_j \\
b_{2j} &= \left[K_5^j + c_x K_6^j - (K_4^j - K_3^j)\frac{s_x'}{s_x L_x}\right]\rho_j^{m+1} \\
b_{3j} &= \left[K_4^j - K_3^j - (K_5^j - K_6^j)\frac{s_x' L_x}{s_x''}\right]\rho_j \\
b_{4j} &= \left[K_4^j - K_3^j - (K_5^j + K_6^j)\frac{s_x' L_x}{s_x''}\right]\rho_j^{m+1} \\
b_{5j} &= (K_2^j - K_1^j)(\rho_j + \rho_j^{m+1}) \\
b_{6j} &= K_2^j \rho_j^{(m+1)/2} \text{ for } m \text{ odd} \\
&= \frac{1}{2}(K_1^j + K_2^j)\rho_j^{1+m/2} \text{ for } m \text{ even}
\end{aligned}\right\} \quad \text{(21)}$$

The value of P^* necessary to sustain the enforced deformation follows by substitution of Eq. 14 into Eq. 17a:

$$P^* = E_x' \sum_{j=1}^{6} (K_2^j - K_1^j)\rho_j C_j \quad (22)$$

To compute the P^*-value for a given P_y^0, first the roots and eigenvectors of Eq. 13 must be found, then solve Eq. 20 and finally evaluate the expression in Eq. 22. The critical value of P_y^0 may then be obtained in a manner analogous to the regula falsi method by repeating the preceding solution for various chosen values of P_y^0 until P^* becomes negligible. This solution has been programmed using complex FORTRAN variables where appropriate, e.g., for ρ_j, K_i^j, C_j, b_{ij}. The characteristic roots, ρ_j, and the eigenvectors have been computed using the standard library subroutines. In practical computation, caution is needed not to use an excessive change in P_y^0, because the lowest critical value could be missed. Nevertheless, after some computing experience, only about four analyses for four different values of P_y^0 were necessary to find the lowest critical value of P_y^0 with about seven digits exact.

NONDIMENSIONAL PARAMETERS AFFECTING BUCKLING LOAD

The critical value, $P^0_{y\,crit}$, cannot depend on all of the parameters appearing in the foregoing equations, but only their nondimensional ratios. Using nondimensional length coordinates $\xi = x/B$ and $\eta = y/H$ in which x and y are the horizontal and vertical cartesian coordinates, and nondimensional displacements $\bar{u} = u/B$, $\bar{v} = v/H$, Eq. 1 for $f_x = f_y = m = 0$ may be brought by algebraic transformations to

$$\frac{H}{B}\,\frac{L_x^2}{r_x^2}\,\frac{k_x}{k_y}\,\Delta_{rr}\bar{u} + 2s_y'\,\Delta_s\phi + s_y''\,\frac{B}{H}\,\Delta_{ss}\bar{u} = 0 \quad (23a)$$

$$\frac{B}{H}\,\frac{L_y^2}{r_y^2}\,\frac{k_y}{k_x}\,\Delta_{ss}\bar{v} - 2s_x'\,\Delta_r\phi + s_x''\,\frac{H}{B}\,\Delta_{rr}\bar{v} = 0 \quad (23b)$$

$$s_y'\,\frac{B}{H}\,\frac{k_y}{k_x}\,\Delta_s\bar{u} - s_x'\,\frac{H}{B}\,\Delta_r\bar{v} + \left(s_x' + \frac{k_y}{k_x}\,s_y'\right)\phi$$

$$+ \frac{s_x c_x}{2n_x^2}\,\Delta_{rr}\phi + \frac{s_y c_y}{2n_y^2}\,\frac{k_y}{k_x}\,\Delta_{ss}\phi = 0 \quad (23c)$$

in which the notations

$$\Delta_{rr}\bar{u} = \frac{\bar{u}_{r+1,s} - 2\bar{u}_{r,s} + \bar{u}_{r-1,s}}{\Delta\xi^2};$$

$$\Delta_{ss}\bar{u} = \frac{\bar{u}_{r,s+1} - 2\bar{u}_{r,s} + \bar{u}_{r,s-1}}{\Delta\eta^2};$$

$$\Delta_r\phi = \frac{\phi_{r+1,s} - \phi_{r-1,s}}{2\Delta\xi};\quad \Delta_s\phi = \frac{\phi_{r,s+1} - \phi_{r,s-1}}{2\Delta\eta} \quad (24)$$

are used for the difference expressions analogous to partial derivatives with respect to ξ and η, and $\Delta\xi = 1/n_x = L_x/B$, $\Delta\eta = 1/n_y = L_y/H$ in which n_x and n_y denote the number of bays and the number of stories per half-wavelength H. Furthermore, $r_y = \sqrt{I_y/A_y}$ and $r_x = \sqrt{I_x/A_x}$ denote the radii of inertia of the cross sections. Boundary conditions (Eq. 7) for the left side of the frame with $P^L = V^L = M^L = 0$ may be transformed as follows:

$$\bar{u}_{0,s} = \bar{u}_{1,s};\quad \frac{H}{B}\,\frac{\bar{v}_{1,s} - \bar{v}_{0,s}}{\Delta\xi} = \frac{\phi_{0,s} + \phi_{1,s}}{2} + \frac{1 - c_x}{1 + c_x}\,\frac{\phi_{1,s} - \phi_{0,s}}{n_x\Delta\xi};$$

$$s_x'(\phi_{0,s} + \phi_{1,s}) = s_x''\,\frac{H}{B}\,\frac{\bar{v}_{1,s} - \bar{v}_{0,s}}{\Delta\xi} \quad (25)$$

At the right side of the frame the boundary conditions are similar. Also, the sinusoidal distributions assumed in the vertical direction are unaffected by the introduction of nondimensional variables. In Eqs. 24 and 25, s_y, c_y, s_y', s_y'' depend on $P^0_{y\,crit}/P_{E_y}$ where $P_{E_y} = \pi^2 EI_y/L_y^2$ = the Euler load for the vertical member. The slenderness L_x/r_x of horizontal beams could be important only if horizontal axial forces were large. Because this is not so in practical frames, the effect of this parameter may be neglected and s_x, c_x, s_x', s_x'' are independent constants while s_y, c_y, s_y', s_y'' depend on P_y^0/P_{E_y}. Inspection of Eqs. 23, 24 and 25 then shows that

$$\frac{P^0_{y\,crit}}{P_{E_y}} \approx f\left(\frac{k_x}{k_y}, \frac{H}{B}, \frac{L_y}{r_y}, n_x, n_y\right) \quad (26)$$

With increasing numbers n_x, n_y of bays and floors, the difference expressions (Eq. 24) approach in the limit partial derivatives $\partial^2\bar{u}/\partial\xi^2$, $\partial^2\bar{u}/\partial\eta^2$, $\partial\phi/\partial\xi$, etc. whose values must be finite. Consequently, the last two terms in Eq. 23c and one term in Eq. 25 become negligible, and it is seen that parameters n_x, n_y completely disappear from Eqs. 23a, 23b, 23c and 25. Furthermore, experience, as well as the formula in Eq. 31 in the sequel, shows that H/B and L_y/r_y are better replaced by parameters Hr_y/BL_y and L_y/r_y, because the effect of the latter then appears to be small so that there are only two important parameters. Thus

$$\frac{P^0_{y\,crit}}{P_{E_y}} \approx f\left(\frac{k_x}{k_y}, \frac{H/B}{L_y/r_y}, \frac{L_y}{r_y}\right) \quad (27)$$

It is interesting to note that in the limit for $n_x\ n_y \to \infty$, Eqs. 23a, 23b and 23c represent differential equilibrium equations of a certain orthotropic elastic continuum, and Eq. 25 represents the appropriate boundary conditions. Term ϕ can then be expressed from Eq. 25c and substituted in Eqs. 23a and 23b. For smaller numbers n_x, n_y, Eqs. 23 and 25 can also be approximated by differential equations which, however, do not correspond to an ordinary orthotropic

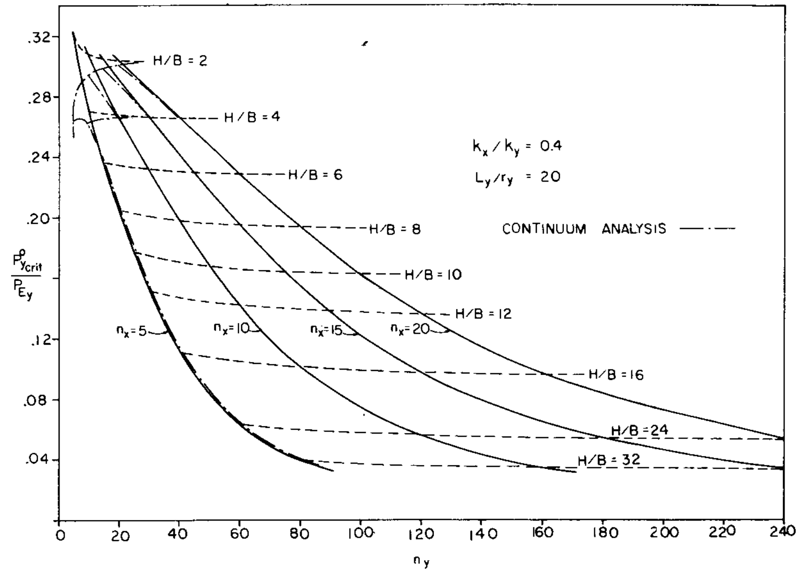

FIG. 3.—CRITICAL LOADS PER COLUMN FOR TYPICAL FRAME PROPERTIES [Solid lines for constant number of bays, n_x; dashed lines for constant height-to-width ratio. (Note the small effect of n_x, n_y at H/B constant.) Dash-dot lines show solution according to continuous approximation from Ref. 4.]

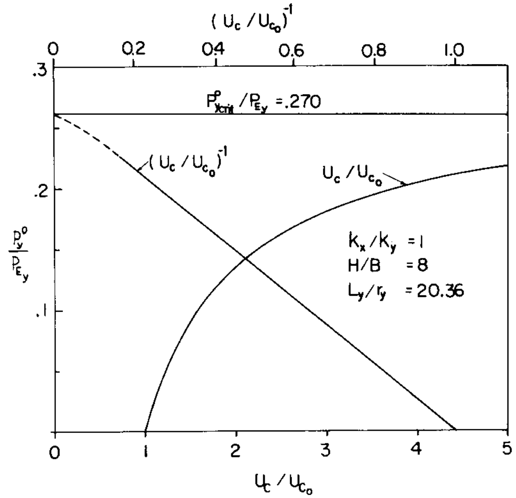

FIG. 4.—PLOT OF U_c/U_{c_o} VERSUS INITIAL COLUMN LOADS P^o_y (U_c = horizontal displacements of typical frame center line caused by fixed lateral load amplitude P^* and U_{c_o} = U_c for P^o_y = 0.)

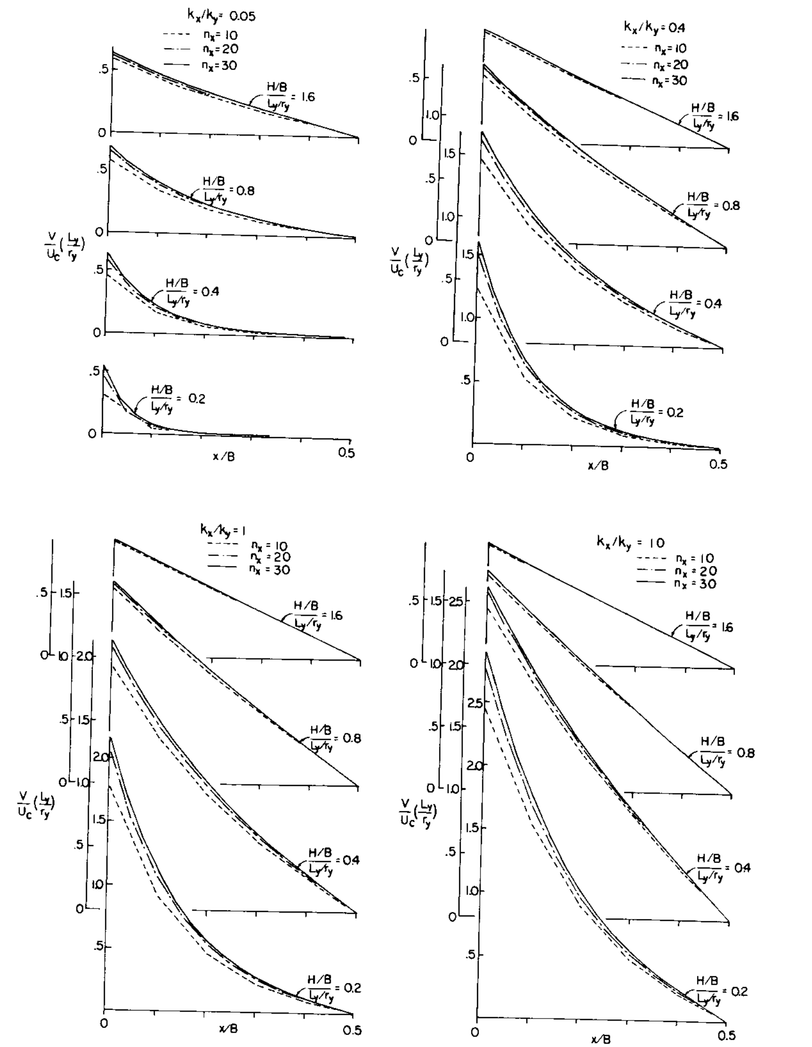

FIG. 5.—DISTRIBUTION OF AMPLITUDE OF VERTICAL JOINT DISPLACEMENTS ACROSS FRAME WIDTH AT BUCKLING FOR VARIOUS BEAM-TO-COLUMN STIFFNESS RATIOS k_x/k_y, FRAME HEIGHT-TO-WIDTH RATIOS H/B, AND COLUMN SLENDERNESS RATIOS L_y/r_{y_0} (Distributions of column extensions are same. For P_y less than buckling value, distributions are appreciably different only if P^o_y/P_{E_y} is not small (not shown).]

continuum but to the so-called micropolar continuum (16). Analogy of this continuum with frames has been established in previous papers (2,3) and application to the present problem will be studied in a separate paper (4).

RESULTS OF NUMERICAL PARAMETRIC STUDY

The solution procedure expounded previously has been programmed in FORTRAN IV, and a great number of frames with various member properties and various numbers of bays n_x and stories n_y, has been analyzed numerically. The results are shown in Figs. 3 to 6.

The weakening influence of the number of floors is seen from the lines for H/B = constant in Fig. 3. The effect of n_x, n_y is negligible if, roughly, n_x or $n_y > 10$ for $H/B > 2$, or $n_x > 5$, $n_y > 10$ for $H/B > 4$. In this case, of course, the continuum approximation mentioned before must be also acceptable. That the effect of the third parameter in Eq. 27 is indeed rather small can be seen from Fig. 6.

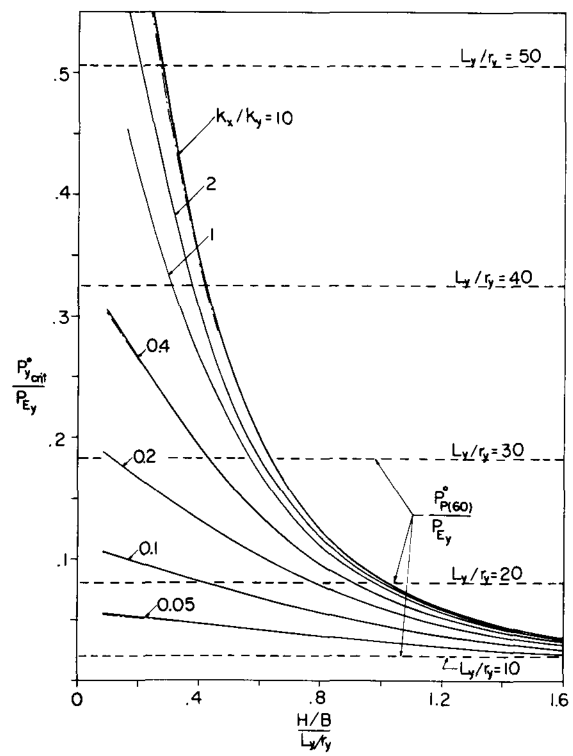

FIG. 6.—CRITICAL LOAD PER COLUMN AS FUNCTION OF RATIO OF FRAME SLENDERNESS H/B TO COLUMN SLENDERNESS L_y/r_y (solid lines) (Dashed lines indicate plastic yield loads for columns with various L_y/r_y having yield limit of 60,000 psi. Dash-dot lines show results according to continuous approximation from Ref. 4.)

The value of the critical axial force for short-wave buckling modes in which axial extensions are negligible lies, for typical building frames without diagonal bracing, between 0.8 P_{E_y} and 0.95 P_{E_y}. The critical forces for the long-wave extensional buckling modes are thus of no interest when $P^o_{y_{crit}}$ is in this range or above it. Such cases have not been included in Figs. 3 to 7. It should be noted, however, that even in such cases the presence of axial force P^o_y may considerably reduce the frame stiffness in long-wave deformation modes. This is of importance especially for analysis of vibrations.

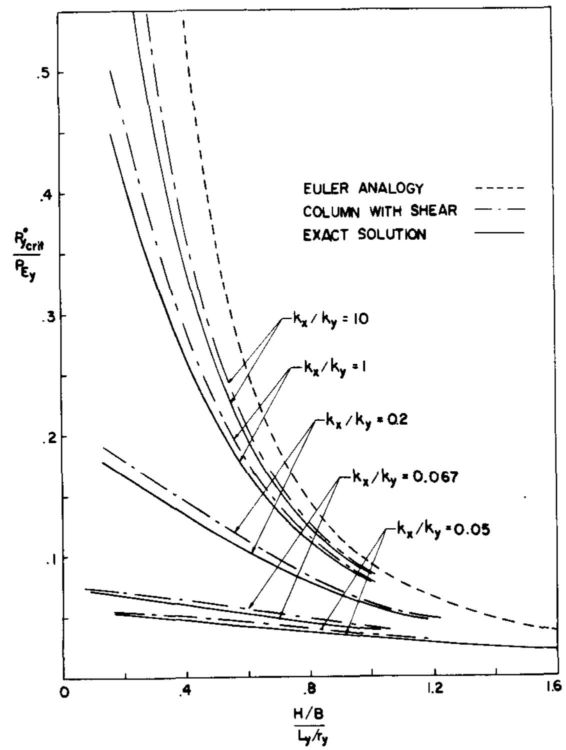

FIG. 7.—COMPARISON OF FRAME CRITICAL LOADS PER COLUMN AS OBTAINED BY PRESENT METHOD; BY ANALOGY TO EULER COLUMN (dashed lines), AND BY ANALOGY TO COLUMN WITH SHEAR (dash-dot lines) (For cases I_y = 6,000 in^4, A_y = 120 sq in., L_y = 144 in., N_x = 20.)

It is found that frames for which long-wave buckling needs to be considered have either small k_x/k_y or are very high and slender with members so stiff that short-wave buckling is precluded, as is typical for stiffening frames of modern tall buildings. In practice, these frames are arranged as a framed-tube to provide also torsional rigidity.

The critical load values are also irrelevant when they exceed the plastic yield load of the columns, P_p. Lines indicating the P_p value for the frames with various L_y/r_y have been included in Figs. 6 for steels with 60,000 psi yield limit. This value is about the upper limit of the high quality structural steels which are now becoming widespread. Obviously, for such steels the range in which long-wave buckling modes are of importance is considerably wider than for the ordinary structural steels.

It is also of interest to determine the relationship of the present solutions to the solutions for buckling of free standing frames, as in Fig. 2(c). The critical axial loads for long-wave buckling of such frames has been solved exactly for a few cases (frames 52 stories high) in a previous paper (3). Comparison of numerical results indicates that the critical load of the free standing frames about equals the critical load of the frames solved previously [Fig. 2(a)] if the height of the free standing frame equals the quarter-wavelength of the buckling modes solved previously, i.e., $H/2$. This agrees with what might be intuitively expected. If the present solution is applied to the free standing frame, the boundary conditions on top are not fulfilled. However, they are fulfilled at least integrally in that the force resultants and the moment resultant of all internal member forces acting on top (a section which corresponds to the quarter-wavelength) are zero. Discrepancy in the boundary conditions ought thus to have, according to the Saint-Venant principle, only local effect decaying rapidly away from the top of free-standing building.

Another simplification with regard to actual situations are the constant values of P_y^0 and the column cross sections along the building height. However, with variable P_y^0, a simple solution would not be feasible. Approximately, the case with variable P_y^0 and variable cross section can be replaced with a case in which P_y^0 and the cross section parameters have constant values corresponding to the averages for the actual structure.

APPROXIMATION BY COLUMN WITH SHEAR

Long ago, Engesser's formula for buckling of columns with shear deformations was successfully applied to predict approximately the buckling loads of built-up columns (20). It is thus of interest to check whether large frames can also be approximately considered as columns with shear.

In analogy with the well-known portal method of approximate frame analysis, it will be assumed that the bending moments in the midlength of all columns and beams are zero, which is equivalent to the assumption that the rotations of all joints in a given floor and the two adjacent floors are all equal. Let a relative horizontal displacement γL_y between the adjacent floors be imposed. Then, taking the initial forces, P_y^0, into account and neglecting P_x^0, the moment equilibrium equations for the interior joints and the boundary joints read: $6\beta k_x \phi + 2k_y s'_y \phi - 2k_y s'_y \gamma = 0$ in which $\beta = 2$ for an interior joint and $\beta = 1$ for an exterior joint. Then, determining the sum, $\Sigma\tilde{V}$, of the corresponding shear forces, $\tilde{V}$, in the columns, each of which is expressed (after substitution for ϕ) as follows: $\tilde{V}/\gamma = (s''_y\gamma - 2s'_y\phi)k_y/L_y = 12(k_y/L_y)$ $[(6/s'_y) + 2k_y/(k_x\beta)] - P_y^0$, the shear rigidity $\tilde{R}$ of the cross section of the whole frame is obtained:

$$\tilde{R} = \frac{\Sigma\tilde{V}}{\gamma} = 12\,\frac{m\kappa k_y}{L_y} \qquad (28a)$$

$$\kappa = \frac{1 - \dfrac{2}{m}}{\dfrac{6}{s'_y} + \dfrac{k_y}{k_x}} + \frac{\dfrac{2}{m}}{\dfrac{6}{s'_y} + \dfrac{2k_y}{k_x}} - \frac{\pi^2}{12}\,\frac{P_y^0}{P_{E_y}} \qquad (28b)$$

in which m = the number of columns. The moment of inertia of the horizontal cross section of the whole building (neglecting the contribution of I_y values) is

$$\tilde{I} = A_y L_x^2\,[1^2 + 2^2 + \ldots + (m-1)^2] - mA_y\left[\frac{L_x(m-1)}{2}\right]^2$$

$$= \frac{m}{12}\,(m^2 - 1)A_y L_x^2 \qquad (29)$$

According to Engesser's formula [p. 133 in Timoshenko and Gere (20)], the critical value $\tilde{P}_{\rm crit}$ of the resultant of initial axial forces in all columns is

$$\tilde{P}_{\rm crit} = mP^0_{y_{\rm crit}} \approx \frac{\tilde{P}_E}{1 + \dfrac{\tilde{P}_E}{\tilde{R}}} \qquad (30)$$

or

$$\frac{P^0_{y_{\rm crit}}}{P_{E_y}} = \left[\frac{12}{1 - m^{-2}}\left(\frac{r_y}{L_y}\,\frac{H}{B}\right)^2 + \frac{\pi^2}{12\kappa}\right]^{-1} \qquad (31)$$

in which $\tilde{P}_E = \pi^2 E\tilde{I}/H^2$ = Euler load for the frame taken as a single column without shear.

Eq. 31 is not an explicit formula for $P^0_{y_{\rm crit}}$, as the latter appears in Eq. 28b for κ. In the cases of interest for long-wave buckling, however, $P^0_{y_{\rm crit}} \ll P_{E_y}$, and so an approximate value of $P^0_{y_{\rm crit}}$ may be obtained, using $P_y^0 \approx 0$, $s'_y \approx 6$ in Eq. 28b. Results of this approximate analysis are shown by dash-dot lines in Fig. 7 in comparison with the exact solutions for a typical frame. It is seen that the predictions are surprisingly accurate and entirely satisfactory for most design purposes, even in cases when the actual distribution of $v_{r,s}$ along the floor is far from a linear one. Still more accurate values can be obtained for high values of $(H/B)/(L_y/r_y)$ by solving Eqs. 31 and 28b exactly, e.g. by the regula falsi method. For low values of $(H/B)/(L_y/r_y)$, solving Eqs. 31 and 28b by an iterative procedure leads to a value for $P^0_{y_{\rm crit}}/P_E$, which is less than exact and in greater error than the first approximation.

It is noteworthy that according to the approximate formula (Eq. 31), only the first two parameters of Eq. 27 affect P_y^0/P_{E_y} with Eq. 28b. This confirms again that the third parameter, L_y/r_y, is thus of lesser importance.

In view of the agreement just mentioned, the analogy with a column exhibiting shear deformation may probably be applied in situations where the exact solution would be too complicated. These include especially the free-standing frames with variable column cross section and variable axial force. The differential equation for buckling of columns with shear (6,20) can be integrated in these cases still relatively easily.

SUMMARY AND CONCLUSIONS

1. Long-wave extensional buckling of large regular planar frames compressed between two rigid half-spaces is solved exactly by the methods of finite difference calculus. Assuming sinusiodal buckling modes, the partial difference equations (Eq. 1) expressing equilibrium of joints are converted to ordinary difference equations (Eq. 10) which may be solved in terms of discrete complex exponentials (Eq. 14). The coefficients of the determinant which has to vanish at the critical state depend on P_y^0 nonlinearly, in a complicated manner involving the well-known nonlinear stability functions and complex roots and eigenvectors of another (6 × 6) matrix which do not possess explicit expressions. The critical values are therefore determined by a numerical procedure analogous to the method regula falsi.

2. In a frame with many bays and stories, the ratio of critical axial force to the column Euler load depends predominantly on the ratio of column and beam bending stiffnesses (k_y/r_y) and the ratio of column slenderness (L_y/r_y) to building slenderness (H/B), and weakly on the column slenderness. If the numbers of bays or stories are smaller, these also have some effect.

3. Results of the numerical parametric study presented herein (Figs. 3 to 7) show that long-wave buckling requires consideration in very tall and slender frames with members so stiff that the well-known short-wave buckling is unimportant, especially if high yield steel is used. For member properties such as those used presently in framed tubes (Sears Tower in Chicago or World Trade Center in New York), the long-wave buckling load is much smaller than the short-wave buckling load usually considered. This fact is also of interest for the analysis of lateral deflections and vibrations with account of stiffness reduction due to axial forces in columns.

4. The critical loads for long-wave buckling may be determined with sufficient accuracy utilizing a column with shear as a model. For this case, approximate formula Eq. 31 is given.

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APPENDIX II.—NOTATION

The following symbols are used in this paper:

B = width of frame;

c_x, c_y = stability functions of P_x^0 or P_y^0 defined by Eq. 2;

E_x', E_y' = EA_x/L_x, EA_y/L_y, in which A_x, A_y = cross-sectional areas;

$F_{1_r}, \ldots, F_{6_r}$ = discrete functions of r defined by Eq. 9;

$H = \pi L_y/\gamma$ = height corresponding to half-wavelength (Fig. 2);

$K_1^j, \ldots, K_6^j$ = jth eigenvector of matrix Eq. 13 corresponding to ρ_j;

k_x, k_y = EI_x/L_x, EI_y/L_y in which I_x, I_y = cross-sectional moments of inertia;

L_x, L_y = length of beam or column;

M^L, M^R, P^L, P^R = moments and axial forces applied on joints at sides [Fig. 2(c)];
m = number of columns per frame width B;
n_x, n_y = number of bays and number of stories per quarter-wavelength $H/2$;
P^* = parameter of lateral load, Eq. 16;
P_x^0, P_y^0 = initial axial force in horizontal and vertical members (positive for compression);
P_{E_x}, P_{E_y} = Euler buckling load of horizontal and vertical members;
R_r = discrete function of r in Eq. 4;
s_x, s_y, s_x', s_y', s_x'', s_y'' = stability functions of P_x^0, P_y^0 defined by Eqs. 2 and 3;
U_r, V_r = discrete functions of r in Eq. 4;
u, v = horizontal and vertical displacements of joints;
V^L, V^R = shear forces applied at left and right boundary joints [Fig. 2(c)];
x, y = horizontal and vertical cartesian coordinates;
$\gamma = \pi L_y/H$ = parameter in Eq. 4;
$\rho_1, \ldots, \rho_6$ = roots of characteristic equation (13); and
ϕ = rotation of joints (positive if counterclockwise).

Subscripts.

c = value of r given by Eq. 18;
r, s = number of joint from left or bottom boundary; and
x, y = labels of quantities referring to x or y directions.

9301 LONG-WAVE EXTENSIONAL BUCKLING OF LARGE FRAMES

KEY WORDS: **Buckling;** Deflection; Finite difference method; **Frames; Lateral stability;** Sidesway; **Stability; Structural engineering; Tall buildings**

ABSTRACT: In very high and slender frames with very stiff members, the safety against long-wave buckling modes involving axial extensions requires investigation. The problem is solved exactly for a planar regular rectangular frame of rectangular boundary, compressed between two sliding rigid half spaces. The problem is reduced to a system fo six linear first-order difference equations which are solved in terms of discrete complex exponentials. Because the coefficients of the determinant to vanish depend on axial loads nonlinearly, by means of explicitly inexpressible complex roots and eigenvectors of another 6 times 6 matrix and the well-known nonlinear stability functions, the critical load is determined numerically by a procedure analogous to the method regula falsi. The ratio of column critical force to column Euler load depends essentially on the ratio of column and beam bending stiffnesses, the ratio of building slenderness to column slenderness, and weakly on column slenderness ratio.

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