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SINGULARITIES OF ELASTIC STRESSES AND OF HARMONIC FUNCTIONS AT CONICAL NOTCHES OR INCLUSIONS

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Abstract—The nature of singularities at the vertex of conical notches and inclusions is found for problems of potential theory and for elastostatic problems of torsion and of axisymmetric stress. A solution in terms of spherical harmonics and a general numerical solution based upon the field equations are used to determine the power dependence of the field quantities upon the distance from the apex of the cone. Eigenvalues representing the exponent are computed for various values of cone angle and for various Poisson ratios.

1. INTRODUCTION

Whereas singularities in two-dimensional problems of potential theory and of elasticity have already been thoroughly examined, little is known of the singularities in three-dimensional problems. Even the singularities at conical notches and inclusions, the simplest of the three-dimensional problems, seem to have escaped attention and will, therefore, be examined in this study. Spherical coordinates will be employed and will be separated in a manner which is analogous to that introduced by Knein[1] (upon suggestion of von Kármán), and later independently developed by Williams[2], and Karp and Karal[3]. Attention will be restricted to problems that either are axisymmetric or are reducible to one-dimensional problems. The axisymmetric elasticity problems for a cone has recently been investigated by Thompson and Little[4], but they considered only cones of acute vertex angle (less than π) in which no singularities arise. Nevertheless, it is easy to check that their general solution applies for all angles, and so it can be used herein.

Two variants of solution will be used. One will be based on spherical harmonics[4]. The other will be a general numerical method which, after some additional refinements[5], is applicable to three-dimensional singular problems in general. A demonstration of this numerical method is a second objective of this study.

2. POTENTIAL THEORY

Consider a three-dimensional harmonic function $u(\theta, \phi, r)$ in spherical coordinates θ, ϕ, r which are centered in the apex of a cone whose axis coincides with the pole. In analogy with the method of Knein[1], Williams[2], and Karp and Karal[3], the radial coordinate r will be separated by assuming the harmonic function in the form

$$u(\theta, \phi, r) = r^{\lambda} U(\theta) \cos k\phi \tag{1}$$

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where k must be an integer in order to guarantee continuity of u. If this expression is substituted into the Laplace equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0 \tag{2}$$

variable r may be eliminated. This yields a second-order ordinary differential equation for function $U(\theta)$,

$$\frac{\mathrm{d}^2 U}{\mathrm{d}\theta^2} + \cot\theta\,\frac{\partial U}{\partial \theta} + \left[\lambda(\lambda+1) - \frac{k^2}{\sin^2\theta}\right] U = 0 \qquad (\theta \neq 0). \tag{3}$$

In the limit $\theta \to 0$, this equation degenerates into a condition of axisymmetry:

$$\partial U/\partial\theta = 0 \quad \text{for } k = 0; \quad \text{or } U = 0 \text{ for } k \neq 0 \qquad (\text{at } \theta = 0). \tag{4}$$

Two types of the boundary condition on the surface of the cone ($\theta = \beta$) will be considered:

$$\text{(a)}\ U = 0 \qquad \text{at } \theta = \beta \quad \text{(inclusion)} \tag{5a}$$

$$\text{(b)}\ \partial U/\partial\theta = 0 \quad \text{at } \theta = \beta \quad \text{(notch)}. \tag{5b}$$

(Note that the boundary condition $U = \text{const.}$ may be reduced to $U = 0$ by a substitution of a new variable for u.)

The boundary conditions on the radial rays will not be specified, and so an infinite number of solutions is naturally expected. That this is indeed so is clear from the fact that equations (2) and (5a) or (5b) represent an eigenvalue problem. Its eigenstates form a complete orthogonal system and, therefore, a solution for any specified boundary conditions on the radial rays is a linear combination of all eigenstates. However, in a sufficiently small neighborhood of the apex of the cone, the eigenstate that corresponds to the root λ of the smallest real part prevails (except in the special cases in which the boundary conditions on radial rays yield a zero coefficient for this eigenstate). Consequently, determination of the smallest $Re(\lambda)$ is of particular interest, especially when $|Re\ \lambda| < 1$ because the gradient is then unbounded near the apex $r = 0$ (singular behavior).

(A) *Solution in terms of Legendre functions*

Introducing the new variable $x = \cos\theta$, it is readily recognized that equation (3) is the Legendre differential equation (for $k = 0$) or the associated one (for $k \neq 0$) [6]. Its solutions are the associated Legendre functions $P_\lambda^k(\cos\theta)$ [6] (automatically satisfying axisymmetry condition 4). The boundary condition (5a) or (5b) requires finding such λ that

$$\text{(a)}\ P_\lambda^k(\cos\beta) = 0 \qquad \text{(inclusion)} \tag{6a}$$

$$\text{(b)}\ \left.\frac{\mathrm{d}P_\lambda^k(\cos\theta)}{\mathrm{d}\theta}\right|_{\theta=\beta} = 0 \quad \text{(notch)}. \tag{6b}$$

The condition (6b) may be brought to the form (see formula 8.733–1 in [6]):

$$(\lambda+1)x_0 P_\lambda^k(x_0) - (\lambda - k + 1)P_{\lambda+1}^k(x_0) = 0, \qquad x_0 = \cos\beta \quad \text{(notch)} \tag{6c}$$

which is more expedient for numerical calculation. The values of λ have been solved from equations (6a) and (6c) numerically (with the aid of a computer), using the *regula falsi* method. This has been particularly simple because it was known in advance (from the numerical method described below) that all roots λ are real. The values of Legendre function, including the associated one, have been computed from its hypergeometric series representation[6].

The smallest λ-values (probably accurate to four digits) are shown in Fig. 1. Singularity

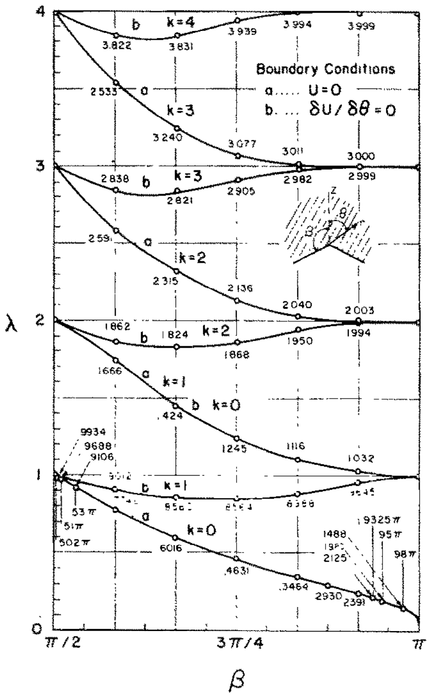

Fig. 1. Four smallest values of λ in potential theory problems.

(i.e. $\lambda < 1$) is obtained only for $\beta > \pi/2$, and for $k = 0$ (axisymmetric state) in case of inclusion, or for $k = 1$ (period 2π) in case of notch. For angles β close to π ($\beta > 0{\cdot}95\pi$) the hypergeometric series converges poorly and an asymptotic expansion[6] is required; from such an expansion it further follows that

$$\text{(a)} \lim_{\beta\to\pi} \lambda = 0, \quad \lim_{\beta\to\pi} \frac{d\lambda}{d\beta} = \infty \quad \text{(inclusion)} \tag{7a}$$

$$\text{(b)} \lim_{\beta\to\pi} \lambda = 1, \quad \lim_{\beta\to\pi} \frac{d\lambda}{d\beta} = 0 \quad \text{(notch)}. \tag{7b}$$

In the limiting case of a line inclusion ($\beta = \pi$) the solution is well known[7]:

$$U(\theta) = C \ln\left(\tan\frac{\theta}{2}\right), \quad \lambda = 0, \tag{8}$$

while for a line notch a homogeneous field ($\lambda = 1$) is the solution.

(B) *Direct numerical method based on field equations*

Interval $\theta \in (0, \beta)$ is subdivided by discrete nodes, uniformly spaced, and equation (3) and the boundary conditions are replaced by their finite difference approximations. If the boundary conditions (4) and (5a) or (5b) are eliminated, one obtains for the nodal values U_i a tridiagonal system of linear algebraic equations which can be brought to the form:

$$\sum_{j=1}^{n} a_{ij} U_j - \mu U_i = 0 \qquad \text{with } \mu = \lambda(\lambda + 1);\ (i = 1, 2, \ldots, n). \tag{9}$$

where coefficients a_{ij} are independent of λ. Thus, the problem is reduced to a standard eigenvalue problem. Standard library subroutines have been used for its numerical solution. After determining μ, whether complex or real, the corresponding value of λ is computed as $\lambda = -\frac{1}{2} + \sqrt{\frac{1}{4} + \mu}$. The solution has been programmed for complex λ, and it has been proved that no complex roots exist. With a step size $\Delta\theta = \beta/48$ the results coincided to four digits with those obtained analytically (Fig. 1). To illustrate the convergence, the λ-values for $\beta/\Delta\theta = 12$, 24, 48 and 96 are 0·46396, 0·46332, 0·46315 and 0·46311 in the case of a notch with $\beta = \frac{3}{4}\pi$ and $k = 0$ (for second-order finite difference formulas). The direct numerical solution involves more algebraic operations than the previous numerical solution based on an analytical approach, but the cost of computation is so small that the difference is undetectable. When searching for complex λ, the direct numerical approach is more simple to program, and even more importantly, all computations are explicit and the computer runs are sure to succeed (whereas the solution of λ from (6a) or (6c) requires tracing in the complex plane the curves of $Re(\lambda) = 0$ and the curves of $Im(\lambda) = 0$, and finding their intersections, for which a computer program that would not necessitate many runs and an interaction of the programmer is difficult to write). It seems that with small $\Delta\theta$ the direct numerical method works satisfactorily even for β quite close to π (0·97π), although for β sufficiently close to π it must fail, because function $U(\theta)$ tends, according to (8), to become singular at $\theta = \pi$.

Physically, the above solutions represent distributions of temperature near a uniformly heated perfectly conducting conical inclusion or a non-conducting conical inclusion; or similar distributions of electric charge, of stream function in flow or seepage problems, etc. The cases for $k = 1$ and $k = 0$ correspond to homogeneous distant field with gradients parallel to $\phi = 0$, $\theta = \pi/2$, and to $\theta = 0$, respectively. The above solution also describes scattering of waves by a reflecting or absorbing cone, because near the singularities the Helmholtz reduced wave equation is equivalent to Laplace equation. The same is true of the Poisson equation.

3. ELASTIC TORSION

As is well known ([8], p. 326), the elasticity problem admits solutions for which $u_r = u_\theta = 0$ (u denotes displacement component in the direction defined by the subscript). These solutions correspond to the case of torsion about axis $\theta = 0$. Displacement u_ϕ then satisfies the equation ([8], p. 326):

$$\nabla^2(u_\phi \cos\phi) = 0. \tag{10}$$

In the case of a rigid conical inclusion, the boundary condition on the surface of a cone is $u_\phi = 0$. Thus, the problem is identical to the previously solved potential theory problem of an inclusion (case a in Fig. 1) with $k = 1$. It is seen that no singularity of stress occurs ($\lambda > 1$).

In the case of a conical notch, shear stress $\sigma_{\phi\theta}$ must vanish at the cone. Using the expression for strain $e_{\phi\theta}$ in spherical coordinates [8, 9], this condition reduces to

$$\partial u_\phi/\partial\theta = u_\phi \cot\beta \tag{11}$$

which gives (according to formula 8·733–1 in [6])

$$\lambda P^1_{\lambda+1}(x) - (2+\lambda)xP^1_\lambda(x) = 0 \qquad (x = \cos\theta). \tag{12}$$

This case has been solved in the same manner as described before and again no stress singularity was found to occur.

4. ELASTICITY

In analogy with Knein's or Williams' approach, or equation (1), displacements u_θ, u_ϕ, u_r will be considered in the form

$$u_r = r^\lambda U_r(\theta), \qquad u_\theta = r^\lambda U_\theta(\theta), \qquad u_\phi = 0 \quad \text{(axisymmetry)}. \tag{13}$$

In case of a rigid conical inclusion, the boundary conditions require that

$$U_\theta = U_\phi = U_r = 0 \quad \text{at } \theta = \beta \quad \text{(inclusion)}. \tag{14}$$

In case of a conical notch, the boundary conditions require stresses $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ to vanish. If these stresses are related according to Hooke's law to strains $\varepsilon_{\theta\theta}$, ε_{rr}, and $\varepsilon_{r\theta}$, and if these are in turn expressed in terms of displacements u_r, u_θ (see [9, 8]) and equation (13) is substituted, variable r disappears and the boundary conditions of a notch take the form

$$\left.\begin{aligned} &\partial U_\theta/\partial\theta = -(1+\nu'+\lambda\nu')U_r - (\nu'\cot\beta)U_\theta \\ &\partial U_r/\partial\theta = (1-\lambda)U_\theta \end{aligned}\right\} \quad \text{at } \theta = \beta \quad \text{(notch)} \tag{15}$$

where $\nu' = \nu/(1-\nu)$, ν = Poisson's ratio.

(A) *Solution in terms of Legendre functions*

The displacements are best expressed in terms of the Papkovich–Neuber potentials, which automatically satisfies Navier's differential equations of equilibrium. This approach has been adopted by Thompson and Little [4]. Although they considered no notches or inclusions ($\beta > \pi/2$), it is easy to check that their solution also applies to these cases. According to equation 2.39 in [4], non-zero solutions of the type (13) exist, in case of a stress-boundary condition, if

$$\begin{aligned} &x\{2c(x^2-1)\lambda^2 + 2c(x^2-1)\lambda + x^2\}[P_\lambda(x)]^2 \\ &\quad + x\{2c(x^2-1)\lambda^2 + 2c(x^2-1)\lambda + 1\}[P_{\lambda-1}(x)]^2 \\ &\quad - 2c\{2x^2(x^2-1)\lambda^2 + (x^2-1)(3x^2-1)\lambda \\ &\qquad + [x^4 + 2(1-\nu)\lambda^2 + 1]\}P_\lambda(x)P_{\lambda-1}(x) = 0, \end{aligned} \tag{16}$$

$$x = \cos\theta, \qquad c = 1/4(1-\nu).$$

In case of displacement boundary condition (14), equations (2.32) and (2.33) from [4] can be shown to yield

$$(1+c\lambda)x[P_\lambda(x)]^2 + c\lambda x[P_{\lambda-1}(x)]^2 - [x^2c(1+2\lambda) + (1-c)]P_\lambda(x)P_{\lambda-1}(x) = 0 \quad (\lambda \neq 0). \tag{17}$$

Similarly as in the potential theory case, λ has been solved from equations (16) and (17) by the regula falsi method. (Legendre functions have been evaluated from their hypergeometric series representation[6].) The results, which seem to be accurate to four digits, are shown in Table 1 and Fig. 2.

Table 1. The smallest value of λ in axisymmetric elastic problems corresponding to Fig. 2

	β						
ν	0.51π	$\frac{7\pi}{12}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	$\frac{11\pi}{12}$	0.97π
Inclusions							
0	0·9793	0·8387	0·6886	0·5416	0·4014	0·2682	0·1760
0·1	0·9809	0·8492	0·7028	0·5536	0·4091	0·2720	0·1777
0·2	0·9830	0·8638	0·7227	0·5700	0·4193	0·2768	0·1798
0·3	0·9861	0·8856	0·7528	0·5939	0·4334	0·2834	0·1827
0·4	0·9911	0·9222	0·8057	0·6325	0·4544	0·2926	0·1866
0·46	0·9957	0·9599	0·8675	0·6709	0·4728	0·3003	0·1899
0·49	0·9988	0·9882	0·9309	0·6993	0·4846	0·3047	0·1916
0·499	0·9989	0·9987	0·9779	0·7099	0·4887	0·3066	0·1927
Notches							
0	0·9706	0·8486	0·8334	0·8798	0·9411	0·9854	0·9983
0·1	0·9676	0·8293	0·8072	0·8577	0·9294	0·9825	0·9980
0·2	0·9645	0·8089	0·7781	0·8318	0·9153	0·9790	0·9977
0·3	0·9614	0·7874	0·7456	0·8012	0·8978	0·9749	0·9973
0·4	0·9584	0·7648	0·7093	0·7644	0·8755	0·9694	0·9971
0·499	0·9553	0·7411	0·6686	0·7200	0·8464	0·9624	0·9969

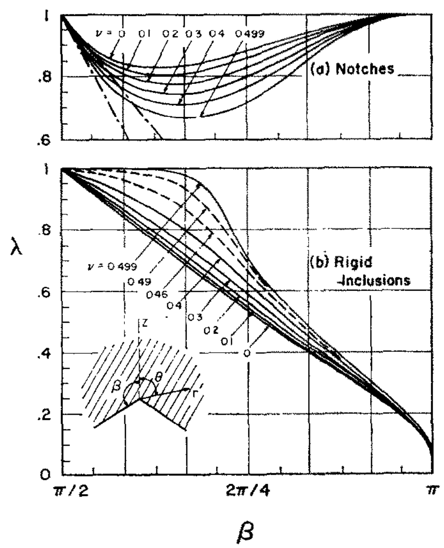

Fig. 2. The smallest value of λ in axisymmetric elastic problems.

(B) *Direct numerical solution based on field equations*

The numerical approach could also be based on Papkovich–Neuber potentials, but with regard to stress boundary conditions it is more convenient to use the three Navier's differential equations of equilibrium in terms of displacements (see Ref. [9], Section 96, p. 141), of which the condition for the ϕ-direction is automatically satisfied. If expressions (13) are substituted into these equations, variable r disappears and one obtains:

$$\frac{\partial^2 U_r}{\partial\theta^2} + \cot\theta\frac{\partial U_r}{\partial\theta} + [\nu''(\lambda-1)-\lambda-1]\frac{\partial U_\theta}{\partial\theta} + \nu''(\lambda-1)(\lambda+2)U_r + [\nu''(\lambda-1)-\lambda-1]\cot\theta\, U_\theta = 0, \tag{18}$$

$$\nu''\frac{\partial^2 U_\theta}{\partial\theta^2} + [\nu''(\lambda+2)-\lambda]\frac{\partial U_r}{\partial\theta} + \nu''\cot\theta\frac{\partial U_\theta}{\partial\theta} + \left[\lambda(\lambda+1) - \frac{\nu''}{\sin^2\theta}\right]U_\theta = 0,$$

where $\nu'' = (1-\nu)/(0{\cdot}5-\nu)$. For $\theta = 0$ these equations degenerate into conditions of axisymmetry:

$$\partial U_r/\partial\theta = 0 \quad \text{and} \quad U_\theta = 0 \qquad \text{at } \theta = 0. \tag{19}$$

For numerical solution, interval $(0, \beta)$ is subdivided by discrete nodes and equations (18), (19) and (14) or (15) are replaced by their finite difference approximations. (To achieve high accuracy, fourth-order approximations [10] have been used.) This results in a banded system of n linear algebraic equations (of band-width 11):

$$\sum_{j=1}^{n} A_{ij}(\lambda)U_j = 0 \qquad (i = 1, 2, \ldots, n) \tag{20}$$

where coefficients $A_{ij}(\lambda)$ are nonlinear functions of λ. When complex roots λ are searched, A_{ij} and U_i must be considered complex; otherwise they may be treated as real. (No advantage is gained by eliminating the boundary conditions from equation 20.) Equation (20) defines a nonlinear generalized eigenvalue problem which cannot be reduced to the standard linear eigenvalue problem (equation 9). However, efficient computer methods exist which could handle this problem (up to a size of several thousands of equations without special difficulties); see [5].

Computer analyses for a subdivision with step $\Delta\rho = \beta/96$ (resulting in a system of 192 equations) have yielded results (Fig. 2, Table 1) that coincide to four digits with the results of the preceding analytical solution, except for the case $\nu = 0{\cdot}499$ for which the error varies between 0·0001 and 0·08. (This is because $\nu'' \to \infty$ for $\nu \to 0{\cdot}5$; but for $\nu = 0{\cdot}5$ the solution could be based on differential equations of equilibrium for an incompressible material.) The numerical solution is about equally easy to program as the analytical one. It involves more arithmetic operations but the computer cost is not excessive, anyhow.

Approaching the cone vertex along any radial ray, the stresses grow to ∞ as $r^{\lambda-1}$. Note that the stress singularity strength $\lambda - 1$ depends on Poisson's ratio. Physically, the axisymmetric singular stress states obtained are excited by combinations of a normal stress parallel to axis $\theta = 0$ at infinity and of equal biaxial normal stresses perpendicular to axis $\theta = 0$ at infinity.

Knowing λ, the eigenstates, when desired, can easily be computed from the formulas 2.32 and 2.33 derived in [4] by setting in these formulas $C'_\lambda = C$ and $D'_\lambda = Cx_0(1 + c\lambda)P_\lambda(x_0)/$

$\lambda P_{\lambda-1}(x_0)$ and taking into account minor changes in notations. (In particular, $k = c$, $\alpha_n = \lambda$, $\mu = x$ and ϕ, θ are interchanged.) According to these formulas,

$$u_r = Cr^\lambda(1 + c\lambda)\left\{xP_\lambda(x) - \frac{x_0 P_\lambda(x_0)}{P_{\lambda-1}(x_0)} P_{\lambda-1}(x)\right\},$$
$$u_\theta = Cr^\lambda\left\{c\frac{\mathrm{d}}{\mathrm{d}x} P_{\lambda-1}(x) - [1 - c(1 + \lambda)]P_\lambda(x) - \frac{(1 + c\lambda)x_0 P_\lambda(x_0)}{\lambda P_{\lambda-1}(x_0)}\frac{\mathrm{d}}{\mathrm{d}x} P_{\lambda-1}(x)\right\}\sin\theta \tag{21}$$

where $x_0 = \cos\beta$, $x = \cos\theta$, C = arbitrary constant. The stresses can similarly be obtained by direct substitution from equations (2.34)–(2.37) in Ref. [4].

Knowledge of the field near the singularity makes it possible to construct a singular finite element for the vertex region of a cone, and thus to solve practical boundary value problems for bodies of finite dimensions.

5. CONCLUSION

Conical notches and inclusions produce singularities of stress or potential gradient whose strength varies from 0 to -1 in dependence on the cone angle. The stress singularity also depends on the Poisson's ratio. The problem can be solved analytically in terms of Legendre functions. A numerical solution based on finite differences in the angular spherical coordinate is also possible and yields equally accurate results. This fact at the same time serves as a check on the validity of this numerical method, which is applicable to a broad class of other problems.

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Абстракт — Определяется природа сингулярностей в вершине конических вырезов и включений, для задач теории потенциала и для упругостатических задач кручения и осесимметрического напряжения. Применяются решения в виде сферических гармоник и общее численное решение, основанные на уравнениях поля, с целью определения зависимости от степени для количеств поля, в зависимости от вершины конуса. Подсчитано собственные значения, которые изобращают показатель степени для разных значений угла конуса и для разных коэффициентов Пуассона.