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## VISCOPLASTICITY OF NORMALLY CONSOLIDATED CLAYS

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### PROBLEM AND METHOD OF APPROACH

Realistic constitutive relationships are needed in the field of soil mechanics to develop representative models for the viscoplastic behavior of soil-structure systems. Although a comprehensive constitutive relation should ideally be based on microscopic material characteristics, the present state of knowledge precludes such a treatment. While the approach (endochronic viscoplasticity) adopted in this study is macroscopic in the general sense, an attempt is made to develop relations and material parameters that relate to the microscopic characteristics and behavior of cohesive soils. The internal state of the material is described by defining state variables that can be qualitatively interpreted in terms of microscopic processes, such as particle dislocations and dislocation arrangements. Endochronic theory initiated from the concept of internal state variables is based on the assumption that there exists a group of such variables that fully defines a continuous free energy. The effect of history-dependence and energy dissipation are handled by means of the internal state variables.

Endochronic theory is an inelastic constitutive relation based on the concept of intrinsic time, which is a nondecreasing scalar variable that depends on increments of inelastic strain, as well as increments of time, and geometrically represents the length of the path traced by the states of the material in the strain-time space of variable metric. Valanis (1971) coined the term "endochronic" and was the first to successfully use endochronic theory to model the cross-hardening behavior of metal alloys. However, state parameters, which are equivalent

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to the intrinsic time parameter, have been introduced beginning the early 1950's and can be traced to the works of Ilyushin, Hill, Rivlin, Pipkin, and others. Modeling inelastic behavior by means of intrinsic time was pioneered by Schapery (1968, 1969), who formulated a theory of continuum thermodynamics by introducing a "reduced" (or intrinsic) time parameter as a nondecreasing scalar variable whose increments depend on both time and strain or stress increments. In fact, it can be shown that endochronic theory is completely equivalent to strain-rate dependent viscoplasticity, provided that the limiting instantaneous response is required to be inelastic (Bazant, 1978).

Valanis' endochronic theory (1971) was later extended by Bazant (1974), Bazant and Bhat (1976), and Bazant and Shieh (1977) to describe the behavior of concrete (especially the strain-softening tendency, hydrostatic pressure sensitivity, and inelastic dilatancy) and by Bazant and Krizek (1976) to describe the inelastic behavior of cohesionless soils. At present, various characteristics of endochronic theory, such as the violation of Drucker's postulate and the shape of the inelastic stiffness locus (Bazant, 1977; Bazant, Bhat, and Shieh, 1976) are being debated in the literature. Although it must be recognized that various aspects of endochronic theory are based on hypotheses rather than physical justifications, the endochronic formulation is adopted chiefly because of its convenience and flexibility in treating unloading, cyclic loading, strain-softening, and cross-hardening. In the case of nonmetallic materials, it is also important to note that there exists no microstructural mechanism that seems to favor any theory other than endochronic theory. This is in contrast to metals, where the plastic slip mechanism and the validity of Schmid's law point toward plasticity, while in soils (as well as concrete) the existence of friction and particle separations (and microcracking) actually precludes classical plasticity which satisfies Drucker's postulate.

The increment of intrinsic time,  $z$ , may be expressed as

$$(dz)^2 = \left(\frac{d\zeta}{Z_1}\right)^2 + \left(\frac{dt}{\tau_1}\right)^2 \dots \dots \dots (1)$$

in which  $Z_1$  and  $\tau_1$  are material constants; and  $dt$  is the external time increment. The parameter  $\zeta$ , called the rearrangement measure, may be visualized as a parameter representing, on a macroscopic scale, the accumulation of microstructural changes that take place as deformation occurs; these changes may lead to either strain softening or strain hardening. The increment  $d\zeta$  is assumed to be a function of only the current state of stress,  $\sigma$ , and strain,  $\epsilon$ , and the cumulative value  $\xi$ ; this function can be formulated in differential form as  $d\zeta = F(\epsilon, \sigma, \xi) d\xi$ , in which the function  $F$  (which must be determined semi-empirically from experimental data) represents hardening and softening behavior and the variable  $\xi$ , termed the distortion measure, is defined as  $d\xi = \sqrt{J_2^{d\epsilon}}$ , in which  $J_2^{d\epsilon}$  is the second invariant of incremental deviatoric strain. The inelastic volumetric strains (densification or dilatancy) caused by shear will be represented by the variable  $\lambda$ , called the densification-dilatancy measure. If it is assumed that the inelastic volume change is due only to shear, which means that inelastic volumetric strains due to a hydrostatic stress change (i.e., consolidation) are excluded from the formulation,  $d\lambda$  can be expressed as  $d\lambda = L(\epsilon, \sigma, \lambda) d\xi$ , in which the function  $L$  must be determined empirically from experimental data.

Eq. 1 may be rewritten as  $Z_1(dz/dt)^2 = (d\zeta/dt)^2 + (Z_1/\tau_1)^2$ , in which  $d\zeta/dt$  is a measure of the strain rate magnitude. When the strain rate is assumed to be large (i.e.,  $d\zeta/dt \gg Z_1/\tau_1$ ),  $Z_1(dz/dt)^2 = (d\zeta/dt)^2$  or  $(dz)^2 = (d\zeta/Z_1)^2$ , which gives a time-independent theory. Most of the present analysis of test data corresponds to this latter assumption, partly because a proper treatment of the time-dependent aspects of soil behavior is beyond the scope of this study. However, it is still appropriate to regard the theory as viscoplasticity because its logical derivation (historically the original one) is based on the assumption that viscosity depends on the strain rate (Bazant, 1978). In particular, the time-dependent term is needed in the case of cyclic tests (Krizek, Ansal, and Bazant, 1978).

**ENDOCHRONIC DEVIATORIC STRESS-STRAIN RELATION**

Restricting attention to initially isotropic clays at small strain, separate equations may be written for the deviatoric and volumetric components of the stress and strain tensors. The deviatoric stress-strain relation may be written as

$$de_{ij} = \frac{s_{ij}}{2G} + \frac{s_{ij}}{2G} dz \dots \dots \dots (2)$$

the strain increments are expressed as a sum of elastic and inelastic components. The dependence of the elastic component on the clay type, stress history, present stress state, and change in the stress and strain states is achieved through the shear modulus by defining it as a function of its initial value and the stress and strain invariants. The elastic strain accounts for the effects of: (1) The applied stress level; and (2) the changes in the elastic properties on the inelastic strain, while the intrinsic time reflects the effects of the clay type, stress history, and variation in the states of stress and strain.

For cohesive soils it may be assumed as a first-order approximation that the resistance to shear stress depends mainly on the interparticle distance, which is directly related to the void ratio and the effective normal stress (including stress history). To include the influence of these factors on the rearrangement measure, strain hardening and strain softening must be considered. Since strain hardening and strain softening is a coupled process for cohesive soils,  $F(\epsilon, \sigma, \zeta)$  will be treated as a single function without separating it into individual strain softening and strain hardening functions. However, it is convenient to consider the function  $F$  as a composite function, in which

$$F(\epsilon, \sigma, \zeta) d\xi = \frac{d\eta}{f(\eta)}; \quad d\eta = F_\eta(\epsilon, \sigma) d\xi \dots \dots \dots (3)$$

To establish the necessary internal relations with respect to the governing factors,  $F_\eta$  will be assumed to have the form

$$F_\eta(\epsilon, \sigma) = F_{\eta_1}(I_1^\epsilon) F_{\eta_2}(I_1^\sigma) F_{\eta_3}(J_2^\epsilon) \dots \dots \dots (4)$$

As previously suggested, both deviatoric and volumetric stresses cause volume change and a change in the interparticle distances, and the magnitude of the effect of the volumetric stress change must be considered to be stress-history dependent. An increase in the volumetric stress always causes a decrease in



$F$  (i.e., the rate of development of inelastic strain), but the amount of the decrease is different for paths along the virgin and reloading curves. Consequently, it is possible to make the same assumptions as in Critical State Theory regarding the virgin-consolidation and swelling lines, and an expression for the change in void ratio as a function of the preconsolidation stress, slopes of the virgin consolidation and swelling lines, and initial and present stresses can be obtained. Accordingly, the dependence of the strain softening-hardening function on the volume change due to the volumetric stresses can be formulated. However, instead of these approximate relations, the same effect can be achieved by expressing  $F_{\eta_1}$  in terms of the first strain invariant as  $F_{\eta_1}(I_1^e) = |1 - a_1 I_1^e|$ , in which  $a_1$  is a positive material parameter. This type of expression requires only the one additional assumption that the respective influences of the volume change and the effective normal stress can be separated.

The frictional aspect of cohesive soils suggests that the strain softening and hardening should depend on the effective confining stress. This is achieved in terms of the first invariants of the effective stress in the form  $F_{\eta_2}(I_1^{\sigma'}) = [0.01 + a_2(I_1^{\sigma'}/p_a)]^{-1}$ , in which  $a_2$  is a positive material parameter and  $p_a$  is atmospheric pressure (expressed in the same units as the stress invariant and introduced for the sole purpose of establishing a dimensionless relationship). The influence of shear stresses is observed to be stress-history dependent; they is on or below the virgin consolidation line. For both cases the important factor is the magnitude of the shear strain, rather than the magnitude of the shear stress. It is possible to express this influence in terms of the applied shear stresses by using some additional assumptions, but this would lead to more approximate and more complicated relations. It is simpler and more logical to define the influence of shear stresses in the form  $F_{\eta_3}(J_2^e) = 1 + a_3 J_2^e$ , in which  $J_2^e$  is the second deviatoric strain invariant and  $a_3$  is the material parameter for the stress-history dependence.

Up to this point the major factors that control strain softening and hardening were considered separately. However, it was observed that there is an upper limit for softening or hardening, depending on the accumulated inelastic strains but independent of the factors that cause them. This stage corresponds to what is known as the critical state, where both strain hardening and strain softening diminish. In the case of strain hardening due to shear stresses, the limit (critical state) is more pronounced. Although  $F_{\eta_3}$  can increase or decrease continuously, such behavior is not normally observed; accordingly, a general limiting relation, which incorporates the critical state concept, can be formulated in terms of accumulated rearrangement; this formulation is developed by considering another variable,  $\eta$ , which represents the continuous rearrangement, such that

$$d\eta = \left[ a + \frac{|1 - a_1 I_1^e| (1 + a_3 J_2^e)}{0.01 + a_2 \frac{I_1^{\sigma'}}{p_a}} \right] d\xi \dots \dots \dots (5)$$

in which  $a$  is a positive material parameter necessary to determine the irreversible strain increment for the critical case where there is no hardening or softening. The limiting function is chosen as

$$f(\eta) = 1 + \frac{\beta_1}{1 + \beta_2 \eta} \dots \dots \dots (6)$$

in which  $\beta_1$  and  $\beta_2$  are positive material parameters that are dependent on the stress history.

All derivations are concerned with the cases where the soil element is under compression. It is commonly accepted that the tensile strength of cohesive soils is negligible in most cases. In multidimensional relations it can be assumed that tension begins when the first stress invariant becomes negative. The sudden failure under tensile stresses can be modeled by Eq. 5, because the value of  $d\eta$  approaches infinity as  $I_1^{\sigma'}/p_a$  approaches  $-0.01/a_2$ , which represents an empirical tensile strength.

**ENDOCHRONIC VOLUMETRIC STRESS-STRAIN RELATIONS**

The volumetric relation may be written as

$$d\epsilon = \frac{d\sigma'}{3K} + d\epsilon''; \quad d\epsilon'' = d\lambda = L(\epsilon, \sigma, \lambda) d\xi \dots \dots \dots (7)$$

in which  $\sigma'$  = the effective volumetric stress (defined as  $\sigma' = \sigma - u$ , where  $u$  = pore pressure and  $\sigma$  = total volumetric stress). As in the deviatoric relation, the strain is expressed as a sum of two components, elastic and inelastic. The bulk modulus, defined as a variable, accounts for the effects of clay type, stress history, and stress path followed for the elastic volumetric strains. The inelastic volumetric strains,  $\epsilon''$ , for cohesive soils may be due to either shear or volumetric stresses. Strains due to shear stresses are deformation dependent, and those due to volumetric stresses are time dependent.

An inelastic volume change requires a change in the volume of the fluid phase (pore water). Since the pore water is much less compressible than the skeleton of clay particles, the volumetric strain is severely restricted unless the specimen is drained and small enough to allow a sufficiently rapid escape of water. For undrained conditions, where no loss of pore water from the soil specimen is permitted, volumetric strains are negligible, and even when drainage is permitted they may often be minor due to the long time period needed for drainage of all but very small specimens. All of these facts may be properly described by the two-phase medium concept (Bazant and Krizek, 1975). For sake of simplicity and because the distinction between the fluid and solid phases in clay is not completely clear (some water is bound to the solid particles), the two-phase medium will be treated in a simplified manner. The effect of the stress in the fluid on the volume change of the solid is not considered separately, as in the complete two-phase medium concept; instead, the pore pressures are expressed directly in terms of the volume change,  $d\epsilon$ , as determined from Eq. 7.

Although the inelastic volume changes is attributed only to shear strains, the function  $L$  will be formulated to include the relatively small volume changes that are due to volumetric stress under multiaxial stress conditions, where shear stresses also exist. However, only situations involving shear stresses and relatively small volumetric stress changes will be considered herein.



The densification-dilatancy function,  $L$ , similar to the strain softening-hardening function,  $F$ , is related to the clay type, stress history, and stress path. These factors must be expressed in terms of stress and strain invariants, and the function  $L$  may be assumed to have the separated form

$$L(\epsilon, \sigma, \lambda) = L_1(I_1^\sigma) L_2(I_1^\sigma) L_3(J_2^\epsilon) L_4(\lambda) \dots \dots \dots (8)$$

Although the volumetric strains considered are caused by primarily shear strains and partly by minor changes in the effective volumetric stress, the inelastic portion of the volumetric strain is dependent on the total strain. Assuming that this dependence is linear, one may write  $L_1(I_1^\sigma) = |1 + C_1 I_1^\sigma|$ , in which  $C_1$  is a material parameter. The volume changes due to shear and volumetric stresses are dependent on the effective confining stress, because of the frictional properties of soils; this dependence may be formulated in terms of the first stress invariant as  $L_2(I_1^\sigma) = (1 + C_2 I_1^\sigma / p_a)^{-1}$ , in which  $C_2$  is a material parameter. It is preferable to express the densification or dilatancy in terms of shear strains, rather than shear stresses, because it is the result of shear distortions and repacking of the soil particles; one suitable relation, which is a function of the deviatoric strain invariant, may be written as  $L_3(J_2^\epsilon) = (1 + C_3 J_2^\epsilon)^{-1}$ , in which  $C_3$  is a material parameter. The trend of volume change given by the densification-dilatancy function can be positive or negative, but the magnitude of the volume increment for a given strain increment at steady-state cyclic loading will decrease monotonically, approaching zero in the limit; this behavior can be achieved by the use of a limiting function in terms of the accumulated inelastic volumetric strain, written as  $L_4(\lambda) = (1 + C_4 \lambda)^{-1}$ , in which  $C_4$  is a material parameter. The combination of the foregoing relations allows the time-independent densification-dilatancy measure to be expressed in the form

$$d\lambda = \frac{C_0 |1 + C_1 I_1^\sigma|}{\left(1 + \frac{C_2 I_1^\sigma}{p_a}\right) (1 + C_3 J_2^\epsilon) (1 + C_4 \lambda)} d\epsilon \dots \dots \dots (9)$$

in which  $C_0$  is another material parameter that is necessary to account for the effects of stress history.

**TWO-PHASE MEDIUM AND PORE PRESSURES**

The concept of a two-phase medium (fluid and solid) is a natural choice for analyzing the undrained behavior of saturated cohesive soils. Since the compressibility of the composing soil grains is about 30 times less than the compressibility of water (Lambe and Whitman, 1969), the assumption of incompressible soil grains will not introduce any significant error. On the other hand, the fluid phase (pore water) must be treated as compressible and the solid structure (matrix of clay particles) is even more compressible than the pore water. Although Mitchell (1975) reported that the bound water has different properties than free pore water, the ratio of bound water to free pore water in most cases is small, so its effect may be neglected; it will be assumed that: (1) The compressibility of free water is the same as that of bound water; (2) both are elastic; and (3) both are capable of carrying only volumetric stresses.

The tendency of the solid structure to change its volume is coupled with

the pore pressure, and the solid structure will deform only as much as the pore water permits. If no drainage is allowed, the volume change of the solid structure will equal the volume change of the pore fluid, and the total volumetric strain can be given by Eq. 7, in which  $d\epsilon = dV_s / 3 V_s$ , in which  $V_s$  = the total volume or the volume of the solid structure. If the volume of the pore water is  $V_f = n V_s$ , in which  $n$  = the porosity, and the compressibility of water,  $C_w$ , is a constant, the pore pressure increment can be expressed as

$$du = C_w \frac{dV_s}{V_f} = C_w \frac{dV_s}{n V_s} = \frac{3 C_w}{n} d\epsilon \dots \dots \dots (10)$$

A more suitable form, which expresses pore pressure in terms of applied total stresses,  $\sigma$ , can be obtained by substituting Eq. 7 for  $d\epsilon$ , in which  $\sigma'$  may be expressed as  $d\sigma' = d\sigma - u$  ( $\sigma$  being the total volumetric stress):

$$du = \frac{C_w K}{nK + C_w} \left( \frac{d\sigma}{3K} + 3 d\epsilon'' \right) \dots \dots \dots (11)$$

From Eq. 10 it is seen that the assumption of an incompressible pore fluid leads to an infinite pore pressure, and the use of this assumption would preclude any meaningful physical explanations for the pore pressure response.

**VARIATION OF ELASTIC MODULI**

The variations of the elastic moduli along the stress path are formulated as functions of the initial magnitudes of the moduli and the ratios of the changes that take place in certain important parameters. The initial moduli may be determined from certain tests or from empirical relations given in the literature. Two important factors that change along a given stress path are the void ratio and the effective normal stress. These variables, in turn, depend on the accumulated densification-dilatancy measure,  $\lambda$ , and the first stress invariant,  $I_1^\sigma$ , which will be adopted to represent the changes in elastic moduli. Since the densification-dilatancy measure is a strain, the relative change in the void ratio can be expressed as

$$\frac{de}{e_0} = \frac{3\lambda_r (1 + e_0)}{e_0} = \frac{3\lambda_r}{n} \dots \dots \dots (12)$$

in which  $e_0$  = the initial void ratio; and  $n$  = the porosity. Experimental evidence indicates that the shear modulus can be expressed as

$$G_r = G_0 \left( 1 + b_1 \frac{I_1^{\sigma'} - I_1^{\sigma'_0}}{I_1^{\sigma'_0}} + b_2 \frac{3\lambda_r}{n} \right) \dots \dots \dots (13)$$

in which  $G_0$  and  $I_1^{\sigma'_0}$  are the initial values of the shear modulus and the first stress invariant, respectively; and  $b_1$  and  $b_2$  are empirically determined material parameters. Based on the suggestion by Lade and Musante (1976, Fig. 32), Poisson's ratio at points  $\nu$  can be estimated from the plasticity of the soil, and the bulk modulus at points along a given stress path can then be calculated as  $K = 2G(1 + \nu) / 3(1 - 2\nu)$ .

## ANALYSIS OF TEST DATA

In fitting test data with the foregoing constitutive equations, the test specimen has been assumed to be in a homogeneous state. Due to the complex nature of the differential equations, a step-by-step integration process is followed, increasing prescribed strains or stresses in small increments and using the same algorithm as that used by Bazant and Bhat (1976). The values from a previous step provide an initial estimate, and inner iterations within a step are used to obtain improved response increments within the step. For the initial interaction of the first loading increment, all incremental values must be estimated or can be taken as zero. Experience indicates that this step-by-step integration method is reasonably stable, and provided loading steps are sufficiently small, convergence is usually achieved in a couple of iterations for both the stress-controlled and strain-controlled procedures. For the stress-controlled procedure, however, the peak point can not be determined exactly and the post-peak behavior cannot be predicted.

Although the behavior of clays depends on many internal and external factors whose effects are complex and interrelated, the current state of knowledge with respect to available testing and exploration techniques justifies an analysis of only the major factors and their respective influences. The internal factors are related to the composition of the soil and primarily may be listed as: (1) The types and the amount of clay minerals; (2) the types of absorbed cations; (3) the shape and size distribution of particles; and (4) the pore fluid chemistry. On the other hand, the external factors represent the environmental conditions or the result of the environmental conditions, such as the water content, density, confining pressure, temperature, and soil fabric. One of the most influential factors is the type and amount of clay minerals present; these minerals exhibit a wide range of engineering properties and exert a strong influence on the plasticity, compressibility, and shear strength of clay soils. However, the difficulty of obtaining realistic quantitative relations for each factor may be partially avoided by using soil parameters (such as the Atterberg limits) that represent the composite behavior.

The compressibility and strength of clay soils are functions of the forces applied to distort or displace the particles or groups of particles; the physical and physico-chemical forces of particle interactions produce the available resistance, but this resistance also depends on the stress history and the stress state. Clay soils manifest a wide variation in fabric and structure; since these characteristics are not readily quantifiable, it is difficult to correlate them with constitutive properties. However, it is an oversimplification to treat the behavior of soils as a combination of two separate phenomena—friction and cohesion. Internal and external factors, as well as drainage conditions and testing technique, control both components. The approach adopted in this study considers different basic factors and their relation to the stress-strain-strength response, rather than using independent cohesion and friction parameters. The material parameters defined and used in the application of endochronic theory to clay are interrelated and represent the composite behavior. Although the proposed constitutive law is derived for clays in general, the following quantitative model is limited to normally and lightly overconsolidated clays. Overconsolidated clays exhibit different constitutive characteristics, and these need to be considered separately.

## SELECTED BASE DATA

At this stage it is necessary to select specific test data for normally consolidated clays to determine and evaluate the validity of the proposed constitutive law. The clays chosen for this study and their basic engineering properties are summarized in Table 1. Data from seven different clays that cover a relatively wide range of composition and behavioral pattern were selected from the literature. One of the basic criteria for the selection was the sample preparation method, which hopefully would not yield a significantly anisotropic fabric. As shown

TABLE 1.—Properties of Clays Used as Base Data Set

Clay type (1)	Liquid limit, as a percentage (2)	Plasticity index, as a percentage (3)	Specific gravity (4)	Poisson's ratio (5)	Consolidation pressure, in kilograms per square centimeter (6)	Dry density, in pounds per cubic foot (7)	Porosity (8)	Reference (9)
Boston Blue Clay	33	13	2.78	0.23	6.0	101.0	0.395	Ladd and Varallay (1965)
Weald Clay Grundite	43	25	2.74	0.26	2.11	104.8	0.387	Parry (1960)
					1.0	88.3	0.487	Lade and Musante (1976)
					1.5	91.5	0.468	
Kaolinite	65	35	2.65	0.28	2.0	94.1	0.454	
					1.05	66.7	0.596	Brewer (1972)
					2.11	69.6	0.579	
Vicksburg Buckshot Clay	57	36	2.69	0.28	4.22	75.4	0.544	
					1.5	91.3	0.456	Donaghe and Townsend (1975)
Eabpl Clay	79	53	2.72	0.29	1.5	77.6	0.543	
					3.0	81.5	0.520	
Kawasaki I Clay (Undisturbed)	80	38	2.75	0.28	3.0	69.8	0.593	Ladd (1964)

in Table 1, data from 12 undrained triaxial compression tests performed at various consolidation pressures were used.

The final form of the equations and the correlations of the parameters with various soil properties are based on this data set. In order to determine the effects of the parameters and their variation with respect to clay types and consolidation stress, mathematical optimization was used. Due to the nature of the formulation and the step-by-step integration, the mathematical algorithm adopted was a finite difference Levenberg-Marquardt routine for solving nonlinear



least-squares problems (developed by T. J. Aird for the International Mathematical and Statistical Subroutine Library package). The material parameters used in the constitutive relations are interrelated, and it is probably possible to have different sets of material parameters values that would give fairly good results; however, the optimization routine can only identify local optimums in the vicinity of the initial estimates. The purpose was to find a set of values such that certain correlations with respect to material properties and the stress and strain state of the clay sample can be established, but this set is not necessarily the global optimum. It was necessary to undertake many stages of optimization to determine the effects of different parameters and to achieve realistic correlations; this process was largely trial and error and based somewhat on previous knowledge about the behavior of clays. Nevertheless, the end result is a set of simple semi-empirical relations that yields the most suitable formulation.

**EMPIRICAL EXPRESSIONS FOR MATERIAL PARAMETERS**

The constitutive law defines two basic material variables (intrinsic time and densification-dilatancy measure) that represent the effects of all major factors on the volumetric and shear response of clays along any stress or strain path. In this study the internal factors are represented by the clay type only. Despite the drawbacks of this approach, there is no other more reliable method. Plasticity will be expressed in terms of the liquid limit and the plasticity index and will be used to indicate the clay type. Since only relatively slow strain rates are considered in the quasistatic tests evaluated herein, the rate dependence will be excluded.

The first material variable, the intrinsic time measure, is expressed by a set of differential equations involving seven material parameters, which are related to various engineering properties, such as plasticity and compressibility. In the rate-independent solution the intrinsic time and the rearrangement measure are directly proportional. The material parameter  $Z_1$  (Eq. 1) is termed the plasticity coefficient and is considered to be dependent only on the clay type and the stress history. The material parameter  $a$  (Eq. 5) is necessary to yield inelastic strains for cases where there is no hardening or softening; it is assumed to be dependent on the clay type, stress history, and void ratio, but can be taken as a constant in Eq. 5, perhaps because  $Z_1$  is a variable. The parameter  $a_1$  (Eq. 5) is the proportionality constant for the assumed direct relationship between the volume change and the rearrangement measure; based on the data set selected,  $a_1$  may be taken as a constant, although it seems more logical for  $a_1$  to depend on the clay type. The material parameter  $a_2$  in Eq. 5 represents the frictional behavior, but in a manner different than the angle of internal friction, because the effective confining stress is important for both frictional behavior and the development of interparticle forces; the data studied indicate that this parameter can be taken as a constant. The parameter  $a_3$  in Eq. 5 represents the component due to the effect of shear distortion, which may lead to densification or dilatancy because of interparticle movements (as rotation, rolling, or sliding). It depends on the internal, as well as external, parameters and will be treated as a variable, called the distortion coefficient, which will be represented by  $a_d$ . For normally consolidated soils, the hardening, which is partly controlled by  $\beta_1$ , has a limited range, because consolidation is excluded; on the other hand, the softening, which is partly controlled by  $\beta_2$ , is relatively significant, especially in the undrained

conditions. Therefore, it is appropriate to treat  $\beta_1$  as a constant and  $\beta_2$  as a material coefficient, called the softening coefficient,  $\beta$ .

As a consequence of the foregoing considerations, the intrinsic time may now be expressed as

$$\Delta \eta = \left[ 4 + \frac{|1 - 500 I_1^e| (1 + a_d J_2)}{0.01 + \frac{0.75 I_1^e}{p_a}} \right] \Delta \xi \dots \dots \dots (14)$$

$$\text{and } \Delta z = \frac{\Delta \eta}{Z_1 \left( 1 + \frac{5n}{1 + \beta \eta} \right)} \dots \dots \dots (15)$$

which involves only three coefficients. The other material variable, the densification-dilatancy measure,  $d\lambda$ , which is the ratio of the inelastic volumetric strain increment to the deviatoric strain increment, was expressed in terms of five material parameters for the rate-independent solution. The material parameter,  $C_0$ , in Eq. 9 is a composite parameter that depends essentially on all major factors; for normally consolidated clays this parameter is termed the densification coefficient,  $C$ , and may be treated as a function of clay type and stress history. The parameters  $C_1$ ,  $C_2$ ,  $C_3$ , and  $C_4$  may be assumed as constants for the same reasons given in the cases of  $a$ ,  $a_1$ ,  $a_2$ , and  $\beta_1$ . Accordingly, the densification-dilatancy measure can be expressed in terms of one material coefficient as

$$\Delta \lambda = \frac{C |1 + 2,500 I_1^e| \Delta \xi}{(1 + 1,000 J_2) \left( 1 + \frac{0.25 I_1^e}{p_a} \right) (1 + 9,000 \lambda)} \dots \dots \dots (16)$$

The material parameters in Eq. 13 are also taken as constants. Based on the quasistatic test data,  $b_1$  and  $b_2$  have been determined and Eq. 13 can be rewritten as

$$G = G_0 \left( 1 + \frac{I_1^{\sigma'} - I_1^{\sigma'_0}}{10 I_1^{\sigma'}} + \frac{3 \lambda}{10 n} \right) \dots \dots \dots (17)$$

**CORRELATION OF PROPOSED MATERIAL PARAMETERS FOR VARIOUS CLAYS**

When various clays are considered, the similarities of their behavior must be reflected in certain relations between the material parameters. Based on the selected data set and certain simplifying assumptions tempered with engineering judgment, it has been found that for normally consolidated clays, such correlations exist for five material parameters, including the initial elastic modulus, while the remaining 10 material parameters can be taken the same for various clays. The correlations for the coefficients have been established with respect to the measurable soil properties.

Two alternative approaches are available to obtain the values of the coefficients; the first is to perform a relatively simple stress-strain test and determine the parameters by an optimization technique, and the second is to establish correlations for each coefficient so that it can be determined without conducting a stress-strain test for the particular clay. These two approaches were combined

to yield two relatively accurate and three approximate correlations for the five material coefficients. Based on the general pattern of the results, it is assumed that the two exact correlations are valid for all normally consolidated clays. The first of these relationships can be expressed as

$$Z_1 = 0.00294 \left( \frac{e_0 \sigma_c}{p_a} \right)^2 - 0.0177 \left( \frac{e_0 \sigma_c}{p_a} \right) + 0.0396 \dots \dots \dots (18)$$

Although the initial void ratio,  $e_0$ , and the consolidation pressure,  $\sigma_c$ , individually may not represent any material properties, their product for normally consolidated clays reflects partially the effects of different clay types and stress histories. The plasticity coefficient,  $Z_1$ , accounts for the rigidity and deformability of the clays, and it can be interpreted simply as a ratio of inelastic strains to elastic strains.

The second relationship describes the distortion coefficient,  $a_d$ ; in this case the ratio of the initial void ratio to the consolidation pressure gave more consistent results, and the proposed correlation is

$$a_d = 153.8 \left( \frac{e_0 p_a}{\sigma_c} \right) + 34.62 \dots \dots \dots (19)$$

This equation has the advantage of partially including the effects of external and internal factors without using the approximate index properties. Since the distortion coefficient,  $a_d$ , represents the strain softening or hardening, it is related primarily to the clay fabric, dry density, and effective forces between particles. Although Eq. 19 is simple, it yields realistic results and enables the rest of the correlations to be developed.

The three approximate correlations for the initial elastic modulus,  $E$ , the densification coefficient,  $C$ , and the softening coefficient,  $\beta$ , are shown in Fig. 1. In general, all three correlations are determined in terms of the liquidity index,  $I_L$ , and the normalized consolidation pressure,  $\sigma_c/p_a$ , which represents the stress history, the rigidity, and partially the effect of the clay type. The variations from the proposed relations are significant in a few cases, but this is expected in view of the limited data base that was used. The reason for choosing the product or ratio of the liquidity index and the consolidation pressure is partially empirical, because they yield relatively good correlations with the coefficients; however, these correlations are approximate and only applicable for the range explored. The liquidity index reflects the rigidity and the stress history; an increase in the liquidity index increases the deformability, thereby increasing the values of  $C$  and  $\beta$  and decreasing the magnitude of the initial elastic modulus. An increase in the consolidation pressure also increases the densification coefficient, the softening coefficient, and the initial elastic modulus, assuming the liquidity index is same. The increase in  $C$  and  $\beta$  can be justified because the presence of larger normal stresses may yield larger distortions and densification. The chosen correlations are capable of representing the indicated phenomena, and they are sufficiently simple to use in practical problems.

Another more reliable method for determining  $E$ ,  $\beta$ , and  $C$  involves performing a relatively simple test, such as a consolidated-undrained triaxial compression test, and obtaining the values by an optimization procedure or trial and error. Since  $E$  can be determined from the initial slope of the stress-strain curve,

only  $\beta$  and  $C$  remain to be evaluated. With this approach,  $Z_1$  and  $a_d$  are determined from the assumed relations and the other three parameters are optimized to achieve the best fit. The optimized theoretical responses are shown in Fig. 2 together with the predicted responses obtained by using the approximate correlations. Although the optimized fits are fairly accurate for all practical purposes, some of the predicted responses are not within the desired limits. This is due partly to the approximate nature of the correlations; the selected

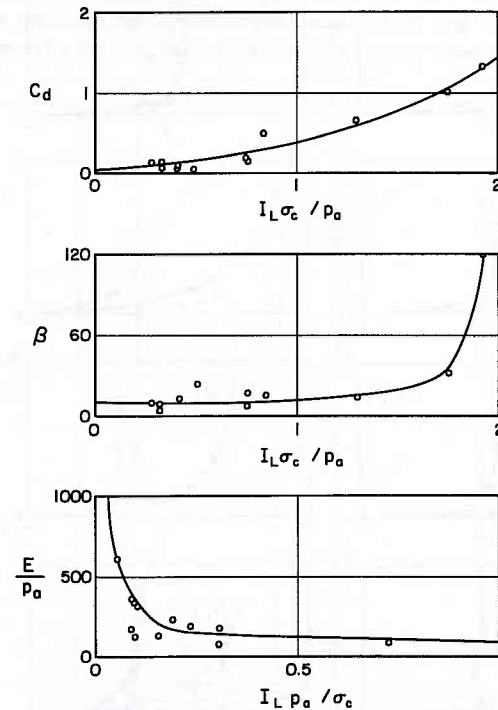


FIG. 1.—Approximate Correlation for Densification Coefficient  $C$ , Softening Coefficient  $\beta$ , and Elastic Modulus  $E$

data set involved seven different clays tested by different researchers under variable conditions.

APPLICABILITY OF PROPOSED MODEL

The proposed constitutive relation has been developed by rational considerations that are applicable to arbitrary stress paths in stress space. Thus, once the proper values of the material parameters are determined, it should be possible to predict the general three-dimensional response of the material. This ability is supported by the fits of data from three-dimensional consolidated-undrained tests performed by Lade and Musante (1976) on remolded Grundite samples (the base data set included conventional triaxial tests on the same clay). The coefficients were determined by optimizing the values of  $E$ ,  $\beta$ , and  $C$ , while



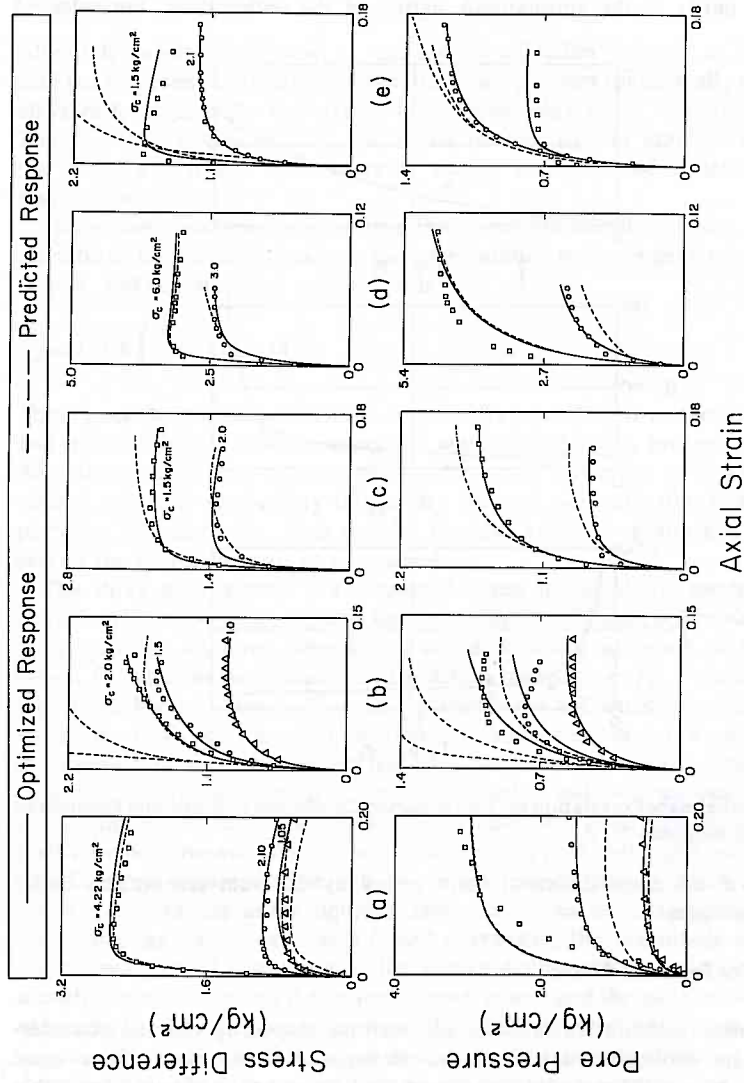


FIG. 2.—Stress-Strain-Pore Pressure Data for Selected Clays with Optimized and Approximate Predicted Responses: (a) Kaolinite (After Brewer, 1972); (b) Grundite (After Lade and Musante, 1976); (c) Eapbl Clay (After Donaghe and Townsend, 1975); (d) Kawasaki I (After Ladd, 1964) Boston Blue Clay (Ladd and Varalloy, 1965); (e) Vicksburg Buckshot Clay (After Donaghe and Townsend, 1975) and Weald Clay (After Parry, 1960)

$Z_1$  and  $a_d$  were calculated from Eqs. 18 and 19. The corresponding predictions for various values of the intermediate principal stress are shown in Figs. 3(a), 3(b), and 3(c). During the tests the value of  $B = (\sigma_2 - \sigma_3)/(\sigma_1 - \sigma_3)$  was kept constant by changing the total major and intermediate principal stresses. The predictions slightly exceed the pore pressures observed, but this is probably due to the inaccuracy of the formulation and the nature of the three-dimensional testing technique. Although the stress-strain relations are fairly accurate, they cannot predict the post-peak behavior. Another prediction, using the value of the coefficients obtained by optimization, is shown in Fig. 3(d). The extension test was performed by keeping the total confining pressure constant and decreasing

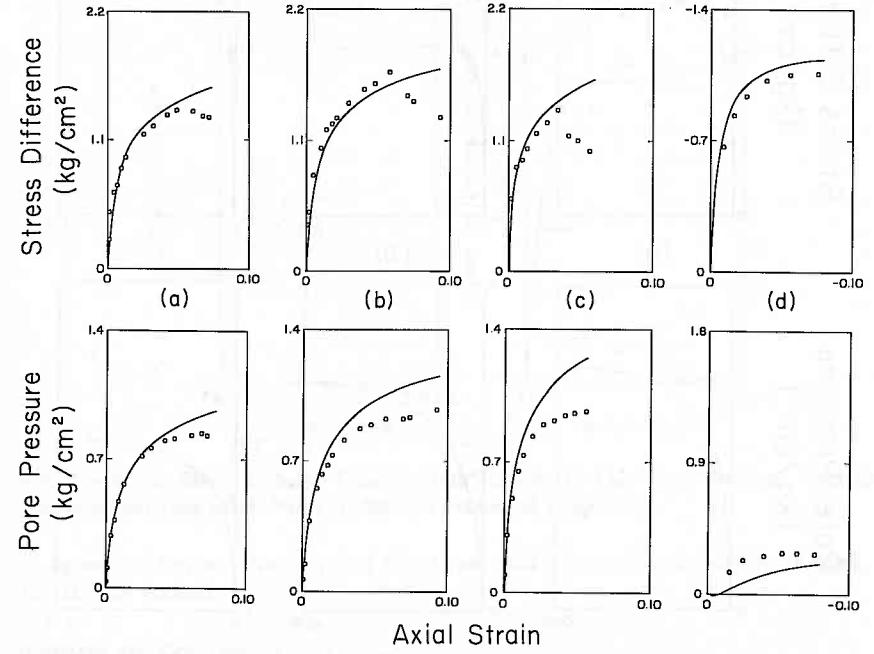


FIG. 3.—Stress-Strain Pore Pressure Data and Predicted Relations Based on Optimized Parameters—Three-Dimensional Tests on Grundite (After Lade and Musante, 1976) with Different Intermediate Principal Stresses: (a)  $B = 0.2$ ; (b)  $B = 0.4$ ; (c)  $B = 0.7$ ; (d) Triaxial Extension Test on Weald Clay (After Parry, 1960)

the axial stress (Parry, 1960). The predicted curve is reasonably accurate for all practical purposes, but the pore pressures are slightly underestimated. An attempt was also made to check the capability of the proposed approximate correlations by predicting the stress-strain-pore pressure behavior of Vicksburg Buckshot clay tested in triaxial compression [Fig. 4(a)] and the remolded Boston Blue clay tested in triaxial compression [Fig. 4(b)] and extension [Fig. 4(c)].

Results from drained triaxial compression tests reported by Parry (1960) were predicted on the basis of the parameters obtained from the undrained tests in the base data set. It was observed that the plasticity coefficient,  $Z_1$ , had to be increased (approximately threefold) for the drained case, mainly because



the effective mean stresses differ significantly from those in the undrained condition. The deviatoric stress is significantly larger in the undrained case for the same value of the stress difference between the major and minor principal stresses, causing more deformation and an underestimation of the stress difference. Thus,  $Z_1$  must be considered dependent on the drainage conditions.

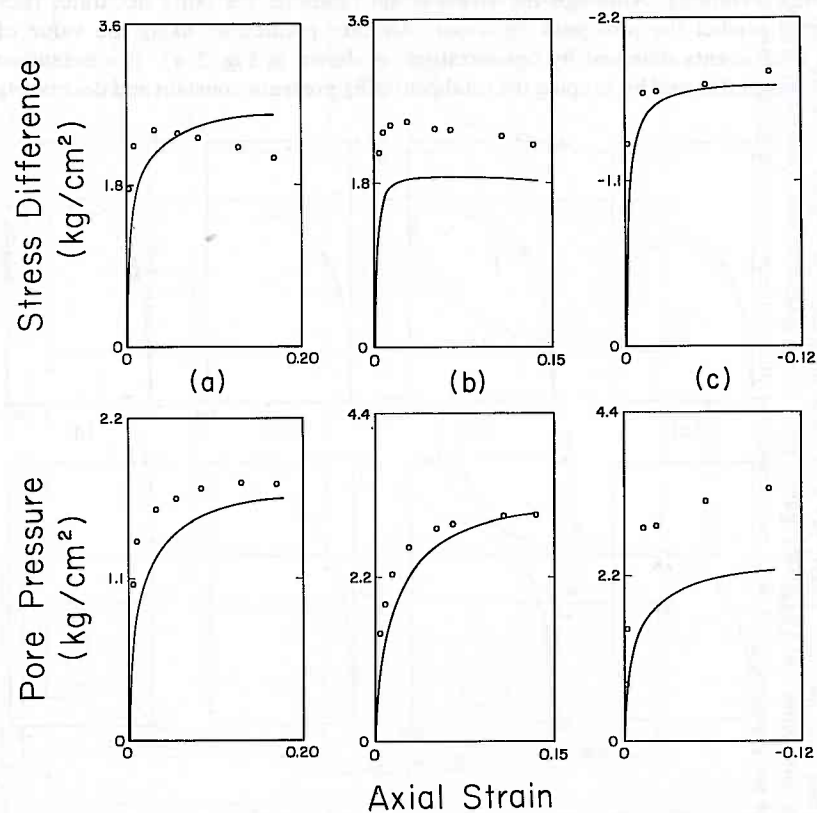


FIG. 4.—Stress-Strain Pore Pressure Data from Various Clays and Predicted Relations from Approximate Parameters: (a) Vicksburg Buckshot Clay (After Donaghe and Townsend, 1975); (b) Triaxial Compression—Boston Blue Clay (After Ladd and Varallay, 1964) (c) Triaxial Extension—Boston Blue Clay (After Ladd and Varallay, 1964)

In Fig. 5 the theoretical curve was obtained by optimizing with respect to  $Z_1$ , while the other four coefficients were kept at the same values.

#### FAILURE CRITERION

The failure criterion is implied by the present theory, and the failure envelopes for a given situation can be constructed numerically by collecting points where

strain changes with no change in stress. However, the failure envelopes depend on the stress and strain paths and cannot be expressed merely in terms of stresses and strains at failure.

#### CYCLIC LOADING

The proposed constitutive relation has been applied for cyclic loading, and good fits of test data (axial strains and pore pressures) from cyclic constant load amplitude triaxial tests on kaolinite have been obtained. This development

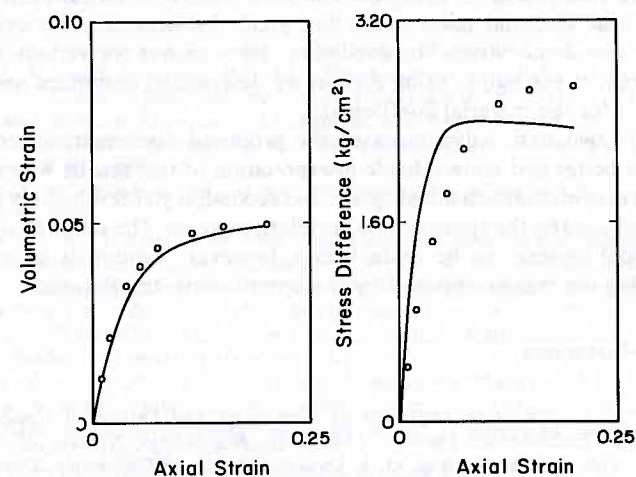


FIG. 5.—Stress-Strain-Volume Change Data for Weald Clay from Drained Triaxial Compression Test (After Parry, 1960) and Predicted Response

is, however, beyond the scope of this paper and is reported separately (Krizek, Ansal, and Bazant, 1978).

#### SUMMARY AND CONCLUSIONS

Three-dimensional constitutive relations for isotropic clays subjected to small strains are derived by using strain-rate-dependent viscoplasticity (endochronic theory). The path length in deviatoric strain space,  $\xi$ , is modified by a hardening-softening function to obtain inelastic strain increments and by other functions to obtain the inelastic densification or dilatancy increments. The change of void ratio and undrained pore pressure is expressed in terms of inelastic volume change, effective volumetric stress change, and water compressibility, by considering the clay as a two-phase medium. The model accounts for: (1) Strain softening and hardening; (2) densification and dilatancy; (3) frictional aspects; (4) strain-rate dependence of the response; and (5) pore pressure buildup due to inelastic volume changes of the solid skeleton. The specific expressions for the material parameters are semi-empirical. A total of 18 material parameters, including the initial shear modulus, are used to model and predict the quasistatic drained or undrained stress-strain-pore pressure behavior of cohesive soils. These

parameters and their relationship to soil type, test type, and stress history are investigated in an effort to attribute to them various physical interpretations in relation to the micromechanics of deformation.

The functions defining the material parameters were identified by fitting the data from 12 undrained triaxial tests performed on seven types of clay. By finding correlations between various material parameters and various characteristic properties of soils, such as initial void ratio, consolidation pressure, and liquidity index, the number of material coefficients appearing in the constitutive relation was reduced from 15 to five without losing significant accuracy. A combined trial-and-error approach and least-square optimization was utilized to identify the material parameters that yield the best fit. The capability of the theory was demonstrated by predicting the response for various other tests with different stress paths, using previously determined optimized and approximate values for the material coefficients.

There are two basic advantages of the proposed mathematical model; first, it permits a better and more reliable interpretation of test results when analyzing soil response in multidimensional space, and second, it yields relatively reasonable predictions based on the approximate correlations given. The range of applicability of the model appears to be quite broad; however, caution is in order when extrapolating the results obtained by the approximate correlations.

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14546 VISCOPLASTICITY OF CONSOLIDATED CLAYS

**KEY WORDS:** Clays; Constitutive equations; Normally consolidated soils; Pore pressure; Stress strain characteristics; Two layer soil system; Viscoplasticity

**ABSTRACT:** A constitutive law is developed to model the behavior of normally consolidated isotropic cohesive soils under multidimensional stress or strain paths. This law represents the endochronic form of viscoplasticity and involves a number of material variables that are defined in terms of semi-empirical expressions and model: (1) Strain softening and hardening; (2) densification and dilatancy; (3) frictional aspects; and (4) strain rate dependence of the response. Furthermore, by considering saturated soils as two-phase media, the model accounts for the development of pore pressures due to the volumetric strain in the solid skeleton. Data reported in the literature are used to demonstrate the applicability of the approach, and approximate correlations between material parameters are established to predict the stress-strain-pore response of normally consolidated clays.

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