RANDOM SHRINKAGE STRESSES IN AGING VISCOELASTIC VESSEL

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ABSTRACT: Shrinkage stresses caused by random variation of environmental humidity in a long cylindrical concrete wall sealed at the internal surface are analyzed, taking creep and aging of concrete into account. The random time-variation of humidity and stress is analyzed by the spectral method using the elastic-viscoelastic analogy. Spatial variations are described by Bessel and Kelvin functions. Numerical examples typical of a 1 m thick reactor containment and a thin shell of 10 cm thickness are given. The environmental humidity varies with a dominant period of one year. The results confirm that, due to low moisture diffusivity, as well as aging, the problem must be analyzed as nonstationary. The mean and the standard deviation are obtained as functions of time as well as location. Random stress fluctuations are found to be significant in all cases and the influence of aging to be strong. In the thicker containment only about 30 percent of the wall thickness ever feels the random variation, while in the thin shell the entire thickness is affected.

STATEMENT OF PROBLEM

The observed effects of shrinkage and creep in concrete structures exhibit a particularly broad random scatter, broader than that of strength and elastic properties. A probabilisitically based design for creep and shrinkage, therefore, is perhaps the most important development needed for eliminating the frequently experienced damages due to these effects.

The cause of the random scatter in creep and shrinkage is basically threefold: (a) Statistical variation of material parameters characterizing creep and shrinkage (8,9); (b) stochastic nature of the creep and shrinkage process in time (15); and (c) the influence of the random variation of environmental humidity and temperature, i.e., the variations of weather. If the production of concrete is very carefully controlled and the constitutive law reliably determined, the third cause is probably the principal one, and we will be occupied with it in this work since no detailed study of this phenomenon seems to have been made so far.

The objective is to calculate random shrinkage stresses in a cylindrical concrete vessel, the limiting case of which is a planar wall. We will take into account the relaxation of random shrinkage stresses caused by creep, but we will have to leave aside the more difficult problem of random pore humidity effect on the creep properties themselves which, in fact, overlaps with the cause (a) stated previously. We will determine for each point of the wall and for each time, the mean stress response and its standard deviation. From these, one then easily can determine the confidence limits on the stress magnitude as well as the probability distribution of the times to reach the cracking stress if the distribution (e.g., the normal distribution) is given. We will employ the method of power response spectra widely used in random vibration theory (16), but we will need to devise some novel approach to cope with the phenomenon of aging, i.e., the variation of creep properties with the age of concrete. The fact that we analyze the random response for age-dependent material properties is the main innovation of this work from the theoretical viewpoint. At the end, we will apply the theory to a nuclear reactor containment and to a thin concrete shell.

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To simplify the solution, we adopt the following assumptions: (a) The structure is assumed to be adequately prestressed so as to prevent cracking of concrete, even after the loss of prestress in concrete which may be locally quite high, as shown in Ref. 6; (b) the moisture diffusivity is independent of pore humidity and age of concrete; (c) the free (unrestrained) shrinkage is proportional to pore humidity, which is an acceptable assumption for humidities over 50%; (d) the creep (constitutive) law obeys the principle of superposition, i.e., is linear, which is an acceptable assumption for service stress levels; (e) the creep Poisson ratio is constant, which is a reasonable assumption for concrete (2,18,23); (f) the effect of simultaneous temperature variation is neglected; (g) the effect of pore humidity variation on creep is neglected; (h) the stochastic nature of the creep (constitutive) properties is neglected, as already mentioned; (i) the strains are small enough to be related linearly to displacements; (j) the material is homogeneous; and (k) the stiffness of the liner is negligible.

Without creep and aging, the problem would be mathematically identical to the problem of thermal stresses produced in an elastic cylinder by random variation of ambient temperature, solved by Heller and co-workers (21). The present solution represents an extension of their work and the preceding ones (19,20). We should observe also that mathematically the problem is identical to that of random thermal stresses in a viscoelastic aging cylinder, and it would be easy to extend the present solution to simultaneous random variation of humidity and temperature.

VARIATION OF HUMIDITY IN CYLINDRICAL WALL

Consider a cylindrical wall of inner radius, a, and outer radius, b, with cylindrical coordinates, r (radius), θ (polar angle), and z (axial distance). Its inner surface is sealed and the outer surface is exposed to an environment of randomly fluctuating humidity (Fig. 1).

To make an analytical solution feasible, we must assume that the diffusion equation governing humidity in concrete is linear, i.e., the diffusivity is independent of pore humidity. This is not a very good assumption for concrete (3,7), but it is an acceptable approximation which is used widely. We also neglect the dependence of diffusivity on age, the possible increase of diffusivity due to cracking, and possible thermal moisture transport due to temperature gradients. These

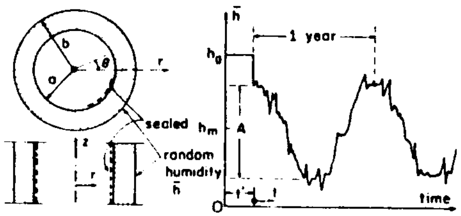

FIG. 1.—Cylindrical Wall Subjected to Random Humidit

simplifications mean that we can consider for the diffusivity a constant value, which should be selected as the time and space average of the actual variable diffusivity.

The frequency response function is obtained by determining the stationary solution of the diffusion equation:

$$\frac{\partial \bar{h}}{\partial t} = C\left(\frac{\partial^2 \bar{h}}{\partial r^2} + \frac{1}{r}\frac{\partial \bar{h}}{\partial r}\right) \quad \text{for} \quad a < r < b \quad \text{(1)}$$

with the following boundary and initial conditions:

$$\bar{h}(b,t) = e^{i\omega t}; \quad \frac{\partial \bar{h}}{\partial r}(a,t) = 0; \quad \bar{h}(r,0) = 0 \quad \text{(2)}$$

The real part of complex variable $\bar{h}$ is the humidity in concrete (relative vapor pressure in the pores); C = moisture diffusivity of concrete (a real number, m²/day); ω = circular frequency of the harmonic input (rad/day); and $i = \sqrt{-1}$.

The solution of the preceding equations is obtained (see, e.g., Ref. 12) as

$$\bar{h}(r,t) = \sum_{n=1}^{\infty} \frac{C\lambda_n^2}{C\lambda_n^2 + i\omega} a_n R_n(r)(e^{i\omega t} - e^{-C\lambda_n^2 t}) \quad \text{(3)}$$

in which a_n and $R_n(r)$ are real numbers and functions given by

$$a_n = \frac{\pi J_1^2(\lambda_n a)}{J_1^2(\lambda_n a) - J_o^2(\lambda_n b)}; \quad R_n(r) = Y_o(\lambda_n b) J_o(\lambda_n r) - J_o(\lambda_n b) Y_o(\lambda_n r) \quad \text{(4)}$$

λ_n are the roots of the equation:

$$J_1(\lambda a) Y_o(\lambda b) - J_o(\lambda b) Y_1(\lambda a) = 0 \quad \text{(5)}$$

in which J_n, Y_n = Bessel functions of the first and second kind of order n. To obtain root λ_n for large n, an asymptotic expression for the roots of Eq. 5 may be used (see Ref. 1). In numerical calculations, this asymptotic expression has been used for all n to obtain the first estimate for an iterative solution of Eq. 5 (for which the bisection method was used). The asymptotic expression gives good results even for small n if $a >> b - a$.

According to Eq. 3, the frequency response function of humidity (a complex variable) is

$$H(r,\omega) = \sum_{n=1}^{\infty} \frac{C\lambda_n^2}{C\lambda_n^2 + i\omega} a_n R_n(r) \quad \text{(6)}$$

Alternatively, substituting $\bar{h}(r,t) = H(r,\omega)e^{i\omega t}$ into the original equations (Eqs. 1–2), the following form for $H(r,\omega)$ is obtained:

$$H(r,\omega) = \frac{K_1(ka\sqrt{i})\, I_o(kr\sqrt{i}) + I_1(ka\sqrt{i})\, K_o(kr\sqrt{i})}{K_1(ka\sqrt{i})\, I_o(kb\sqrt{i}) + I_1(ka\sqrt{i})\, K_o(kb\sqrt{i})} \quad \text{(7)}$$

in which $k = \sqrt{(\omega/C)}$; and I_n, K_n = modified Bessel functions of the first and the second kind of order n. The modified Bessel functions with imaginary argument can be expressed (1) as

$$(i)^n I_n(z\sqrt{i}) = ber_n z + i\, bei_n z; \quad (i)^{-n} K_n(z\sqrt{i}) = ker_n z + i\, kei_n z \quad \text{(8)}$$

in which $ber_n z$, $bei_n z$, $ker_n z$, and $kei_n z$ are the Kelvin functions of order n.

The impulse response function of humidity, $h(r,t)$ may be obtained as the Fourier transform of the frequency response function (16), i.e.:

$$h(r,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} H(r,\omega)\, e^{i\omega t}\, d\omega = \sum_{n=1}^{\infty} C\lambda_n^2 a_n R_n(r) e^{-C\lambda_n^2 t} \quad (t \geq 0) \quad \text{(9)}$$

Since the random environmental humidity history may be expanded into Fourier series (or Fourier integral), we may restrict attention to the following boundary and initial conditions:

$$\bar{h}(b,t) = h_m + \phi(t); \quad \phi(t) = A\, e^{i(\omega_o t + \phi_o)}; \quad \frac{\partial \bar{h}}{\partial r}(a,t) = 0; \quad \bar{h}(r,0) = h_o \quad \text{(10)}$$

i.e., the wall is initially at uniform humidity, h_o, and the outer surface is subjected to a harmonic change of humidity. The inner surface is sealed as before. h_m = the mean value of the environmental humidity such as the annual mean value; ϕ_o = initial phase; ω_o = given circular frequency; and A = amplitude. In reality, of course, $\phi(t)$ is given as a series of harmonic functions. In that case, however, the solution can be obtained separately for each term of the series and the results then may be superimposed, which is straightforward.

h_o, h_m, ω_o, and ϕ_o are considered to be deterministic (real numbers), while A (real number) is a random variable characterized by the following expectation and variance:

$$E\{A\} = 0; \quad E\{|A|^2\} = \sigma_o^2 \quad \text{(11)}$$

in which E denotes the expectation; and σ_o is the standard deviation. The deterministic data on the boundary give the following mean value of pore humidity (real function):

$$\eta(r,t) = h_m + (h_o - h_m)\sum_{n=1}^{\infty} a_n R_n(r)\, e^{-C\lambda_n^2 t} \quad \text{(12)}$$

The stochastic part of the data gives a zero mean and a nonzero variance. The autocorrelation function and the spectral density of $\phi(t)$ (complex variables) are given as

$$R_{\phi\phi}(t_1,t_2) = \begin{cases} \sigma_o^2\, e^{i\omega_o}(t_1 - t_2) & \text{for} \quad t_1, t_2 \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{(13)}$$

$$S_{\phi\phi}(\omega) = 2\pi\sigma_o^2\,\delta(\omega - \omega_o) \quad (14)$$

in which $\delta(\omega)$ is the delta function. Then, the autocorrelation function of humidity, $R_{hh}(t_1,t_2)$, and the cross-correlation function of humidity, $R_{h\phi}(t_1,t_2)$ (complex variables), can be obtained (25) as follows:

$$R_{hh}(t_1,t_2) = \int_{-\infty}^{\infty} R_{h\phi}(t_1,t_2-\tau)\,h(\tau)\,d\tau;$$

$$R_{h\phi}(t_1,t_2) = \int_{-\infty}^{\infty} R_{\phi\phi}(t_1-\tau,t_2)\,h(\tau)\,d\tau \quad (15)$$

in which $h(\tau)$ = impulse response function of humidity = $h(r,\tau)$. Substituting Eqs. 9 and 13 into Eq. 15

$$R_{hh}(t_1,t_2) = \sigma_o^2 e^{i\omega_o(t_1-t_2)}[H(r) - Q(r,t_1)][H^*(r) - Q^*(r,t_2)] \quad (16)$$

in which $H(r)$ = frequency response function = $H(r,\omega_o)$:

$$Q(r,t) = \sum_{n=1}^{\infty}\frac{C\lambda_n^2}{C\lambda_n^2 + i\omega_o}a_nR_n(r)\,e^{-(C\lambda_n^2+i\omega_o)t} \quad (17)$$

and asterisks denote complex conjugates. The standard deviation is then obtained as

$$\sigma(r,t) = \sqrt{R_{hh}(t,t)} = \sigma_o\,|H(r) - Q(r,t)| \quad (18)$$

For $t \to \infty$, the variance, σ^2, approaches a stationary value

$$\sigma(r,\infty) = \sigma_o\,|H(r)| \quad (19)$$

since the transient term, $Q(r,t)$, tends to zero. This transient term is due strictly to the initial condition and, in contrast to stresses, has nothing to do with the aging of concrete because the effect of age on diffusivity (3,7) is neglected. (However, using a Dirichlet series expansion of C as a function of t, similar to Eq. 36 in the sequel, it would be possible to consider such an effect in our approach.)

Variation of Stresses and Deformations in Cylindrical Wall

Due to the humidity variation analyzed so far, shrinkage strain, ϵ_o, is produced in the cylindrical wall. Because it is distributed nonuniformly, shrinkage stresses are produced. These are then relaxed due to creep of concrete.

We will consider concrete to be an aging, linearly viscoelastic material with a time-invariant creep Poisson's ratio, ν. For the solution, it is convenient to characterize the viscoelastic properties by means of the relaxation function, $E_R(t,t')$, the determination of which, from the compliance function (creep function), $J(t,t')$, is well-known (10). No restriction on the form of relaxation function or the corresponding compliance function need be made. However, the dependence of the creep properties on the random or even mean pore humidity cannot be considered without rendering the present analytical approach impossible. Thus, the relaxation function must be chosen so as to represent the average

creep properties of the cross section for the mean environmental humidity.

By virtue of the linearity of the constitutive relation, the elastic-viscoelastic analogy (3,13,14,17,22,26) is applicable and represents the easiest method of solution. We have to use, of course, the generalized form of this analogy that is applicable to aging materials (3,22,26) (and cannot be stated in terms of Laplace transform, in contrast to classical viscoelasticity). According to this analogy, the Young's modulus, E_c, of concrete may be replaced in the elastic solution by the Volterra integral operator, $\underline{E}$, for creep of concrete, which is defined by the following stress-strain relation:

$$s_{ij}(t) = \frac{1}{1+\nu}\underline{E}\,e_{ij}(t) = \frac{1}{1+\nu}\left[E_c(t)e_{ij}(t) - \int_0^t \frac{\partial E_R(t,\tau)}{\partial\tau}e_{ij}(\tau)d\tau\right];$$

$$S_{kk}(t) = \frac{1}{1-2\nu}\underline{E}\,\epsilon_{kk}(t) = \frac{1}{1-2\nu}\left[E_c(t)\,\epsilon_{kk}(t) - \int_0^t \frac{\partial E_R(t,\tau)}{\partial\tau}\epsilon_{kk}(\tau)d\tau\right] \quad (20)$$

in which $E_R(t,\tau)$ = relaxation function = normal stress in concrete at age t caused by a unit normal strain imposed at age τ and held constant afterwards; $s_{ij} = S_{ij} - (1/3)\,\delta_{ij}S_{kk}$ = deviatoric part of stresses, S_{ij}; $e_{ij} = \epsilon_{ij} - (1/3)\,\delta_{ij}\epsilon_{kk}$ = deviatoric part of strains, ϵ_{ij} (considered to be small); and δ_{ij} = Kronecker's delta (S_{ij}, ϵ_{ij} = real variables). Note that, by definition

$$E_c(t) = E_R(t,t) \quad (21)$$

Latin lower case subscripts refer to local cartesian coordinates, x_i (i = 1, 2, 3 or $i = r, \theta, z$), and repetition of subscripts implies summation.

The cylinder is assumed to be sufficiently long so that each plane normal to the cylinder axis is in a state of plane strain. To simplify the solution, the stiffness of the liner that seals the inner surface is assumed to be negligible, and so the radial stress, S_{rr}, vanishes at both surfaces. This makes the stress boundary conditions of concrete wall homogeneous, which is a crucial property that makes possible the analytical solution based on elastic-viscoelastic analogy. Otherwise (i.e., when there is a liner of nonzero stiffness), it would be necessary to solve the Volterra integral equations numerically, e.g., by the collocation method. In practical nuclear containments, the stiffness of the liner is normally less than 10% of the concrete wall stiffness, which justifies our neglect of the liner stiffness.

Replacing E_c in the known elastic solution (11) of a hollow infinite cylinder without a liner by operator $\underline{E}$, we get

$$S_{rr}(r,t) = \frac{E_c(t)}{(1-\nu)r^2}\left[\frac{r^2-a^2}{b^2-a^2}\int_a^b r\,\epsilon_o(r,t)\,dr - \int_a^r r\,\epsilon_o(r,t)\,dr\right]$$

$$-\frac{1}{(1-\nu)r^2}\left[\frac{r^2-a^2}{b^2-a^2}\int_a^b r\int_o^t \epsilon_o(r,\tau)\frac{\partial E_R(t,\tau)}{\partial\tau}d\tau\,dr\right.$$

$$\left.-\int_a^r r\int_o^t \epsilon_o(r,\tau)\frac{\partial E_R(t,\tau)}{\partial\tau}d\tau\,dr\right] \quad (22)$$

$$S_{\theta\theta}(r,t) = \frac{E_c(t)}{(1-\nu)r^2}\left[\frac{r^2+a^2}{b^2-a^2}\int_a^b r\,\epsilon_o(r,t)dr + \int_a^r r\,\epsilon_o(r,t)dr - r^2\,\epsilon_o(r,t)\right]$$

$$-\frac{1}{(1-\nu)r^2}\left[\frac{r^2+a^2}{b^2-a^2}\int_a^b r\int_o^t \epsilon_o(r,\tau)\frac{\partial E_R(t,\tau)}{\partial\tau}d\tau\,dr\right.$$

$$\left.+\int_a^r r\int_o^t \epsilon_o(r,\tau)\frac{\partial E_R(t,\tau)}{\partial\tau}d\tau\,dr - r^2\int_o^t \epsilon_o(r,\tau)\frac{\partial E_R(t,\tau)}{\partial\tau}d\tau\right] \quad (23)$$

$$S_{zz}(r,t) = \nu\,[S_{rr}(r,t) + S_{\theta\theta}(r,t)] - \left[E_c(t)\,\epsilon_o(r,t)\right.$$

$$\left.-\int_o^t \epsilon_o(r,\tau)\frac{\partial E_R(t,\tau)}{\partial\tau}d\tau\right] \quad (24)$$

We must now relate humidity and shrinkage. To make an analytical solution feasible, we will employ a linear relation. This is an acceptable approximation within a relatively broad humidity range (24), 0.5–1.0, and so

$$\epsilon_o(r,t) = -\kappa\,[h_o - \bar h(r,t)] \quad \text{for} \quad 0.5 \le \bar h \le 1.0 \quad (25)$$

in which h_o (a real number) is the initial reference value of humidity $\bar h(r,t)$. Coefficient κ (a real number) may be determined from experimental data (24).

Since $\bar h$ is complex, ϵ_o is a complex number and the free (unrestrained) shrinkage strain is considered to be the real part of ϵ_o. Then, because the stresses, S_{rr}, $S_{\theta\theta}$, and S_{zz}, are linear functionals of ϵ_o (Eqs. 22–24), they must be considered as complex variables, and so must be the strains e_{ij} and ϵ_{kk} (because of Eq. 20). The actual stresses and strains are considered to be the real parts of these complex variables. Substituting the mean value of humidity, Eq. 12, into Eq. 25, and then Eq. 25 into Eqs. 22–24, we get the mean values of stresses (real variables):

$$\eta_r(r,t) = \kappa\,(h_o - h_m)\sum_{n=1}^{\infty} a_n C_{rn}(r)\,[E_c(t)\,e^{-C\lambda_n^2 t} - L_2(t)];$$

$$\eta_\theta(r,t) = \kappa\,(h_o - h_m)\sum_{n=1}^{\infty} a_n C_{\theta n}(r)\,[E_c(t)\,e^{-C\lambda_n^2 t} - L_2(t)];$$

$$\eta_z(r,t) = \kappa\,(h_o - h_m)\left\{\sum_{n=1}^{\infty} a_n C_{zn}(r)\,[E_c(t)e^{-C\lambda_n^2 t} - L_2(t)]\right.$$

$$\left.+[E_c(t) - L_1(t)]\right\} \quad (26)$$

in which $$C_{rn}(r) = \frac{1}{(1-\nu)r^2}\left[\frac{r^2-a^2}{b^2-a^2}\int_a^b r\,R_n(r)\,dr - \int_a^r r\,R_n(r)\,dr\right];$$

$$C_{\theta n}(r) = \frac{1}{(1-\nu)r^2}\left[\frac{r^2+a^2}{b^2-a^2}\int_a^b r\,R_n(r)\,dr + \int_a^r r\,R_n(r)\,dr - r^2R_n(r)\right];$$

$$C_{zn}(r) = \nu\,[C_{rn}(r) + C_{\theta n}(r)] - R_n(r) \quad (27)$$

$$L_1(t) = \int_o^t \frac{\partial E_R(t,\tau)}{\partial\tau}d\tau = E_R(t,\tau) - E_R(t,0);$$

$$L_2(t) = \int_o^t \frac{\partial E_R(t,\tau)}{\partial\tau}e^{-C\lambda_n^2\tau}\,d\tau \quad (28)$$

The integrals in Eqs. 27 are evaluated as follows:

$$\int_a^r r\,R_n(r)\,dr = \frac{1}{\lambda_n}\{Y_o(\lambda_n b)\,[r\,J_1(\lambda_n r) - a\,J_1(\lambda_n a)]$$

$$-J_o(\lambda_n b)\,[r\,Y_1(\lambda_n r) - a\,Y_1(\lambda_n a)]\}; \quad \int_a^b r\,R_n(r)\,dr = \frac{2}{\pi\lambda_n^2} \quad (29)$$

The impulse response functions for stresses may be obtained by substituting the impulse response function of humidity, Eq. 9, into Eqs. 22–24:

$$h_j(r,t) = \kappa\sum_{n=1}^{\infty} C\lambda_n^2\,a_n\,C_{jn}(r)\,[E_c(t)\,e^{-C\lambda_n^2 t} - L_2(t)] \quad (j = r,\theta,z) \quad (30)$$

The autocorrelation functions of stresses (complex variables) are then calculated as

$$R_{j\phi}(t_1,t_2) = \int_{-\infty}^{\infty} R_{\phi\phi}(t_1-\tau,\,t_2)\,h_j(\tau)\,d\tau \quad (31)$$

$$R_{jj}(t_1,t_2) = \int_{-\infty}^{\infty} R_{j\phi}(t_1,\,t_2-\tau)\,h_j(\tau)\,d\tau;$$

$$= \sigma_o^2\,e^{i\omega_o(t_1-t_2)}\,P_j(r,t_1)\,P_j^*(r,t_2) \quad (j = r,\theta,z) \quad (32)$$

in which $$P_j(r,t) = \kappa\sum_{n=1}^{\infty} C\lambda_n^2\,a_n\,C_{jn}(r)\int_o^t [E(\tau)e^{-C\lambda_n^2\tau}$$

$$-L_2(\tau)]\,e^{-i\omega_o\tau}\,d\tau \quad (j = r,\theta,z) \quad (33)$$

The standard deviation is obtained as

$$\sigma_j(r,t) = \sigma_o|P_j(r,t)| \quad (j = r,\theta,z) \quad (34)$$

Given a relaxation function, $E_R(t,\tau)$, the mean values and the standard deviations for stresses may now be readily calculated. In order to express the relaxation function, it is most effective to use the aging Maxwell chain model (3,10), which was shown to be generally capable of accurately approximating test data. For this model (3,10)

$$E_R(t,\tau) = \sum_{\mu=1} E_\mu(\tau)\,e^{-(t-\tau)/\tau_\mu} + E_\infty(\tau) \quad (35)$$

in which $E_\mu(\tau)$, $E_\infty(\tau)$ = age-dependent moduli of the Maxwell chain; and τ_μ

= relaxation times, which should be spaced uniformly in log-time with a spacing not exceeding log 10, i.e., $\tau_\mu = \tau_1\, 10^{\mu-1}$ ($\mu = 1, 2, \ldots m$). The approximation is then good within the time range $0.5\,\tau_1 \le \tau \le \tau_m$.

As a novel step that makes it possible to solve the problem analytically with the consideration of aging, we now expand time functions $E_\mu(\tau)$ and $E_\infty(\tau)$ into a series of real exponentials called Dirichlet series:

$$E_\mu(\tau) = E_{o_\mu} + \sum_{j=1}^{N+1} \hat{E}_{j\mu}\, e^{-\tau/\tau_j'}; \quad E_\infty(\tau) = E_{o\infty} + \sum_{j=1}^{N+1} \hat{E}_{j\infty}\, e^{-\tau/\tau_j'} \qquad (36)$$

in which τ_j' may, but need not, be chosen the same as τ_μ in Eq. 35. Similarly to the τ_μ, a suitable choice is $\tau_j' = \tau_1'\, 10^{j-1}$, and the time range is $0.5\,\tau_1' \le \tau \le \tau_{N+1}'$. Coefficients E_{o_μ}, $E_{o\infty}$, $\hat{E}_{j\mu}$, and $\hat{E}_{j\infty}$ are constants and may be found by minimizing the sum of square deviations of Eqs. 36 from given functions $E_\mu(\tau)$ and $E_\infty(\tau)$. The method to do this is well-known (3,10) and is briefly described in Appendix I. A Dirichlet series expansion of this type (Eq. 36) was already used in Eq. 5 of Ref. 5 to enable analytical integration for an age-dependent Kelvin chain model. Note that Eqs. 36 are valid only within a certain range of τ, depending on the number of terms $(N + 1)$.

In calculations, the expressions for $E_\mu(\tau)$ and $E_\infty(\tau)$, given in Ref. 10 for Ross Dam test data, were used; they consist of a sum of power functions of τ (see Eqs. 45), each of which was approximated by Dirichlet series according to the formula given on p. 1064 (Eq. 29) of Ref. 4. The sum of these Dirichlet series then leads to Eqs. 36.

Using Eq. 35, $P_j(r,t)$ ($j \equiv r, \theta, z$) now can be expressed in the following form:

$$P_j(r,t) = P_{j\infty}(r) - Q_j(r,t) \quad (j = r, \theta, z) \qquad (37)$$

in which $P_{j\infty}(r) = P_j(r,\infty)$ and

$$P_{r\infty}(r) = \frac{E_o\kappa}{(1-\nu)r^2}\left[\frac{r^2 - a^2}{b^2 - a^2}\int_a^b rH(r,\omega_o)\,dr - \int_a^r rH(r,\omega_o)\,dr\right];$$

$$P_{\theta\infty}(r) = \frac{E_o\kappa}{(1-\nu)r^2}\left[\frac{r^2 + a^2}{b^2 - a^2}\int_a^b rH(r,\omega_o)\,dr + \int_a^r rH(r,\omega_o)\,dr - r^2 H(r,\omega_o)\right];$$

$$P_{z\infty}(r) = \nu\,[P_{r\infty}(r) + P_{\theta\infty}(r)] - \kappa\, E_o\, H(r,\omega_o) \qquad (38)$$

$$E_o = \sum_{\mu=1}^{m} \frac{E_{o_\mu}\,\tau_\mu\, i\,\omega_o}{1 + \tau_\mu i\omega_o} + E_{o\infty} \qquad (39)$$

and $$Q_j(r,t) = \kappa \sum_{n=1}^{\infty} \frac{C\lambda_n^2}{C\lambda_n^2 + i\omega_o}\, a_n C_{jn}(r)\, E_n(t) \quad (j = r, \theta, z) \qquad (40)$$

$$E_n(t) = e^{-i\omega_o t}\left\{\left(\sum_{\mu=1}^{m} E_{o_\mu} + E_{o\infty}\right) e^{-C\lambda_n^2 t}\right.$$

$$- \sum_{\mu=1}^{m} \frac{E_{o_\mu}}{(1 + i\omega_o\tau_\mu)\left(\dfrac{1}{\tau_\mu} - C\lambda_n^2\right)}\left[\left(\frac{1}{\tau_\mu} + i\omega_o\right) e^{-C\lambda_n^2 t} - (C\lambda_n^2 + i\omega_o)\, e^{-t/\tau_\mu}\right]\Bigg\}$$

$$- (C\lambda_n^2 + i\omega_o)\left\{\sum_{j=1}^{N+1}\left(\sum_{\mu=1}^{m} \hat{E}_{j\mu} + \hat{E}_{j\infty}\right)\frac{1 - e^{-(C\lambda_n^2 + 1/\tau_j' + i\omega_o)t}}{C\lambda_n^2 + \dfrac{1}{\tau_j'} + i\omega_o}\right.$$

$$- \sum_{\mu=1}^{m}\sum_{j=1}^{N+1} \frac{\left(1 - \dfrac{\tau_j'}{\tau_\mu}\right)\hat{E}_{j\mu}}{1 + C\lambda_n^2\tau_j' - \dfrac{\tau_j'}{\tau_\mu}}\left[\frac{1 - e^{-(C\lambda_n^2 + 1/\tau_j' + i\omega_o)t}}{C\lambda_n^2 + \dfrac{1}{\tau_j'} + i\omega_o} - \frac{1 - e^{-(1/\tau_\mu + i\omega_o)t}}{\dfrac{1}{\tau_\mu} + i\omega_o}\right]$$

$$\left. - \sum_{j=1}^{N+1} \frac{\hat{E}_{j\infty}}{1 + C\lambda_n^2\tau_j'}\left[\frac{1 - e^{-(C\lambda_n^2 + 1/\tau_j' + i\omega_o)t}}{C\lambda_n^2 + \dfrac{1}{\tau_j'} + i\omega_o} - \frac{1 - e^{-i\omega_o t}}{i\omega_o}\right]\right\} \qquad (41)$$

Therefore, the autocorrelation functions (complex variables) and the standard deviations of stresses may be expressed as follows:

$$R_{jj}(t_1,t_2) = \sigma_o^2\, e^{i\omega_o(t_1 - t_2)}\,[P_{j\infty}(r) - Q_j(r,t_1)][P_{j\infty}^*(r) - Q_j^*(r,t_2)] \qquad (42)$$

$$\sigma_j(r,t) = \sigma_o|P_{j\infty}(r) - Q_j(r,t)| \quad (j = r, \theta, z) \qquad (43)$$

in which asterisks denote complex conjugates.

The solution for a nonaging material can be easily obtained as a special case by setting $\hat{E}_{j\mu}$ and $\hat{E}_{j\infty}$ zero, and $E_{o_\mu} = E_\mu(\tau)$, $E_{o\infty} = E_\infty(\tau)$. In this case, the asymptotic values of the standard deviation for very long times become

$$\sigma_j(r,\infty) = \sigma_o|P_{j\infty}(r)| \quad (j = r, \theta, z) \qquad (44)$$

since the transient term $Q_j(r,t)$, approaches zero as $t \to \infty$. These asymptotic values correspond to the steady-state solution, which could be more directly obtained by characterizing the nonaging viscoelastic behavior in terms of the complex modulus. The complex modulus method is inapplicable here due to the presence of the transient terms which are a result of both the aging and the initial condition.

The preceding method of solution can be easily adapted to a planar wall. In that case, the solution is obtained by replacing Bessel functions with trigonometric functions and the roots, λ_n, with multiples of π. However, since a planar wall, whose middle surface is prevented from warping, is a limiting case of a hollow cylinder as the inner radius tends to infinity, the solution of a planar wall can be obtained by substituting a large value of a such that $a >> b - a$.

The present method of handling the aging creep under random humidity can be used for any structure with homogeneous stress boundary conditions for which an analytical elastic solution is available, and even for arbitrary structures in conjunction with the finite element method in space.

The present solution is also applicable to stresses due to random temperature. Here, however, the creep and aging properties are even more sensitive to temperature than to humidity, which would probably make the neglect of this effect more questionable.

Numerical Examples and their Implications

Consider an infinitely long cylindrical wall with inner radius 20 m and thickness 1 m (Example 1). These are typical values for a nuclear reactor containment shell. We choose the following physical constants: moisture diffusivity $C = 0.3 \times 10^{-4}$ m^2/day, Poisson's ratio $\nu = 0.18$, and shrinkage constant $\kappa = 1.6 \times 10^{-3}$. The initial humidity in concrete is taken as $h_o = 1.0$. For simplicity of calculation, we consider the power spectrum of the random environmental humidity to be the delta function, characterized by mean value $h_m = 0.7$, period $T = 365$ days, standard deviation $\sigma_o = 0.2$, and initial phase $\phi_o = 0.0$.

We consider two ages of concrete at loading: 28 days and 40 yr. The former is a realistic value, while the latter is considered for comparison as a case in which aging is negligible. The coefficients for the Maxwell chain relaxation function are taken from Ref. 10 as determined from the test data for Ross Dam. Also, $\tau_1 = 5$ days and $N = 6$ in Eq. 47 (Appendix I). Also, for comparison, we will also consider a thinner wall with inner radius 10 m, thickness 0.1 m, and the same physical constants (Example 2).

A computer program has been written (in complex arithmetic) to obtain the solution. Fig. 2(a) shows the calculated time variation of humidity. All plots of time variation (for both examples) are given for the point at 5 cm from the outer surface at which the wall is significantly affected by the random environmental humidity. The mean value, η, gradually approaches the mean value, h_m, which happens faster in the thinner wall. The standard deviation, σ, exhibits initial transient behavior and then it approaches a stationary value.

The distribution of humidity across the wall is shown in Fig. 2(b). All distribution diagrams are plotted for the time t = 10 yr. It is seen that a constant humidity state of 70% has already been reached in the thin wall, while the humidity at the center of the thick wall is still 98%. However, in both cases, the standard deviation at each point has almost attained the stationary value within the accuracy of drawing. Thus, it also represents the frequency response function. In the case of the thin wall, the entire wall is affected by the humidity change, while in the case of the thick wall, only 30% of the thickness is affected.

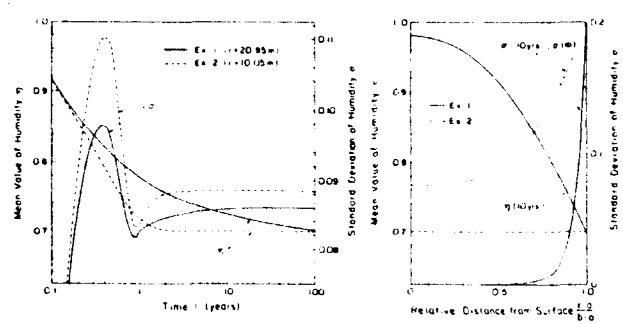

FIG. 2.—(a) Time Variation of Humidity; (b) Distribution of Humidity at t = 10 yr

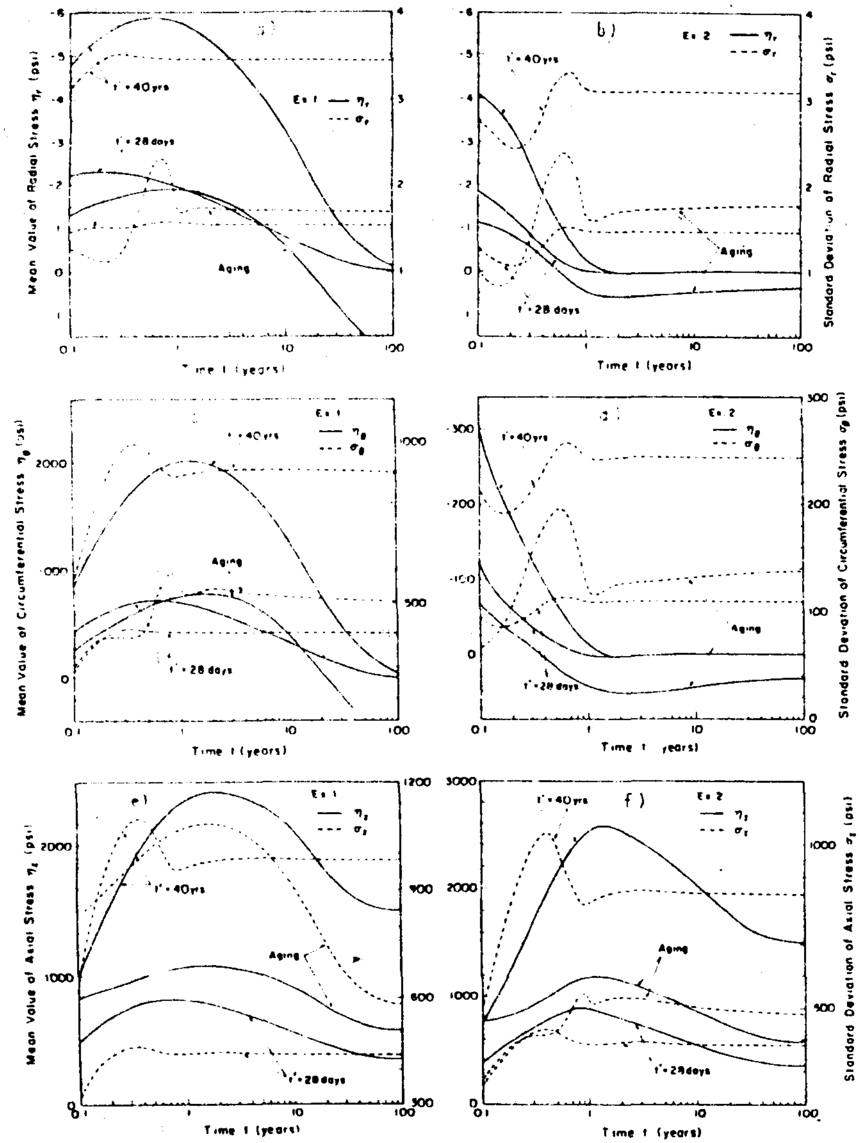

FIG. 3.—Time Variation of Stresses: a, c, e—Containment (1 m thickness); b, d, f—Thin Shell (0.1 m thickness). (1 psi = 6.89 kPa)

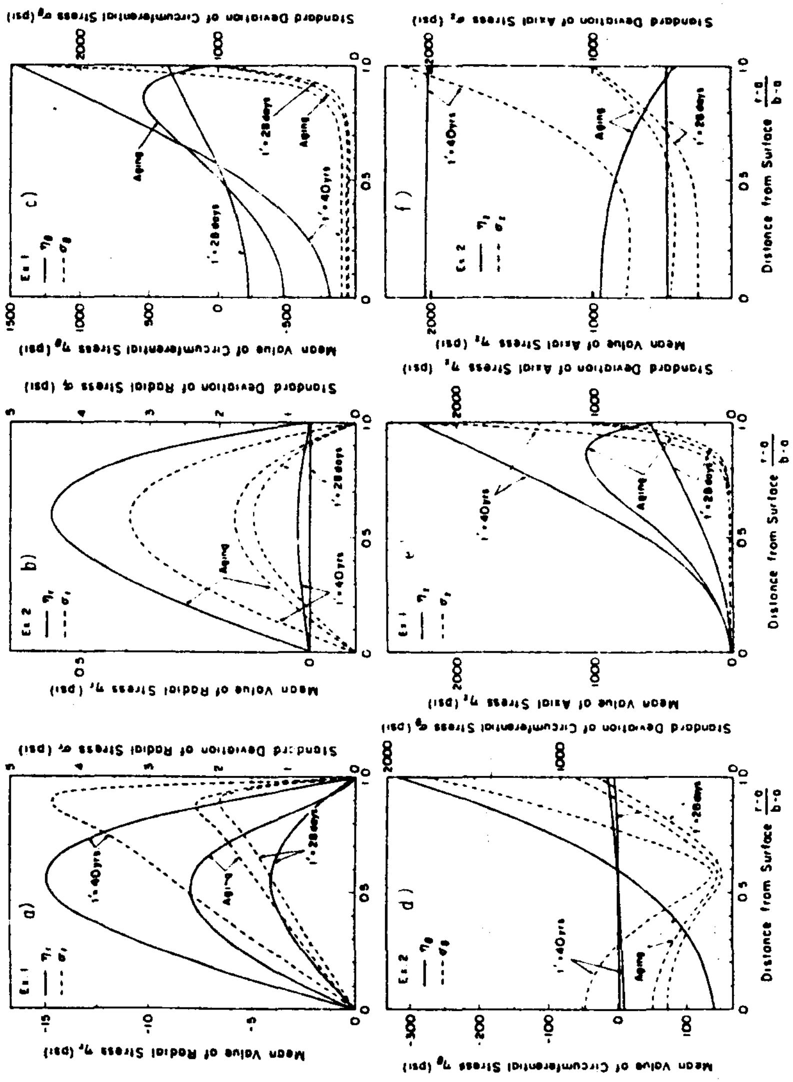

FIG. 4.—Distribution of Stresses at t = 10 yr: a, c, e—Containment (1 m thickness); b, d, f—Thin Shell (0.1 m thickness). (1 psi = 6.89 kPa)

The variation of stresses is shown in Fig. 3(a–f). Two kinds of calculations are made: (a) Aging concrete loaded at t' = 28 days (labeled as "Aging" in the figures); and (b) nonaging material with constant creep (or relaxation) properties pertaining to either the concrete of age t' = 28 days, or to the concrete of age t' = 40 years (labeled by the age t' in the figures), as if the aging stopped at these ages.

In all cases, nonaging material for higher t' gives larger stresses because it is stiffer. We observe that the aging concrete behaves somewhere between the two cases of nonaging material, and this may be exploited in practice for approximate bounds.

The transient behavior of the aging concrete is more complicated than that of nonaging material. It is noteworthy that for aging concrete, the standard deviation does not reach a stationary value during the lifetime of the structure (40 yr), while it does so for nonaging material. We must admit, however, that because of the limited time range of expressions for $E_\mu(t)$ and $E_x(t)$, taken from Ref. 10, the results might be less accurate for times over 10 yr than for short times.

The distribution of stresses is shown in Fig. 4(a–f). Again, the behavior of aging concrete is between the two cases of nonaging concrete with creep properties frozen at low or high age. For nonaging concrete, the standard deviation of stresses at t = 10 yr cannot be distinguished from the stationary value, i.e., the frequency response function, within the accuracy of drawing.

The calculated stresses reach high values sufficient to cause cracking (as shown previously by a deterministic analysis in Ref. 6). Our analysis is, therefore, complete and realistic only if sufficient prestress is superimposed on the concrete structure to prevent cracking. For nuclear structures this normally would be the case. Otherwise, cracking would have to be considered, in which case, however, the problem is nonlinear and the present solution method cannot be applied.

Summary and Conclusions

Shrinkage stresses caused by random variation of environmental humidity in a long cylindrical concrete wall sealed at the internal surface are analyzed. The structure is assumed to be sufficiently prestressed so as to prevent cracking. The pore humidity is assumed to be governed by a linear diffusion equation, and free shrinkage is assumed linearly to depend on pore humidity. The spatial solution of the diffusion equation involves Bessel and Kelvin functions. Concrete creep properties are considered age-dependent and are characterized by a linear superposition integral and a constant creep Poisson ratio. The creep law is converted to a form corresponding to the Maxwell chain model with age-dependent moduli. Neglect of the stiffness of the interior liner makes it possible to use the elastic-viscoelastic analogy for aging creep, which consists of replacing the elastic modulus with a Volterra integral operator of nonconvolution type. To treat the random fluctuations, the spectral method is used. The solution yields, for pore humidity and stresses, the frequency response functions, the impulse response functions, the autocorrelation (and cross-correlation) functions and spectral densities, the mean values, and standard deviations. Numerical examples typical for a nuclear reactor containment, as well as a thin shell, are given, assuming the dominant humidity period to be one year. The main conclusions are as follow:

1. Due to aging, as well as to the initial condition and the low value of diffusivity of moisture (as compared to that of heat), random variations with a dominant period of one year must be treated as nonstationary for the entire lifetime. Thus, the standard deviations and other statistical measures are time-dependent.
2. Solution for age-dependent creep properties is made possible by a novel step—the Dirichlet series expansion of the Maxwell chain moduli as functions of the age of concrete.
3. Significant random stresses are produced in both cases. In the thin shell, the entire thickness is affected, while in the thicker containment, only about 30% of the thickness ever feels the random variation.
4. The influence of aging is strong. Assuming nonaging creep properties that correspond either to the starting age, 28 days, or to the final age, 40 yr, approximate bounds are obtained; but these bounds are rather far apart. A stationary state is approached after 40 yr, but not when aging is considered.

Acknowledgment

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Appendix I.—Some Mathematical Details

Dirichlet Series Coefficients for Maxwell Chain Model.—The coefficients for Dirichlet series expansion (Eqs. 36) of Maxwell chain moduli as functions of age, τ, are readily available (4,10). In Ref. 10, $E_\mu(\tau)$ and $E_\infty(\tau)$ were expressed as a series of power functions:

$$E_\mu(\tau) = E_{o_\mu} + \sum_{k=1}^{4} E_{k\mu} f_k(\tau); \quad E_\infty(\tau) = E_{o\infty} + \sum_{k=1}^{4} E_{k\infty} f_k(\tau) \quad \text{(45)}$$

$$f_1(\tau) = \tau^{1/8}; \quad f_2(\tau) = \tau^{1/4}; \quad f_3(\tau) = \tau^{1/2}; \quad f_4(\tau) = \tau^{3/4} \quad \text{(46)}$$

Coefficients $E_{k\mu}$ and $E_{k\infty}$ are tabulated in Ref. 10 for many typical creep data. To express $E_\mu(\tau)$ and $E_\infty(\tau)$ in the form of Dirichlet series (also called Prony series), it is sufficient to expand each power function, $f_k(\tau)$, into such a series. This is particularly simple because an explicit formula for the Dirichlet series expansion of a power function has been determined (4):

$$f_k(\tau) = \tau^{n(k)} = \sum_{j=1}^{N+1} A_j(k)\, e^{-\tau/\tau'_j} \quad \text{(47)}$$

in which $A(k) = -\alpha[n(k)]\,\tau_j'^{n(k)}$ for $j < N$;

$$A_N(k) = -1.2\,\alpha[n(k)]\,\tau_N'^{n(k)}; \quad A_{N+1}(k) = \beta[n(k)]\,\tau_N'^{n(k)} \quad \text{(48)}$$

$$\tau'_j = 10^{j-1}\,\tau'_1\ (j \le N); \quad \tau'_{N+1} = 10^6\,\tau'_N \quad \text{(49)}$$

Here $n(k) = 1/8,\ 1/4,\ 1/2$, and $3/4$ for $k = 1,2,3,4$. $\alpha(n)$ and $\beta(n)$ are coefficients given in Ref. 4:

$$\alpha = 0.2638; \quad \beta = 1.1076; \quad \text{for} \quad n = \frac{1}{8};$$

$$\alpha = 0.4731; \quad \beta = 1.1748; \quad \text{for} \quad n = \frac{1}{4};$$

$$\alpha = 0.7240; \quad \beta = 1.2005; \quad \text{for} \quad n = \frac{1}{2};$$

$$\alpha = 0.7947; \quad \beta = 1.1207; \quad \text{for} \quad n = \frac{3}{4} \quad \text{(50)}$$

Substituting Eq. 47 into Eqs. 45, one gets the coefficients $\hat{E}_{j\mu}$ and $\hat{E}_{j\infty}$ of Eqs. 36, which represent the aging properties of creep (or relaxation):

$$\hat{E}_{j\mu} = \sum_{k=1}^{4} E_{k\mu} A_j(k); \quad \hat{E}_{j\infty} = \sum_{k=1}^{4} E_{k\infty} A_j(k) \quad \text{(51)}$$

Note that the origin of our time scale, t, starts from the moment, t', at which the outer surface is exposed to the environmental humidity. To take the age of concrete into account, we need to multiply Eqs. 51 by e^{-t'/τ_j}.

Using Eqs. 51, we can evaluate $L_1(t)$ and $L_2(t)$ (Eqs. 28):

$$L_1(t) = \sum_{\mu=1}^{m} [E_\mu(t) - E_\mu(0)e^{-t/\tau_\mu}] + [E_\infty(t) - E_\infty(0)] \quad \text{(52)}$$

$$L_2(t) = \sum_{\mu=1}^{m} \left\{ \frac{E_{o_\mu}}{1 - C\lambda_n^2\tau_\mu}(e^{-C\lambda_n^2 t} - e^{-t/\tau_\mu}) + \sum_{j=1}^{N+1} \frac{\left(1 - \dfrac{\tau_j}{\tau_\mu}\right)\hat{E}_{j\mu}}{1 + C\lambda_n^2\tau'_j - \dfrac{\tau'_j}{\tau_\mu}} \left[e^{-(1/\tau'_j + C\lambda_n^2)t} - e^{-t/\tau_\mu} \right] \right\} + \sum_{j=1}^{N+1} \frac{\hat{E}_{j\infty}}{1 + C\lambda_n^2\tau'_j}[e^{-(1/\tau'_j + C\lambda_n^2)t} - 1] \quad \text{(53)}$$

Integrals of Kelvin Functions.—In Eq. 38, one must evaluate the integral $\int_a^r rH(r,\omega)dr$, in which

$$H(r,\omega) = \frac{1}{D}[K_1(ka\sqrt{i})\,I_o(kr\sqrt{i}) + I_1(ka\sqrt{i})\,K_o(kr\sqrt{i})] \quad \text{(54)}$$

$$I_o(kr\sqrt{i}) = ber\ kr + i\ bei\ kr; \quad K_o(kr\sqrt{i}) = ker\ kr + i\ kei\ kr;$$

$$D = K_1(ka\sqrt{i})\,I_o(kb\sqrt{i}) + I_1(ka\sqrt{i})\,K_o(kb\sqrt{i}) \quad \text{(55)}$$

Then $\int_a^r rH(r,\omega)dr = \frac{1}{D}\left[K_1(ka\sqrt{i})\left(\int_a^r r\, ber\, kr\, dr + i\int_a^r r\, bei\, kr\, dr\right)\right.$

$$\left. + I_1(ka\sqrt{i})\left(\int_a^r r\, ker\, kr\, dr + i\int_a^r r\, kei\, kr\, dr\right)\right] \quad \text{(56)}$$

For each of the foregoing integrals, the following expressions are available (1):

$$\int_a^r r\, ber\, kr\, dr = -\frac{1}{k\sqrt{2}}[r(ber_1 kr - bei_1 kr) - a(ber_1 ka - bei_1 ka)];$$

$$\int_a^r r\, bei\, kr\, dr = -\frac{1}{k\sqrt{2}}[r(bei_1 kr + ber_1 kr) - a(bei_1 ka + ber_1 ka)];$$

$$\int_a^r r\, ker\, kr\, dr = -\frac{1}{k\sqrt{2}}[r(ker_1 kr - kei_1 kr) - a(ker_1 ka - kei_1 ka)];$$

$$\int_a^r r\, kei\, kr\, dr = -\frac{1}{k\sqrt{2}}[r(kei_1 kr + ker_1 kr) - a(kei_1 ka + ker_1 ka)] \quad \text{(57)}$$

Since k is quite large (because of small diffusivity, C), an asymptotic expansion (see Ref. 1) is used to evaluate these Kelvin functions appearing on the right-hand side.

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Appendix III.—Notation

The following symbols are used in this paper:

A = amplitude of environmental humidity;
a, b = inner and outer radius of wall;
C = diffusivity of water in concrete;
E = expectation;
$\underset{\sim}{E}, E_c$ = creep operator and Young's modulus of concrete;
$E_R(t,\tau)$ = relaxation function of concrete;
$E_\mu(\tau)$ = relaxation moduli of Maxwell chain at time τ (μ = 1, 2 ... m);
e_{ij} = deviatoric part of ϵ_{ij};

$H(r)$ = frequency response function of humidity;
$\bar{h}$ = complex variable whose real part is pore humidity (relative vapor pressure) in pores of concrete;
$h_j (j = r, \theta, z)$ = impulse response function for stresses;
h_m = mean environmental humidity;
h_o = initial pore humidity of concrete at age t_o;
$h(r, t)$ = impulse response function;
$R_{hh}(t_1, t_2)$ = autocorrelation function of $\bar{h}$ (r, t);
$R_{h\phi}$ (t_1, t_2) = cross-correlation function of $\bar{h}$ (r, t);
$R_{j\phi}, R_{jj}$ = cross-correlation and autocorrelation functions for stresses;
$R_{\phi\phi}(t_1, t_2)$ = autocorrelation function of ϕ (t) (complex);
r = radial coordinate;
S_{ij}, s_{ij} = stress components and their deviatoric parts;
$S_{\phi\phi}(\omega)$ = spectral density of ϕ (t);
t = current time = duration of exposure to environmental humidity;
t' = age of concrete at time $t = 0$ (start of drying);
ϵ_{ij} = strain components (linearized);
ϵ_o = free (unrestrained) shrinkage;
$\eta(r, t)$ = mean value of pore humidity;
$\eta_r, \eta_\theta, \eta_z$ = mean values of normal stresses;
κ = shrinkage constant (Eq. 25);
λ_n = roots of Eq. 5;
ν = Poisson's ratio of concrete;
σ_j = standard deviations of normal stress components;
σ_o = standard deviation of $\bar{h}$;
τ = previous time;
τ_j' = coefficients of Dirichlet series for E_μ $(j = 1, \ldots, N + 1)$ (Eq. 47);
τ_μ = relaxation times $(\mu = 1, \ldots, m)$;
ϕ_o = phase angle of environmental humidity;
$\phi(t)$ = cyclic component of $\bar{h}$ (complex); and
ω = circular frequency of component of environmental humidity.