Comment on Orthotropic Models for Concrete and Geomaterials

By Zdeněk P. Bažant,[1] F. ASCE

Abstract: Incrementally linear constitutive equations that are characterized by an orthotropic tangential stiffness or compliance matrix have recently become widely used in finite element analysis of concrete structures and soils. It does not seem to be, however, widely appreciated that such constitutive equations are limited to loading histories in which the prinicipal stress directions do not rotate, and that a violation of this condition can sometimes have serious consequences. It is demonstrated that in such a case the orthotropic models do not satisfy the form-invariance condition for initially isotropic solids, i.e., the condition that the response predicted by the model must be the same for any choice of coordinate axes in the initial stress-free state. An example shows that the results obtained for various such choices can be rather different. The problem cannot be avoided by rotating the axes of orthotropy during the loading process so as to keep them parallel to the principal stress axes, first, because this would imply rotating against the material, the defects that cause material anisotropy, such as microcracks, and, second, because the principal directions of stress and strain cease to coincide. The recently popular cubic triaxial tests do not give information on loading with rotating principal stress directions.

Introduction

The statistical scatter of the properties of concrete, and especially soils, is distinctly larger than that of metals, polymers, and most other materials. Thus, it is not surprising to see a strong and certainly justified tendency to keep the mathematical models simple. It is probably for this reason that the incrementally linear constitutive relations that are characterized by an orthotropic tangential stiffness or compliance matrix, called the orthotropic models, have recently become very popular and have been widely used in finite element analysis of concrete structures and soils (1–4,9–14,16–27,29–42,44–48,50–52). In this approach, one tries to figure out the variation of tangential moduli or compliances directly, without the aid of abstract concepts such as loading surfaces (potentials), flow and normality rules, stability postulates, work inequalities, path-dependence, intrinsic time, etc.

It does not seem, however, to be widely appreciated that such constitutive equations are limited to loading histories in which the principal stress directions to not rotate, and that a violation of this condition, typical of finite element applications, can sometimes have serious consequences. The objective of this paper, which is based on a 1979 report (5), is to examine these problems in detail and illustrate them by an example.

The popularity of the orthotropic models may have been aided by the recent exaggerated emphasis on the so-called "true" triaxial tests that utilize cubic specimens loaded by normal stresses on their faces. We will see that these tests are incapable of revealing precisely those important triaxial properties which are the cause of trouble with the orthotropic models.

[1]Prof. of Civ. Engrg., and Dir., Center for Concrete and Geomaterials, Northwestern Univ., Evanston, Ill. 60201.

Note.—Discussion open until November 1, 1983. To extend the closing date one month, a written request must be filed with the ASCE Manager of Technical and Professional Publications. The manuscript for this paper was submitted for review and possible publication on May 10, 1982. This paper is part of the Journal of Engineering Mechanics, Vol. 109, No. 3, June, 1983. ©ASCE, ISSN 0733-9399/83/0003-0849/$01.00. Paper No. 18014.

Are Orthotropic Models Tensorially Invariant?

Over a broad range of triaxial behavior, concrete and soils may be characterized by incrementally linear stress-strain relations, also called hypoelastic (49):

$$d\boldsymbol{\sigma} = \mathbf{C}\,d\boldsymbol{\epsilon} \quad \text{or} \quad d\boldsymbol{\epsilon} = \mathbf{D}\,d\boldsymbol{\sigma} \qquad (1)$$

which, in the component form, reads

$$d\sigma_{ij} = C_{ijkm}\,d\epsilon_{km} \quad \text{or} \quad d\epsilon_{ij} = D_{ijkm}\,d\sigma_{km} \qquad (2)$$

Here $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$ = column matrices of the six stress and strain components; $\mathbf{C}$ = a 6 × 6 tangential stiffness matrix of the material (tangential moduli matrix), $\mathbf{D}$ = a 6 × 6 tangential compliance matrix of the material; σ_{ij}, ϵ_{ij}, C_{ijkm}, D_{ijkm} = tensorial components of $\boldsymbol{\sigma}$, $\boldsymbol{\epsilon}$, $\mathbf{C}$ and $\mathbf{D}$ referred to cartesian coordinates x_i (i = 1, 2, 3); repetition of subscripts implies summation.

If the material is inelastic, matrices $\mathbf{C}$ and $\mathbf{D}$ must be considered to depend on $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$. Determination of this dependence, which causes $\mathbf{C}$ and $\mathbf{D}$ to exhibit the stress-induced (or strain-induced) anisotropy, is the main purpose of the theories of incremental plasticity or hypoelasticity and represents a complex problem. This is because we deal with a fourth-rank tensor ($\mathbf{C}$ or $\mathbf{D}$) which must be a tensorially invariant functional of the histories of two second-rank tensors ($\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$), satisfying the conditions of isotropy of the material with regard to the initial state.

In the orthotropic models one introduces a simplification by assuming that $\mathbf{C}$ and $\mathbf{D}$ have an orthotropic form, i.e.

$$\begin{Bmatrix} d\sigma_{11} \\ d\sigma_{22} \\ d\sigma_{33} \\ d\sigma_{12} \\ d\sigma_{23} \\ d\sigma_{31} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{21} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{31} & C_{32} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} d\epsilon_{11} \\ d\epsilon_{22} \\ d\epsilon_{33} \\ d\epsilon_{12} \\ d\epsilon_{23} \\ d\epsilon_{31} \end{Bmatrix} \qquad (3)$$

or

$$\begin{Bmatrix} d\epsilon_{11} \\ d\epsilon_{22} \\ d\epsilon_{33} \\ d\epsilon_{12} \\ d\epsilon_{23} \\ d\epsilon_{31} \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & 0 & 0 & 0 \\ D_{21} & D_{22} & D_{23} & 0 & 0 & 0 \\ D_{31} & D_{32} & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \begin{Bmatrix} d\sigma_{11} \\ d\sigma_{22} \\ d\sigma_{33} \\ d\sigma_{12} \\ d\sigma_{23} \\ d\sigma_{31} \end{Bmatrix} \qquad (4)$$

The appearance of C and D with just two subscripts (i.e., $C_{\alpha\beta}$ where α and $\beta = 1, \ldots, 6$) indicates the matrix components of $\mathbf{C}$ and $\mathbf{D}$. Obviously, $D_{11} = D_{11\,11}$, $D_{15} = D_{11\,23}$, $D_{44} = D_{12\,12}$, etc. Matrices C_{ij} and D_{ij} usually are chosen as symmetric, but this is not necessary (33) and will be unimportant for our argument.

To account for the decrease of material stiffness as the stress and strain increases, the dependence of the components of $\mathbf{C}$ and $\mathbf{D}$ on σ_{ij} and ϵ_{ij} is introduced in the orthotropic models directly, usually intuitively and without recourse to loading functions, flow rule, etc. Sometimes σ_{ij} and ϵ_{ij} are assumed to appear in the elements of matrix $\mathbf{C}$ and $\mathbf{D}$ in a manner exhibiting orthotropic symmetry. Sometimes, in the erroneous belief that this would achieve tensorial invariance, only the invariants of stress and strain are allowed to appear in the elements of matrix $\mathbf{C}$ or $\mathbf{D}$. We will see, however, that no matter how this dependence is chosen, the orthotropic form of $\mathbf{C}$ and $\mathbf{D}$ in Eqs. 3 and 4 cannot in general satisfy tensorial invariance.

It is important to realize the difference from an orthotropic material, i.e., a material where the orthotropy is built in, due to the initial microstructure, rather than stress-induced. An orthotropic material has to be invariant only with regard to 90° rotations and reflections of coordinate axes (43). We consider isotropic inelastic materials. They may exhibit stress- or strain-induced anisotropy but are isotropic in their initial stress-free state, and that is not the same as an anisotropic material. The induced orthotropy, in particular, is not of a fixed direction but can have any direction in the material depending on the direction of the principal stresses that induced it.

Thus, before the loading starts we can choose, due to initial isotropy, any coordinate system as the material reference frame. It is one basic principle of continuum mechanics of inelastic solids that the material reference frame must be kept attached to the material (i.e., be rotated with the material during the deformation process). We must, however, get the same states of stress and strain for the same loading history regardless of which directions of the coordinates we chose initially.

Form—Invariance Conditions

Consider two cartesian coordinate systems: the original coordinates x_i ($i = 1, 2, 3$) and new rotated coordinates x_i'. The coordinate transformation is $x_j' = c_{ij}x_i$ where $c_{ij} = \cos(x_i', x_j)$ = matrix of direction cosines of x_i in coordinates x_i'. The transformation of the stress tensor is $\sigma'_{km} = c_{ij}c_{jm}\sigma_{ij}$ and the transformation of the compliance tensor is $D'_{pqrs} = c_{ip}c_{jq}c_{kr}c_{ms}D_{ijkm}$. Denote as $\mathbf{D}(\boldsymbol{\sigma})$ or $D_{ijkm}(\boldsymbol{\sigma})$ the compliance matrix or tensor that is evaluated on the basis of stress tensor $\boldsymbol{\sigma}$ with components σ_{ij}. Because the material is isotropic in the stress-free initial state, $\mathbf{D}(0)$ must have the same form as for isotropic elastic material. Since the isotropy group of transformations includes all rotations, the condition of tensorial form-invariance requires (43) that, for any coordinate rotation, the tensor $\mathbf{D}(\boldsymbol{\sigma}')$ determined on the basis of transformed stresses σ'_{ij} must be the same as the transformation of tensor $\mathbf{D}(\boldsymbol{\sigma})$ determined on the basis of original stresses σ_{ij} (Ref. 49 or p. 420 of Ref. 43);

$$D_{pqrs}(\boldsymbol{\sigma}') = D'_{pqrs} \quad \ldots\ldots (5)$$

where $D'_{pqrs} = c_{ip}c_{jq}c_{kr}c_{ms}D_{ijkm}(\boldsymbol{\sigma})$, $\quad \sigma'_{km} = c_{ik}c_{jm}\sigma_{ij}$ $\quad \ldots\ldots (6)$

To satisfy this form-invariance condition (Eq. 5), D_{ijkm} must be a tensor polynomial in σ_{ij}, and the most general form of a symmetric tensor $\mathbf{D}(\boldsymbol{\sigma})$ allowed by material isotropy may be written as

$$D_{ijkm} = D^{(0)}_{ijkm} + D^{(1)}_{ijkm} + D^{(2)}_{ijkm} + D^{(3)}_{ijkm} + D^{(4)}_{ijkm} \quad \ldots\ldots (7)$$

in which
$$D^{(0)}_{ijkm} = A_1\delta_{ij}\delta_{km} + \frac{1}{2}A_2(\delta_{ik}\delta_{jm} + \delta_{jk}\delta_{im}) \quad \ldots\ldots (8a)$$

$$D^{(1)}_{ijkm} = A_3\sigma_{ij}\delta_{km} + A_4\sigma_{km}\delta_{ij} + A_5(\sigma_{jk}\delta_{mi} + \sigma_{jm}\delta_{ki} + \sigma_{ik}\delta_{mj} + \sigma_{im}\delta_{kj}) \quad \ldots\ldots (8b)$$

$$D^{(2)}_{ijkm} = A_6\sigma_{ij}\sigma_{km} + A_7(\sigma_{im}\sigma_{jk} + \sigma_{ik}\sigma_{jm}) + A_8\delta_{ij}\sigma_{kr}\sigma_{rm} + A_9\sigma_{ir}\sigma_{rj}\delta_{km} + A_{10}(\delta_{ik}\sigma_{jr}\sigma_{rm} + \delta_{jk}\sigma_{ir}\sigma_{rm} + \delta_{jm}\sigma_{ir}\sigma_{rk} + \delta_{im}\sigma_{jr}\sigma_{rk}) \quad \ldots\ldots (8c)$$

$$D^{(3)}_{ijkm} = A_{11}\sigma_{ij}\sigma_{kr}\sigma_{rm} + A_{12}\sigma_{ir}\sigma_{rj}\sigma_{km} + A_{13}(\sigma_{ir}\sigma_{rk}\sigma_{jm} + \sigma_{ir}\sigma_{rm}\sigma_{jk} + \sigma_{jr}\sigma_{rk}\sigma_{im} + \sigma_{jr}\sigma_{rm}\sigma_{ik}) + A_{14}(\sigma_{ik}\sigma_{jr}\sigma_{rm} + \sigma_{jk}\sigma_{ir}\sigma_{rm} + \sigma_{im}\sigma_{jr}\sigma_{rk} + \sigma_{jm}\sigma_{ir}\sigma_{rk}) \quad \ldots\ldots (8d)$$

$$D^{(4)}_{ijkm} = A_{15}\sigma_{ir}\sigma_{rj}\sigma_{ks}\sigma_{sm} + A_{16}(\sigma_{ir}\sigma_{rk}\sigma_{js}\sigma_{sm} + \sigma_{jr}\sigma_{rk}\sigma_{is}\sigma_{sm} + \sigma_{ir}\sigma_{rm}\sigma_{js}\sigma_{sk} + \sigma_{jr}\sigma_{rm}\sigma_{is}\sigma_{sk}) \quad \ldots\ldots (8e)$$

Equations 8*a*, and 8*b*, were used to model concrete and soils in Ref. 15. Coefficients $A_1, A_2, A_3, \ldots$ are functions of the invariants of stress and strain. Note that the third and higher powers of tensor σ_{ij}, such as $\sigma_{ir}\sigma_{rs}\sigma_{sj}$ or $\sigma_{ir}\sigma_{rs}\sigma_{sp}\sigma_{pj}$, do not appear in Eqs. 7–8 since, according to Cayley-Hamilton's theorem, they can be expressed as linear combinations of σ_{ij} and $\sigma_{ir}\sigma_{rj}$ with coefficients that depend on the invariants of σ_{ij}. Also note that terms like $A_0\sigma_{rr}$ or $A_0\sigma_{rs}\sigma_{rs}$ need not appear either, since σ_{rr} and $\sigma_{rs}\sigma_{rs}$ are invariants, and $A_1, \ldots, A_{16}$ are assumed to depend on the invariants.

When D_{ijkm} depends on ϵ_{ij}, the dependence must be of the same form as that on σ_{ij} in Eqs. 7–8. An analogous form, but with many more terms, is required when $\mathbf{D}$ depends upon both σ_{ij} and ϵ_{ij}. Similar expressions hold for $\mathbf{C}$.

The orthotropic compliance matrix in Eq. 4 involves none of the σ_{ij}-dependent terms listed in Eqs. 8*b*–8*e*. If it did and if, e.g., $\sigma_{12} \neq 0$, then according to Eqs. 7–8 the term $D^{(1)}_{1112}$, contributing to D_{14}, would be nonzero whereas in Eq. 4 it is zero. We see that if any of the σ_{ij}-dependent terms $D^{(1)}_{ijkm}$ or $D^{(2)}_{ijkm}$ is present, then all components of matrix $\mathbf{D}$ are nonzero if $\sigma_{12}, \sigma_{23}, \sigma_{13}$ are nonzero. Therefore, if the zeros are placed where they are shown in Eq. 4 (or Eq. 3), matrix $\mathbf{D}$ (or $\mathbf{C}$) must be of the form of Eq. 8*a*. This form is equivalent to the well-known isotropic compliance matrix:

$$\mathbf{D} = \mathbf{D}^{(0)} = \begin{bmatrix} E^{-1} & -\nu E^{-1} & -\nu E^{-1} & 0 & 0 & 0 \\ & E^{-1} & -\nu E^{-1} & 0 & 0 & 0 \\ & & E^{-1} & 0 & 0 & 0 \\ & \text{sym.} & & (2G)^{-1} & 0 & 0 \\ & & & & (2G)^{-1} & 0 \\ & & & & & (2G)^{-1} \end{bmatrix} \quad \ldots\ldots (9)$$

in which $E^{-1} = A_1 + A_2$, $\nu E^{-1} = -A_1$, $(2G)^{-1} = A_2$, E, G and ν being variable Young's modulus, shear modulus and Poisson ratio. However, a compliance matrix of isotropic form, even if variable, cannot closely describe the real behavior, because the incremental properties of stressed concrete or soil in different directions are not the same.

We must therefore conclude that if the material reference axes are attached to the material, it is in general inadmissible to restrict the tangential compliance or stiffness matrix to an orthotropic form, with zeros placed as shown in Eqs. 3 and 4. The only case when this is admissible is when the principal stresses σ_1, σ_2, and σ_3 are of the same directions as the principal strains ϵ_1, ϵ_2, and ϵ_3 and do not rotate as the material deforms. Only in this case it is possible to orient the coordinate axes x_1, x_2, and x_3 in such a manner that all shear components of stress and strain increments vanish for any loading increment. Then, however, 6 × 6 matrices are unnecessary and one may write (33):

$$\begin{Bmatrix} d\epsilon_1 \\ d\epsilon_2 \\ d\epsilon_3 \end{Bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \begin{Bmatrix} d\sigma_1 \\ d\sigma_2 \\ d\sigma_3 \end{Bmatrix} \qquad (10)$$

It thus appears that the orthotropic models are unsuitable for general finite element programs because it is not possible to guarantee that the principal stress directions would not rotate during the loading process.

Note that the problem is not avoided by coordinate transformation from x_i to x_i', in which the orthotropic matrix transforms to a matrix with nonzero coefficients relating the increments of normal strains and shear stresses (e.g., $d\epsilon_{11}'$ and $d\sigma_{12}'$). This is because the orthotropic symmetry properties do not change with the coordinate transformation and still apply after transformation with regard to axes x_i which are inclined relative to x_i'.

Is the Lack of Invariance a Serious Problem?

Many experts have no doubt been aware of the lack of tensorial invariance, but they did not expect it could cause discrepancies of more than a few percent. We must therefore also examine the severity of the problem by an example. At the same time, an example will illustrate the problem.

Assuming the initial Poisson ratio to be $\nu = 0.2$, the initial compliance matrix is

$$\mathbf{D}^\circ = \frac{1}{E} \begin{bmatrix} 1 & -0.2 & -0.2 & 0 & 0 & 0 \\ & 1 & -0.2 & 0 & 0 & 0 \\ & & 1 & 0 & 0 & 0 \\ & & & 1.2 & 0 & 0 \\ & \text{sym.} & & & 1.2 & 0 \\ & & & & & 1.2 \end{bmatrix} \qquad (11)$$

Consider that a uniaxial compressive stress $\sigma_{11} = -T$ (all other $\sigma_{ij} = 0$) is first applied on the material (Fig. 1(a)). We shall assume that T is quite

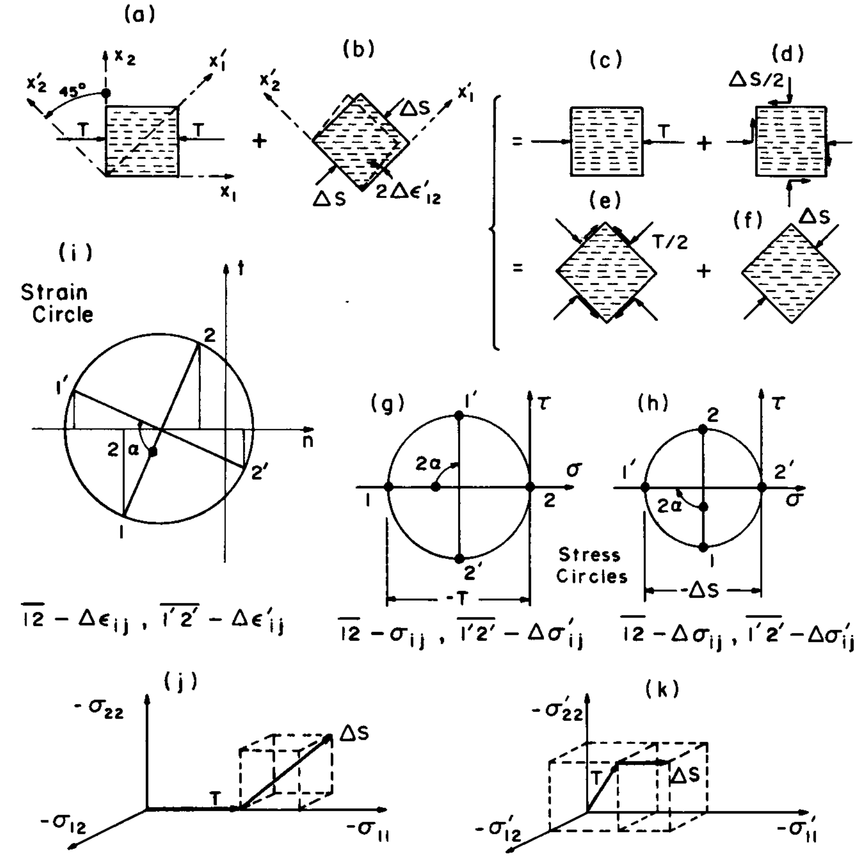

FIG. 1.—Diagrams Illustrating Lack of Tensorial Invariance

large, say 0.9 of the strength, so that the axial compliance D_{11} is much larger than the lateral compliances D_{22}, D_{33} (Fig. 1(j)).

Assume a concrete of strength $f_c' = 4{,}650$ psi (32 MPa), initial Young's modulus $E = 4.7 \times 10^6$ psi (32,400 MPa), strain at peak stress in uniaxial compression test $\epsilon_{c_u} = -0.00215$, and consider the unixial stress $\sigma_{11} = -T = -0.9\, f_c'$ as the stress state just before the loading increment. On the basis of one particular recent orthotropic model for concrete among those referenced here, the following tangential compliance matrix was evaluated in Ref. 5 from the foregoing data:

$$\mathbf{D} = \frac{1}{E} \begin{bmatrix} 1.87 & -0.31 & -0.31 & 0 & 0 & 0 \\ -0.31 & 1.0 & -0.2 & 0 & 0 & 0 \\ -0.31 & -0.2 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.62 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \qquad (12)$$

It does not matter to which particular model this matrix corresponds since the results of the analysis that follows are about the same for all orthotropic models which can represent the difference in incremental stiffness in various directions. What is important in Eq. 12 for our subsequent result is that D_{11}/D_{22} is much larger than 1.0, or that D_{44}/D_{22} is much larger than (1.0 + 0.31). These features are true of all orthotropic models since their purpose is to describe the deviations from an incremental isotropic matrix. (The values of D_{55} and D_{66} in Eq. 12 are not needed for our calculations.)

Consider now alternatively coordinate axes x_1', x_2', and x_3' that are rotated by 45° about axis x_3 (Fig. 1*a*). According to Mohr's circle (see points 1′ and 2′ in Fig. 1(*g*)), the uniaxial compressive stress T then appears as

$$\sigma_{11}' = \sigma_{22}' = -\frac{T}{2}, \quad \sigma_{12}' = \frac{T}{2}, \quad \sigma_{33}' = \sigma_{13}' = \sigma_{23}' = 0 \quad \text{(13)}$$

Using these stress values (with $T = 0.9 f_c'$) and applying again the same orthotropic model as that from which Eq. 12 was evaluated, we now find that, for coordinates x_i', the compliance matrix is

$$\mathbf{D}' = \frac{1}{E}\begin{bmatrix} 1.10 & -0.25 & -0.22 & 0 & 0 & 0 \\ -0.25 & 1.10 & -0.22 & 0 & 0 & 0 \\ -0.22 & -0.22 & 1.0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.35 & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \quad \text{(14)}$$

Since the material is isotropic, tensorial transformation (Eq. 5) of matrix **D** (Eq. 12) into the rotated coordinates must yield matrix **D**′. The matrix resulting from this transformation is, however, altogether different from **D**′, the most significant difference being that large non-zero coefficients are obtained in place of the zeros in Eq. 14. We leave it up to the interested reader to check this himself, but we now demonstrate the discrepancy in a simpler way by superimposing upon the uniaxial compression T in the direction x_1 a uniaxial compressive stress increment $\Delta\sigma_{11}' = -\Delta S$ at 45°-inclined direction x_1' (all other $\Delta\sigma_{ij}' = 0$); see Fig. 1(*b*) and 1(*f*) and the stress paths in Fig. 1(*j*) and 1(*k*). According to the Mohr's circle (points 1 and 2 in Fig. 1(*h*)), this increment is equivalent to (Fig. 1(*d*)):

$$\Delta\sigma_{11} = \Delta\sigma_{22} = \Delta\sigma_{12} = -\frac{\Delta S}{2}, \quad \Delta\sigma_{33} = \Delta\sigma_{23} = \Delta\sigma_{13} = 0 \quad \text{(15)}$$

According to the matrix **D** (Eq. 12), this produces strain increments

$$\Delta\epsilon_{11} = \left(-\frac{\Delta S}{2}\right)\frac{1}{E}(1.87 - 0.31) = -\frac{0.78}{E}\Delta S$$

$$\Delta\epsilon_{22} = \left(-\frac{\Delta S}{2}\right)\frac{1}{E}(-0.31 + 1.0) = -\frac{0.34}{E}\Delta S$$

$$\Delta\epsilon_{12} = \left(-\frac{\Delta S}{2}\right)\frac{1.62}{E} = -\frac{0.81}{E}\Delta S \quad \text{(16)}$$

Using the Mohr's circle (see points 1′, 2′ in Fig. 1(*i*)) we may transform these strains to the rotated axes x_i':

$$\Delta\epsilon_{11}' = \frac{1}{2}(\Delta\epsilon_{11} + \Delta\epsilon_{22}) + \Delta\epsilon_{12}, \quad \Delta\epsilon_{22}' = \frac{1}{2}(\Delta\epsilon_{11} + \Delta\epsilon_{22}) - \Delta\epsilon_{12},$$

$$\Delta\epsilon_{12}' = \frac{1}{2}(\Delta\epsilon_{22} - \Delta\epsilon_{11}) \quad \text{(17)}$$

which yields $\Delta\epsilon_{11}' = -\frac{1.37}{E}, \quad \Delta\epsilon_{22}' = \frac{0.25}{E}\Delta S, \quad \Delta\gamma_{12}' = \frac{0.44}{E}\Delta S$ ..... (18)

in which $\Delta\gamma_{12}' = 2\Delta\epsilon_{12}'$ = increment of shear angle. On the other hand, using matrix **D**′ (Eq. 14) we obtain

$$\Delta\epsilon_{11}' = -\frac{1.10}{E}, \quad \Delta\epsilon_{22}' = \frac{0.25}{E}\Delta S, \quad \Delta\gamma_{12}' = 0 \quad \text{(19)}$$

These strains must be the same as those in Eq. 18, and they are not.

For example, the shear strain increment $\Delta\gamma_{12}'$ calculated according to the orthotropic model in the rotated axes is zero (Eq. 19), but when it is calculated according to the orthotropic model for the original axes it has a significant non-zero value (Eq. 18), amounting to 40% of the axial strain $\Delta\epsilon_{11}'$ obtained for ΔS in the rotated axes. Thus, the lack of tensorial invariance does indeed have serious consequences. The results significantly depend on our choice of coordinates, which violates the principle of objectivity.

It should also be pointed out that the existing orthotropic models generally underestimate the difference in compliances D_{11} and D_{22} (or D_{12} and D_{23}) in Eq. 12. According to plastic-fracturing or endochronic models, greater differences are obtained near the peak point of uniaxial stress-strain diagram. It should be noted that the value of $\Delta\gamma_{12}'$ in Eq. 18 would then be larger, making the discrepancy still greater.

It is interesting to check whether there exists an orthotropic model for which these discrepancies would be insignificant. In general terms we have

$$\Delta\epsilon_{11} = -D_{11}(\boldsymbol{\sigma})\frac{\Delta S}{2} - D_{12}(\boldsymbol{\sigma})\frac{\Delta S}{2},$$

$$\Delta\epsilon_{22} = -D_{21}(\boldsymbol{\sigma})\frac{\Delta S}{2} - D_{22}(\boldsymbol{\sigma})\frac{\Delta S}{2}, \quad \Delta\epsilon_{12} = D_{44}(\boldsymbol{\sigma})\frac{\Delta S}{2} \quad \text{(20)}$$

the transformation of which according to Eq. 17 yields

$$\Delta\epsilon_{11}' = \Delta\epsilon_m' + D_{44}(\boldsymbol{\sigma})\frac{\Delta S}{2}, \quad \Delta\epsilon_{22}' = \Delta\epsilon_m' - D_{44}(\boldsymbol{\sigma})\frac{\Delta S}{2}$$

$$\Delta\epsilon_m' = -[D_{11}(\boldsymbol{\sigma}) + D_{22}(\boldsymbol{\sigma}) + D_{12}(\boldsymbol{\sigma}) + D_{21}(\boldsymbol{\sigma})]\frac{\Delta S}{4}$$

$$\Delta\epsilon'_{12} = [D_{11}(\boldsymbol{\sigma}) - D_{22}(\boldsymbol{\sigma}) + D_{12}(\boldsymbol{\sigma}) - D_{21}(\boldsymbol{\sigma})]\frac{\Delta S}{4} \quad (21)$$

On the other hand, the direct use of Eq. 14 gives

$$\Delta\epsilon'_{11} = D_{11}(\boldsymbol{\sigma}')\Delta S, \quad \Delta\epsilon'_{22} = D_{21}(\boldsymbol{\sigma}')\Delta S, \quad \Delta\epsilon'_{12} = 0 \quad (22)$$

By equating these values to the expressions in Eq. 21 we find that the results could be the same for both coordinate choices only if

$$D_{22}(\boldsymbol{\sigma}) - D_{11}(\boldsymbol{\sigma}) = D_{12}(\boldsymbol{\sigma}) - D_{21}(\boldsymbol{\sigma}) \quad (23)$$

$$D_{11}(\boldsymbol{\sigma}) + D_{22}(\boldsymbol{\sigma}) + D_{12}(\boldsymbol{\sigma}) + D_{21}(\boldsymbol{\sigma}) = -2[D_{11}(\boldsymbol{\sigma}') + D_{21}(\boldsymbol{\sigma}')] \quad (24)$$

$$D_{44}(\boldsymbol{\sigma}) = D_{11}(\boldsymbol{\sigma}') - D_{21}(\boldsymbol{\sigma}') \quad (25)$$

In case of symmetry ($D_{12} = D_{21}$), Eq. 23 requires that $D_{22}(\boldsymbol{\sigma}) = D_{11}(\boldsymbol{\sigma})$, i.e., that the normal stiffnesses in x_1 and x_2 directions be the same. Equation 25 is then equivalent to the relation between G, E and ν for isotropic materials. Thus, we see that tensorial invariance can be achieved with the orthotropic model only if the material is incrementally isotropic. However, this would make it impossible to model the real behavior.

It also follows from Eq. 23 that, as long as D_{11} significantly differs from D_{22}, the discrepancies due to the choice of the coordinate system are large. The discrepancies can be removed only if we abandom incremental orthotropy and assume general stress-induced anisotropy using nonzero values for the remaining coefficients of the matrices, e.g., for D_{41} ($= D_{12,11}$).

Note also that is is impossible to circumvent the condition in Eqs. 4 and 5 by stipulating that matrix **D** or **C** may be evaluated only on the basis of the stress tensor components referred to the coordinate axes of the principal stress applied first. What would we then do if an extremely small stress σ_{11}, say $10^{-6} f'_c$, were followed by stress increment $\Delta\sigma'_{11} = 0.9 f'_c$ in the inclined direction? Is this not equivalent to the first stress being $\sigma'_{11} = 0.9 f'_c$?

Further, serious discrepancies exist between Eqs. 17 and 18 for the values of $\Delta\epsilon'_{11}$ as well as $\Delta\epsilon'_{22}$. These are more sensitive to the numerical values in Eqs. 12 and 13 and substantially differ from model to model. They are significant for any orthotropic model for concrete, as its user may check.

Similar serious discrepancies can be demonstrated for other simple nonproportional loading paths. The interested reader may for example calculate the responses in x_i and x'_i coordinates for these loadings: (1) Uniaxial stress $\sigma_{11} = -T$ followed by shear stress $\Delta\sigma_{12} = \Delta S$; (2) biaxial compression $\sigma_{11} = \sigma_{22} = -T$ followed by $\Delta\sigma_{13} = \Delta S$, $\Delta\sigma_{11} = \Delta\sigma_{33} = -\Delta S$; (3) uniaxial compression $\sigma_{11} = -T$ followed by hydrostatic pressure $\Delta\sigma_{11} = \Delta\sigma_{22} = \Delta\sigma_{33} = -p$; and (4) shear stress $\sigma_{12} = T$ followed by uniaxial compression $\Delta\sigma_{11} = -\Delta S$, etc.

Analogous calculations can be made when the stiffness rather than compliance formulation (Eq. 3) is used. In that case we would consider a uniaxial strain ϵ_{11} followed by uniaxial strain $\Delta\epsilon'_{11}$ in the rotated coordinates and we would obtain a similar magnitude of the discrepancies.

To sum up, our calculations not only illustrate the lack of tensorial invariance, but also reveal that its consequences can be serious.

The initial uniaxial compressive stress $\sigma_{11} = -T$, considered in the foregoing example, produces in rock or concrete a system or microcracks whose planes exhibit a prevalent orientation parallel to axis x_1; see Fig. 1(*a*). The situation when we subsequently apply the skew uniaxial stress $\Delta\sigma'_{11} = -\Delta S$ is pictured in Fig. 1(*b*), and we see that ΔS produces a tangential stress on the weakened crack planes. So, ΔS must produce a shear strain on these planes (Fig. 1(*d*)), which does not allow the deformation to be symmetric with respect to axis x'_1 (Fig. 1(*b*)). Thus, normal stress $\Delta\sigma_{11}$ produces shear strain $\Delta\epsilon_{12}$, which is not reflected in the orthotropic models.

Generally, due to the location of zeros in the orthotropic incremental compliance or stiffness matrix, the cross effects are not present, i.e., the normal stress or strain increments produce no shear strain or stress increments, and the shear stress or strain increments produce no normal strain or stress increments. Yet, these cross effects are important for modeling the inelastic dilatancy of concrete or the compaction of soils, the hydrostatic pressure sensitivity, and other phenomena.

It is normally stipulated that the orthotropic models do not cover unloading. It is, however, not so simple to leave out an unloading criterion. For illustration, consider that a uniaxial stress $\sigma_{11} = -T$ is followed by a shear stress increment $\Delta\sigma_{12} = \Delta S$. Not only do we get different results applying the orthotropic model in the two coordinate systems, but we further face an ambiguity in deciding what is unloading. None of the stresses decreases, and so one might assume that we have no unloading, a case for which the model is intended. However, in the rotated axes, $\Delta\sigma_{12}$ appears as $\Delta\sigma'_{11} = \Delta S$, $\Delta\sigma'_{22} = -\Delta S$ (all other $\Delta\sigma'_{ij} = 0$), and because $\sigma'_{11} = \sigma'_{22} = -T/2$, we see that $|\sigma'_{11}|$ first increases and then decreases, which would be regarded as unloading.

Similar examples of ambiguity, such that a loading for all strain components in one coordinate system appears to involve unloading for some strain components in another coordinate system, can be found for most loading paths in which the principal stress directions rotate. To formulate a criterion for unloading that is the same in any coordinate system and avoids the ambiguity just exemplified, one must obviously use conditions that are invariant with regard to coordinate transformation, i.e., consist of functions of stress invariants. This leads, naturally, to loading functions (loading surfaces) and indicates that their use is inevitable, unless we restrict ourselves to stress histories in which the principal directions do not rotate. (The endochronic theory, too, implies a certain loading function (6,7), although it has originally been derived without introducing one.)

Can Invariance be Achieved by Rotating the Axes of Orthotropy?

Some structural analysts say they avoid the lack of invariance by keeping the orthotropy axes always oriented in the direction of principal stresses, rotating them against the material as principal stress directions rotate, and using the incremental shear moduli (D_{44}) only for the first infinitesimal shear stress increment away from the principal stress reference frame. There are, however, certain limitations with this approach.

First, one must not forget the principal strain directions. They evidently coincide with the principal stress directions as long as these do not rotate. Consider now the first stress increment $\Delta\sigma_{ij}$ which causes the principal stress directions to rotate. In the plane (x_1, x_2), the principal directions of stress and of strain then rotate by the angles

$$\Delta\theta_\sigma = \frac{\Delta\sigma_{12}}{\sigma_{11} - \sigma_{22}}, \quad \Delta\theta_\epsilon = \frac{\Delta\epsilon_{12}}{\epsilon_{11} - \epsilon_{22}} \quad \text{(26)}$$

if $|\Delta\sigma_{12}| << |\sigma_{11} - \sigma_{22}|$, $|\Delta\epsilon_{12}| << |\epsilon_{11} - \epsilon_{22}|$. Now, we should note that, in general, $\Delta\theta_\sigma \neq \Delta\theta_\epsilon$ except when the tangential shear modulus D_{44} $(= \Delta\sigma_{12}/\Delta\epsilon_{12})$ is

$$D_{44} = \frac{\sigma_{11} - \sigma_{22}}{\epsilon_{11} - \epsilon_{22}} \quad (\text{for } \epsilon_{11} \neq \epsilon_{22}) \quad \text{(27)}$$

In none of the existing orthotropic models, the tangential shear modulus is given by this expression, and so the rotations of the principal directions of stress and of strain are not the same. Therefore, as soon as the principal stress directions start to rotate, they also cease to coincide with the principal strain directions. Should then the axes of orthotropy be kept parallel to the principal directions of stress, or of strain? Evidently, they cannot be kept parallel to both.

Second, consider the physical microstructural aspects. The defects which are produced in the microstructure by the previous loading history and are the source of a change in tangential stiffness consist of microcracks, or plastic and frictional slips on certain planes, or grain rearrangements. These defects are locked within the material once they form. Rotation of the axes of orthotropy would physically imply rotating such defects against the material, which is obviously impossible. So, the lack of invariance cannot be avoided by rotating the axes of orthotropy.

We must realize, however, that there exists a certain special case to which the last objection does not apply. This is the case of classical incremental plasticity, for which the incremental stress-strain relation for loading can be written as

$$d\epsilon_{ij} = D_{ijkm} d\sigma_{km}, \quad D_{ijkm} = D^{el}_{ijkm} + \frac{1}{h}\frac{\partial F}{\partial \sigma_{ij}}\frac{\partial F}{\partial \sigma_{km}} \quad \text{(28)}$$

in which D^{el}_{ijkm} = isotropic tensor of elastic moduli, h = function of stress invariants, and $F = F(\boldsymbol{\sigma})$ is the plastic potential (loading surface). If the coordinate axes are rotated at each loading stage so as to coincide with the principal directions of tensor $\partial F/\partial\sigma_{ij}$, then D_{ijkm} differs from D^{el}_{ijkm} only when $i = j = k = m$, which means that the matrix of D_{ijkm} is orthotropic. The directions of orthotropy, however, coincide with the principal stress directions only if tensor $\partial F/\partial\sigma_{ij}$ has the same principal stress directions as σ_{ij}. This is true only if the loading surface is quadratic in σ_{ij}, i.e., if it is of von Mises or Drucker-Prager type.

Physically, the incremental orthotropy exhibited by these classical plasticity theories means that the microstructural defects which cause inelastic behavior are assumed to depend only on the current σ_{ij}, and be independent of the current ϵ_{ij}, as well as of the histories of σ_{ij} and ϵ_{ij} (loading path), i.e., be path-independent. Such an assumption may be quite acceptable for plasticity but hardly for concrete and geomaterials

for which the microstructural defects are mainly microcracks (or grain slips, losses of contacts between grains). The location, size and orientation of microcracks does not depend only on the current stress.

If the current state of microstructural defects depended only on the current stress and strains, the response could be also described by the total strain theory (i.e., Hencky's deformation theory), for which, in the case of isotropy, $s_{ij} = 2G(\boldsymbol{\sigma}, \boldsymbol{\epsilon})\, e_{ij}$ and

$$ds_{ij} = 2G de_{ij} + 2e_{ij} dG, \quad dG = \frac{\partial G}{\partial \epsilon_{mm}} d\epsilon_{mn} + \frac{\partial G}{\partial \sigma_{mn}} d\sigma_{mn} \quad \text{(29)}$$

$$d\sigma_{kk} = 3K d\epsilon_{kk} + 3\epsilon_{kk} dK, \quad dK = \frac{\partial K}{\partial \epsilon_{mn}} d\epsilon_{mn} + \frac{\partial G}{\partial \sigma_{mn}} d\sigma_{mn} \quad \text{(30)}$$

Now, consider that the material axes are rotated so as to coincide with the principal directions of tensor $\partial G/\partial\sigma_{ij}$. Then, however, $d\sigma_{11}$ produces in general a nonzero $d\epsilon_{12}$ because ϵ_{12} is in general nonzero, the principal directions of e_{ij} being different. If, instead, the material axes are rotated so as to coincide with the principal directions of ϵ_{ij}, $d\sigma_{12}$ again produces in general nonzero $d\epsilon_{11}$ because $\partial G/\partial\sigma_{12}$ is in general nonzero. So, the total strain theories do not conform to the assumption of incremental orthotropy.

Limitations of Cubic Triaxial Tests

The broad use of orthotropic models seems to be a consequence of the recent exaggerated emphasis on the cubic triaxial tests as opposed to the classical cylindrical triaxial tests. In cubic specimens (Fig. 2(*a*)), the principal stress axes cannot be made to rotate during the loading process, and by virtue of symmetry the principal directions of stress and strain are forced to coincide. From the foregoing analysis we see that these tests have serious limitations. They do serve the purpose of measuring the effect of the intermediate principal stress, but at the same time they miss other effects which are usually more important.

Thus, it will be necessary to concentrate on different types of tests. One attractive test specimen is a cylinder subjected to axial load, lateral external and internal pressure, and torsion (Fig. 2(*b*)). In this test one can induce any combination of principal stresses, and moreover one can make the principal stress direction have any angle with the specimen axis and rotate, either continuously or abruptly, during the loading process.

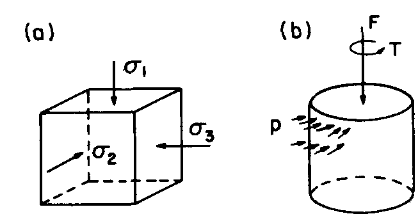

FIG. 2.—Critical Test and Triaxial-Torsional Test

Summary and Conclusions

Examined are incrementally linear (hypoelastic) constitutive equations, called orthotropic models, in which the stress and strain increments are related by a stiffness or compliance matrix that depends on the stresses and strains but is restricted to an orthotropic form. Although these materials exhibit stress-induced anisotropy, they are isotropic in their initial stress-free state. This means that any coordinate system, attached to the material, may be chosen as the material reference frame at the outset, and the response to the same loading worked out in *any* coordinate system must be the same. The conslusions are:

1. When one adheres to the requirement that the material references axes must not be rotated against the material during the deformation process, the orthotropic models are not tensorially invariant. One obtains different results depending on the choice of coordinate axes. Therefore the formulation is unobjective.
2. The lack of tensorial invariance can sometimes cause serious discrepancies. An example in which the shear angle increment is obtained as zero for calculations in one coordinate system and as 40% of the maximum normal strain increment for calculations in a rotated coordinate system is given for one practical orthotropic model.
3. One severe limitation is that orthotropic models neglect the cross effects, i.e. the normal strain increments caused by the shear stress increments and the shear strain increments caused by the normal stress increments. These effects are important for the modeling of dilatancy or compaction due to shear and hydrostatic pressure sensitivity of shear.
4. The physical mechanism that gives rise to these cross terms consists in the fact that the microstructural defects caused by stress, such as microcracking, exhibit some prevalent orientation. If the principal direction of the stress increment is skew (between 0° and 90°) with regard to the prevalent orientation of the defects caused by previous stress history, a generally anisotropic incremental stiffness or compliance matrix must be expected, with all elements of the matrix being nonzero.
5. Orthotropic models should, therefore, be restricted to loading histories in which the principal stress or strain directions do not rotate relative to the material during the loading process or when they rotate they do so only by a negligibly small angle. The coordinate axes in each finite element cannot then be oriented arbitrarily but must be oriented in the principal stress directions.
6. The lack of invariance can hardly be avoided by rotating the orthotropy axes against the material so as to keep them coinciding with the principal stress directions. This is because the orthotropy axes should just as well coincide with the principal strain directions, which rotate differently than the principal stress directions.
7. From the physical viewpoint, rotation of the material reference frame against the material implies rotating the microstructural defects which causes the degradation of tangential stiffness. This is inadmissible for defects such as microcracks or grain rearrangements in sands. It would be admissible only if the material followed classical incremental plasticity with von Mises (or Drucker-Prager) loading surface, but this is not a good model for concrete and geomaterials.

8. The orthotropic models are unsuitable for finite element programs.
9. The recently popular cubic triaxial tests do not provide information on material response when the principal stress directions rotate during the loading process. For this important purpose, other test specimens are needed. One possibility is offered by a cylindrical specimen subjected to axial load, lateral fluid pressure and torsion.

Final Remark.—The orthotropic models are not the only ones which have recently been criticized. Other models were critized, e.g., for their lack of uniqueness, stability, and continuity of response, and still others for their inability to represent observed material behavior which appears to violate uniqueness, stability and continuity. No perfect model exists free from criticism (6,7). Violation of tensorial invariance is, however, a more severe problem, since the model ceases to be objective, in particular, independent of the analyst's choice of reference frame. Objectivity is the first requirement for any mathematical model of a physical phenomenon.

Acknowledgment

Partial support under National Science Foundation Grant ENG 78-07777 to Northwestern University is gratefully acknowledged.

Appendix.—References

1. Al-Mahaidi, "Nonlinear Finite Element Analysis of Reinforced Concrete Deep Members," thesis presented to Cornell University, at Ithaca, N.Y., in 1979, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.
2. Argyris, J. H., Faust, G., Szimmat, J., Warnke, E. P., and Willam, K. J., "Finite Element Ultimate Load Analysis of Three-Dimensional Concrete Structures," Report ISD (Institute for Statics and Dynamics), Technische Universität, Stuttgart, West Germany, Aug., 1974.
3. Bashur, F. K., and Darwin, D., "Nonlinear Biaxial Law for Concrete," *Journal of the Structural Division*, ASCE, Vol. 104, No. ST1, Jan., 1978, pp. 157–170.
4. Bathe, K. J., and Ramaswamy, S., "On Three-Dimensional Nonlinear Analysis of Concrete Structures," *Nuclear Engineering and Design*, Vol. 52, No. 3, May, 1979, pp. 385–409.
5. Bažant, Z. P., "Critique of Orthotropic Models and Triaxial Testing of Concrete and Soils," *Structural Engineering Report, No. 79-10/640c*, Northwestern University, Evanston, Ill., Oct., 1979.
6. Bažant, Z. P., "Endochronic Inelasticity and Incremental Plasticity," *International Journal of Solids and Structures*, Vol. 14, 1978, pp. 691–714.
7. Bažant, Z. P., "Work Enequalities for Plastic-Fracturing Materials," *International Journal of Solids and Structures*, Vol. 16, 1980, pp. 873–901.
8. Bažant, Z. P., and Kim, S. S., "Plastic-Fracturing Theory for Concrete," *Journal of the Engineering Mechanics Division*, ASCE, Vol. 105, No. EM3, June, 1979, pp. 407–428.
9. Boulon, M., et al., "Soil-Structure Coupling. Nonlinear Rheological Relationships and Boundary Conditions in Soil Mechanics," *Computers and Structures*, Vol. 9, 1978, pp. 293–303.
10. Boulon, M., Chambon, R., and Darve, F., "Loi rhéologique incrémentale pour les sols et application par la methode des elements finis," *Revue Française Géotechnique*, Vol. 2, 1977, pp. 5–22.
11. Boulon, M., et al., "Comportement d'un ecran et d'un pieu. Essais, calculs," *Proceedings*, International Congress of Soils Mechanics and Engineering

Foundation, Vol. 2, 1977, pp. 423–426.
12. Chambon, R., Darve, F., and Tratapel, G., "Mesures et calculs avec loi non-linéaire pour un modèle expérimental," Comptes-Rendus du Symposium Franco-polonais de Cracovie, Krakow, Poland, 1977.
13. Chambon, R., and Renoud-Lias, B., "Incremental Nonlinear Stress-Strain Relationship for Soil and Integration by Finite Element Method," *Proceedings*, 3rd International Conference on Numerical and Analytical Methods in Geomechanics, Wittke, W., ed., held in Aachen in Rotterdam, The Netherlands, Apr., 1979, pp. 405–413.
14. Connor, J. J., and Sarne, Y., "Nonlinear Analysis of Prestressed Concrete Reactor Pressure Vessels," T. A. Jaeger, ed., *Proceedings*, 4th International Congress on Structural Mechanics in Reactor Technology, Vol. H., Paper H2/2, London, England, Sept., 1975.
15. Coon, M. D., and Evans, R. J., "Incremental Constitutive Laws and their Associated Failure Criteria with Application to Plain Concrete," *International Journal of Solids and Structures*, Vol. 8, 1972, pp. 1169–1180.
16. Darve, F., "Contribution a la détermination de la Loi rhéologique incrémentale des sols," Thèse de Docteur-Ingénieur, Institut de Mécanique, Grenoble, France, 1976.
17. Darve, F., "Une Formulation Incrémentale des lois rhéologiques. Application aux Sols," Thèse de doctorat d'ét at, Institut de Mécanique de Grenoble, Grenoble, France, in 1978.
18. Darve, F., Boulon, M., Chambon, R., "Loi Rhéologique Incrémentale des Sols," *Journal de Mécanique*, Paris, France, 1978, Vol. 17, No. 5, pp. 679–716.
19. Darve, F., Desrues, J., and Jacquet, M., "Les surfaces de rupture en mécanique des sols en tant qu'instabilité de déformation," *Cahiers du Groupe Français de Rhéologie*, Paris, France, Vol. 5, No. 3, 1980, pp. 93–106.
20. Darve, R., and Flavigny, E., "Simulation de la liquéfaction des Sols avec un loi incrémentale," *Proceedings*, Journée de Rhéologie 1979, "Cycles dans les Sols," Ecole Nationale des Travaux Publics de l'Etat, Vaux-en-Velin, France.
21. Darve, F., and Labanieh, S., "Incremental Constitutive Law for Sands and Clays: Simulations of Monotonic and Cyclic Tests," *International Journal for Numerical and Analytical Methods in Geomechanics*, Vol. 6, 1982, pp. 243–275.
22. Darve, F., Labanieh, S., and Chambon, R., "Incremental Stress-Strain Relationship for Cohensionless Soils," *Proceedings*, ASCE Conference on Numerical Methods in Geomechanics, held at Virginia Polytechnic Institute, Blacksburg, Va., ed. by C. S. Desai, 1976, Vol. I, pp. 264–269.
23. Darwin, D., "A Biaxial Stress-Strain Model for Concrete," *Proceedings*, 3rd Engineering Mechanics Division Specialty Conference of ASCE, held at Austin, Texas, Sept., 1979, pp. 441–444.
24. Darwin, D., Pecknold, D. A., "Analysis of Cyclic Loading of Plane R/C Structures," *Computers and Structures*, Vol. 7, Pergamon Press, Inc., New York, N.Y., 1977, pp. 137–147.
25. Darwin, D., and Pecknold, D. A. W., "Analysis of R/C Shear Panels Under Cyclic Loading," *Journal of the Structural Division*, ASCE, Vol. 102, No. ST2, Feb., 1976, pp. 355–369.
26. Darwin, D., and Pecknold, D. A., "Nonlinear Biaxial Law for Concrete," *Journal of the Engineering Mechanics Division*, ASCE, Vol. 103, No. EM2, Apr., 1977, pp. 229–241.
27. Elwi, A. A., and Murray, D. W., "A 3D Hypoelastic Concrete Constitutive Relationship," *Journal of the Engineering Mechanics Division*, ASCE, Vol. 105, No. EM4, 1979, pp. 623–641.
28. Fung, Y. C., *Foundations of Solid Mechanics*, Prentice Hall, Englewood Cliffs, N.J., 1975.
29. Gallix, R., Liu, T. C., and Lu, S. C. H., "Crack Analysis of Multicavity Prestressed Concrete Reactor Vessels," 4th International Congress on Structural Mechanics in Reactor Technology, London, England, Vol. H, Paper H4/2, T. A. Jaeger, ed., Sept., 1975
30. Geistefeld, H., "Material Law for Concrete Under Multiaxial Stress," *Transactions*, 4th International Conference on Structural Mechanics in Reactor Technology, Vol. H, Paper H5/2, San Francisco, Calif., Aug., 1977, T. A. Jaeger and B. A. Boley, eds.

31. Goldsheider, M., "Spannungen in Sand bei Räumlicher Monotoner Verformung," Dissertation, Universität Karlsruhe, Karlsruhe, West Germany, 1972.
32. Gudehus, G., "Elastoplastische Stoffgleichungen für Trockenen Sand," Ingenieur-Archiv, Vol. 42, 1973, pp. 151–169.
33. Gudehus, G., Goldsheider, M., and Winter, H., "Mechanical Properties of Sand and Clay and Numerical Integration Methods: Some Sources of Errors and Bounds of Accuracy," *Finite Elements in Geomechanics*, G. Gudehus, ed., John Wiley and Sons, Inc., New York, N.Y., 1977.
34. Isenberg, J., and Adham, S., "Analysis of Orthotropic Reinforced Concrete Structures," Journal of the Structural Engineering Division, *Proceedings*, ASCE, Vol. 96, No. ST12, Dec., 1970, pp. 2607–2624.
35. Kabir, A. F., "Nonlinear Analysis of Reinforced Concrete Panels, Slabs and Shells for Time-Dependent Effects," *Report No. 76-6*, Structural Engineering and Structural Mechanics, Univ. of California, Berkeley, Calif., Dec., 1976.
36. Lanier, J., "Etude expérimentale des lois de comportement en grandes déformations à l'aide d'une presse réellement tridimensionnelle," *Cahiers du Groupe Français de Rhéologie*, Paris, France, Vol. 4, 1976, No. 2, pp. 53–60.
37. Link, J., "Eine Formulierung des Zweiaxialen Verformungs und Bruchverhaltens von Beton und deren Anwendung anf die wirklichkeitsnahe Berechnung von Stahlbetonplatten," *Deutscher Ausschuss Für Stahlbeton*, Heft 270, W. Ernst & Sohn, West Berlin, West Germany, 1976.
38. Link, J., "Numerical Analysis for Oriented Biaxial Stress-Strain Relation and Failure Criterion of Plain Concrete," *Transactions*, 3rd International Conference on Structural Mechanics in Reactor Technology, Paper H1/2, London, England, Sept., 1975.
39. Link, J., Schafer, H., and Melhorn, G., "Eine Formulierung des Zweiachsigen Bruch und Verformungsverhaltens von Beton," Beton und Stahlbetonbau, West Germany, Sept., 1974.
40. Liu, T. C. Y., Nilson, A. H., and Slate, F. O., "Biaxial Stress-Strain Relations for Concrete," *Journal of the Structural Division*, ASCE, Vol. 98, No. ST5, May, 1972, pp. 1025–1034.
41. Liu, T. C. Y., Nilson, A. H., and Slate, F. O., "Stress-Strain Response and Fracture of Concrete in Uniaxial and Biaxial Compression," *Journal of the American Concrete Institute*, Vol. 69, No. 5, May, 1972, pp. 291–295.
42. Maher, A., and Darwin, D., "Microscopic Finite Element Model of Concrete," *Proceedings*, First International Conference on Mathematical Modeling, St. Louis, Mo., 1979, pp. 1705–1714.
43. Malvern, L. E., Introduction to the Mechanics of a Continuous Medium, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1969.
44. Mueller, G., "Numerical Problems in Nonlinear Analysis of Reinforced Concrete," SESM Report No. 77-5, University of California, Berkeley, Calif., Sept., 1977.
45. Rajagopal, K. R., "Nonlinear Analysis of Reinforced Concrete Beams, Beam-Columns and Slabs by Finite Elements," thesis presented to Iowa State University, at Ames, Iowa, in 1976, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.
46. Robinet, J. C., DiBenedetto, H., Cormet, A., and Mathieu, P., "Loi de Comportement Non-Linéaire en Écriture Incrémentale pour les Sols," *Proceedings*, Journeé de Rhéologie, *Cycles dans les Sols*, Ecole Nationale les Travaux Publics de l'État, Vaux-en-Velin, France, 1979.
47. Romstad, K. M., Taylor, M. A., and Herrmann, L. R., "Numerical Biaxial Characterization for Concrete," *Journal of the Engineering Mechanical Division, Proceedings*, ASCE, Vol. 100, No. EM5, Oct., 1974, pp. 935–948.
48. Sarne, Y., "Material Nonlinear Time-Dependent Three-Dimensional Finite Element Analysis of Reinforced and Prestressed Concrete Structures," thesis presented to the Massachusetts Institute of Technology, at Cambridge, Mass.,

in 1974, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.
49. Truesdell, C., "Hypo-elasticity," *Journal of Rational Mechanics Analysis*, Vol. 4, 1955, pp. 83–133.
50. Robinet, J. C., "Prédiction du comportement d'une argile sur chemins triaxiaux," Journal de Rhéologie, Ecole Nationale des Travaux Publics de l'Etat, Paris, France, 1978, pp. 55–98.
51. VanGreunen, J., "Nonlinear Geometric, Material and Time-Dependent Analysis of Reinforced and Prestressed Concrete Slabs and Panels," SESM Report No. 79-3, University of California, Berkeley, Calif., Oct., 1979.
52. Vuaillat, P., "Propriétés visqueuses d'une argile. Expériences et formulation incrémentale," Thése de docteur-ingénieur, Institut de Mécanique de Grenoble, France, 1980.