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# NOTES

## LIMITATIONS OF STRAIN-HARDENING MODEL FOR CONCRETE CREEP

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One simple way to approximately describe the decay of creep rate at constant uniaxial stress $\sigma$ is to assume that the creep rate $\dot{\varepsilon}_C$ is a function of load duration $\tau$, or the current creep strain $\varepsilon_C$ or the current age t; thus, in general, $\dot{\varepsilon}_C = F(\sigma, \varepsilon_C, \tau, t)$. The decay of the creep rate as a function of load duration is called time-hardening, the decay of the creep rate as a function of $\varepsilon_C$ is called strain-hardening, and the decay as a function of age is called aging (1,2). Aging is in some way reflected in all existing concrete creep laws, but not strain-hardening or time-hardening. Time-hardening has the advantage that it does not destroy linearity of the stress-strain relation, while strain-hardening does (if the stress is variable and unknown). A form of time-hardening corresponding to an aging Maxwell model was introduced in the 1930's by Glanville and Dischinger and is called the rate-of-creep (or Dischinger) method (3).

L'Hermite (4,5) was apparently the first to introduce strain-hardening for concrete. He proposed a special form of function F such that:

$$\dot{\varepsilon}_C = \sigma(\varepsilon_C^\infty - \varepsilon_C)\left(\frac{k_1}{t} + k_2\right) \tag{1}$$

in which $\varepsilon_C^\infty$, $k_1$, $k_2$ = constants, $t = t' + \tau$ = current age, $t'$ = age at loading. He then showed that his strain-hardening creep law can give a relatively good description of the creep curves of concrete at constant stress for various ages $t'$ at loading. As for time-variable stress, L'Hermite did not consider the strain-hardening law to apply, and with Mamillan (4) he concluded that a hereditary creep law, such as a law based on the principle of superposition, is necessary to describe the observed creep behavior adequately.

Recently, however, a creep formulation which is equivalent to the concept of strain-hardening was proposed by Acker and Lau (7,8,9) for general stress histories and has been adopted for the French code. The purpose of this note is to demonstrate the equivalence of this formulation to strain-hardening and to point out certain serious limitations as far as creep at time-variable stress is concerned.

Acker and Lau's model does not specify function F explicitly, but function F is implied in their model. They characterize creep by an empirical function $\Phi$ which describes the concrete creep curves for various values of constant stress $\sigma$, and for various ages at loading $t'$ as well as various environmental humidities $h$ and temperatures $T$:

$$\varepsilon_C(t) = \Phi(\sigma,\tau,t,h,T). \tag{2}$$

The creep rate at constant stress corresponding to Eq. 2 is:

$$\dot{\varepsilon}_C(t) = \Phi_{,t}(\sigma,\tau,t,h,T) \tag{3}$$

where $\Phi_{,t} = \partial\Phi/\partial t$.

Consider now arbitrary stress history $\sigma(t)$. When stated in mathematical terms, the idea of Acker and Lau was to use Eqs. 2 and 3 as two simultaneous equations which are valid for the current stress $\sigma(t)$ at time t. This idea means that $\tau$ in Eqs. 2 and 3 becomes a parameter. This parameter is called by Acker and Lau the equivalent time, $\tau$, and is defined as the duration of a constant stress equal to the current stress $\sigma(t)$ during which the creep strain produced is the same as the current creep strain $\varepsilon_C(t)$. This definition obviously also requires t to represent the current actual age of concrete, and h and T to represent the current humidity and temperature (this is not explicitly stated in Refs. 7, 8 and 9, but no other alternative seems workable).

The aforementioned definition of $\tau$ means that $\tau$ is to be solved from Eq. 1 in which $\sigma$, h and T are considered with their current values. Thus, denoting as $\Psi$ the inverse of function $\Phi$ with respect to $\tau$, we may write:

$$\tau = \Psi[\sigma(t),\ \varepsilon_C(t),\ t,\ h(t),\ T(t)]. \tag{4}$$

Substituting this value into Eq. 3 and introducing the notation:

$$[\Phi_{,t}[\sigma(t),\tau,t,h(t),T(t)]_{\tau=\Psi[\sigma(t),\ldots]} = F[\sigma(t),\varepsilon_C(t),t,h(t),T(t)] \tag{5}$$

where F is a certain monotonic and smooth function of $\varepsilon_C$, we find that Acker and Lau's formulation is equivalent to the differential equation:

$$\dot{\varepsilon}_C = F(\sigma,\ \varepsilon_C,\ t,\ h,\ T) \tag{6}$$

at variable $\sigma(t)$. The solution of this differential equation for $\sigma$ = const. is Eq. 2.

The fact that the creep rate $\dot{\varepsilon}_C$ in Eq. 6 depends on the total accumulated creep strain $\varepsilon_C(t)$ (in addition to a dependence on t, h and T) means that Acker and Lau's creep formulation is equivalent to a strain-hardening creep law. Conversely, any strain-hardening formulation may be easily stated in the form used by Acker and Lau. For example, Eq. 1 of L'Hermite, which is linear if the stress is prescribed, can be integrated for constant stress $\sigma$ to yield:

$$\Phi(\sigma,\tau,t,h,T) = \left\{\frac{1}{E(t')} + \varepsilon_C^{\infty}\left[1 - \left(\frac{t'}{t}\right)^{k_1} e^{-k_2(t-t')}\right]\right\}\sigma \tag{7}$$

where $E(t')$ = elastic modulus at loading age $t'$, and $\tau = t - t'$. The use of this function in the manner of Acker and Lau is then, of course, equivalent to the direct use of Eq. 1 in structural analysis.

Strain-hardening has found wide applications for creep of metals (1,2), for which the dependence of $\Phi$ on $\sigma$ (or of F on $\sigma$) is always nonlinear. For concrete, the dependence of $\Phi$ on $\sigma$ may be assumed to be linear in the service stress range (as in Eq. 1), and then the function $\Phi(\ldots)/\sigma$ represents the compliance function J(t,t').

The strain-hardening model of Acker and Lau was shown (7) to give relatively good predictions for creep after a partial unloading. However, this model gives poor predictions for other basic stress histories:

1. All strain-hardening models including that of Lau and Acker predict no creep recovery at all; see Fig. 1 in which the test data are taken from Roll (10), Kommendant et al. (11), and Mullick (12) and are summarized e.g. in Ref. 13.

2. All strain-hardening models also underpredict concrete creep for increasing stress histories. This may be illustrated for a two-step history by the test data of Kimishima and Kitahara (14) in Fig. 2 (summarized in Ref. 13). By contrast, the linear superposition gives a reasonably good prediction, and the BTC model from Ref. 11 an excellent prediction. The reason for the underprediction in Fig. 2 is clear from the graphical construction of the response curve according to Lau et al. (7), based directly on their formulation of equivalent time. Curve c is the creep curve at the new stress $\sigma_2$ = 800 psi which gives, at time $t_1$, the same creep strain $\varepsilon_C$ as the curve a at the initial stress $\sigma_1$ = 400 psi. Curve c is determined by finding first curve b which gives, at time $t_1$, the creep strain $\varepsilon_C \sigma_1/\sigma_2 = \varepsilon_C/2$. Of course

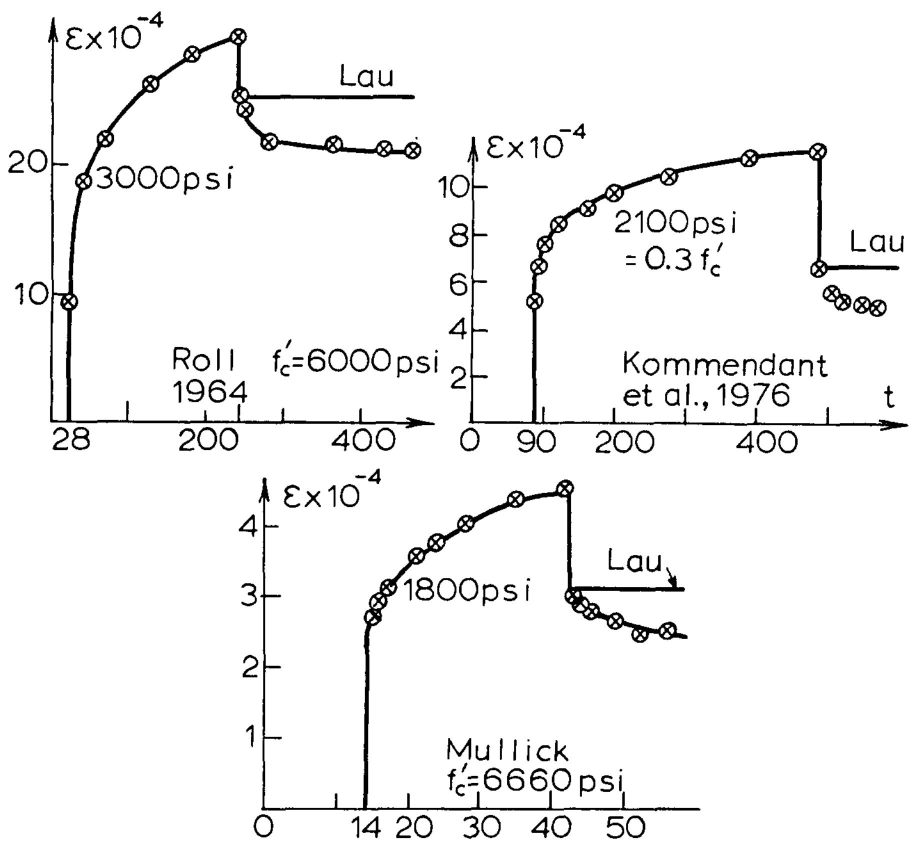


FIG. 1
Some Typical Test Data on Creep Recovery (8,9,10) Compared to the Prediction of the Strain-Hardening Model by Lau et al. (7).

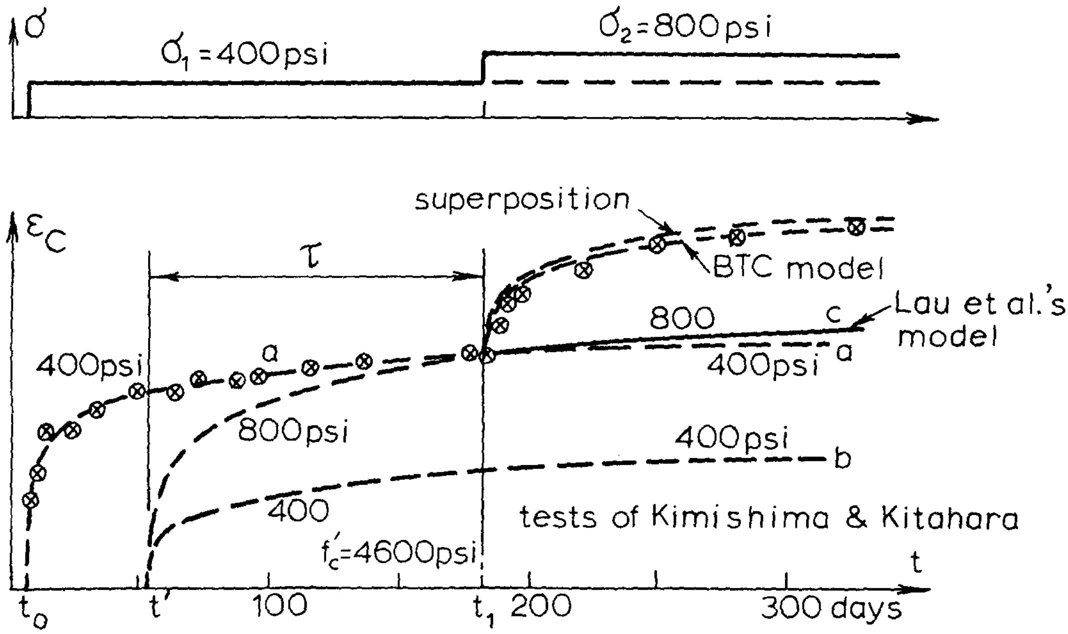


FIG. 2
Typical Test Data for Creep at Two-step Stress History (14) Compared to the Prediction of the Strain-Hardening Model by Lau et al. (7).

the determination of the equivalent time $\tau = t_1 - t'$ for curve b inevitably involves some error, but this makes little difference since for all t'-values between 40 days and 150 days the error in prediction is large.

3. According to Eq. 5 or Eqs. 2-3, all the stress histories which produce, at age $t_1$, the same creep strain $\varepsilon_C$ would have to lead to equal creep rates right after time $t_1$. In reality, though, these creep rates can be quite different.

From the foregoing analysis, it appears that the strain-hardening constitutive law does not give a good representation of concrete creep at variable stress. Its errors in describing creep at variable stress, as seen in Figs. 1 and 2, are approximately as large as those of the classical rate-of-creep method (Dischinger method), and are larger than those of the improved Dischinger method, used in the current CEB-FIP Model Code, or the current ACI Recommendation. At the same time, these existing formulations are easier to use in structural analysis than Acker and Lau's model because they are linear. Acker and Lau (7) demonstrated by an example a nonlinear solution for statically indeterminate beams by the layered approach. Their solution is, however, more involved than a linear solution according to the aforementioned models, especially if the age-adjusted effective modulus method is used.

The concept of strain-hardening, while inadequate if considered alone, might nevertheless prove useful in combination with other concepts, e.g., the principle of superposition. A combination of strain-hardening and a superposition principle has been studied at Northwestern University (13), and one such formulation provided a rather good description of broad range of test data.

Another debatable aspect of the formulation proposed for the French code (7,8,9) is the form of the formulas for function Φ; see pp. 230-231 of Ref. 3.

This aspect, however, is a separate issue, independent of strain-hardening.

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