Solidification Theory for Concrete Creep. II: Verification and Application

By Zdeněk P. Bažant,[1] Fellow, ASCE, and Santosh Prasannan,[2] Student Member, ASCE

Abstract: The theory that was formulated in the preceding paper is verified and calibrated by comparison with important test data from the literature pertaining to constant as well as variable stress at no (or negligible) simultaneous drying. Excellent agreement is achieved. The formulation describing both elastic and creep deformations contains only four free material parameters, which can be identified from test data by linear regression, thus simplifying the task of data fitting. For numerical structural analysis, the creep law is approximated in a rate-type form, which corresponds to describing the solidified matter by a Kelvin chain with nonaging elastic moduli and viscosities. This age-independence on the microlevel makes it possible to develop for the present model a simple version of the exponential algorithm.

Introduction

After developing in the preceding paper a new theory based on a simplified picture of the micromechanics of creep in a solidifying material, the writers proceed in this paper to verify and calibrate this theory by comparison with various important test data from the literature. All definitions and notations from the previous paper are retained.

Rate-Type Creep Law and Rheologic Model

As is well known, the efficiency of numerical creep analysis of structures requires converting an integral-type creep law to a rate-type form. The form of rate-type creep law can always be visualized by a spring-dashpot model. Although there are infinitely many possible arrangements of springs and dashpots, it has been shown that the most general creep behavior can be described by the Maxwell chain, or the Kelvin chain.

In the case of concrete, a complication arises from the fact that the elastic moduli and viscosities of the springs and dashpots are, in general, age-dependent. Owing to this property, the differential equations describing the Kelvin chain are of the second order, while those describing the Maxwell chain are of the first order. For this as well as other reasons (RILEM 1986; Bažant 1982), the Maxwell chain has been preferred, even though the Kelvin chain is more directly related to creep tests at constant stress. This situation, however, is reversed by the present formulation. Because of the new idea of using in the constitutive relation a nonaging creep law for the solidified matter (hydrated cement) and expressing aging by means of a change of volume $v(t)$ (Fig. 1 of Part I), it is possible to associate the rheologic model with the behavior of an element of the solidified matter rather than concrete as a whole. If this is done, then the rheologic model is nonaging, i.e., its spring moduli and viscosities are independent of time, as in classical linear viscoelasticity. Exploiting this fact brings about a major simplification in numerical creep analysis and makes the Kelvin chain more advantageous than the Maxwell chain.

Denoting the strain of the μth unit of the Kelvin chain (Fig. 1 of Part I) as γ_μ, the differential equations for a nonaging Kelvin chain are

$$E_\mu \gamma_\mu + \eta_\mu \dot{\gamma}_\mu = \sigma, \qquad \gamma = \sum_{\mu=1}^{N} \gamma_\mu \quad (1)$$

in which E_μ and η_μ are the elastic moduli and viscosities of the μth unit. Integrating these equations for constant stress σ applied at age t', we obtain

$$\gamma(t) = \sigma \sum_{\mu=1}^{N} \frac{1}{E_\mu}(1 - e^{-(t-t')/\tau_\mu}), \qquad \tau_\mu = \frac{\eta_\mu}{E_\mu} \quad (2)$$

in which τ_μ are constants called the retardation times. This series, called a Dirichlet (or Prony) series, can closely approximate various creep curves. In particular, for the creep curve given by Eq. 11 of Part I

$$\ln(1 + \xi^n) \simeq \sum_{\mu=1}^{N} A_\mu(1 - e^{-\xi/\tau_\mu}), \qquad \xi = t - t' \quad (3)$$

Trying to determine τ_μ from test data is known to lead to an ill-conditioned equation system. Therefore, τ_μ must be chosen, and a suitable choice is

$$\tau_\mu = \tau_1 10^{\mu-1} \qquad (\mu = 1, 2, \ldots N) \quad (4)$$

Constants A_μ may, in general, be found by the method of least squares, and then $E_\mu = 1/q_2 A_\mu$ and $\eta_\mu = E_\mu \tau_\mu$. For the present case, a formula giving A_μ has been found; it is shown in Eq. 17. The response of the Kelvin chain is then approximately equivalent, not only for constant stress but also for variable stress, because $\gamma(t)$ due to variable stress is obtained from $\Phi(t - t')$ by principle of superposition.

The effect of temperature may be introduced similarly as in the previous rate-type models; $\eta_\mu = \tau_\mu E_\mu$ in Eq. 1 is replaced by $\tau_\mu E_\mu f(T)$ where $f(T)$ depends on temperature as indicated by the activation energy theory. Aside from that, variation of temperature requires that $v(t)$ be replaced with $v(t_e)$ where t_e is the equivalent hydration period.

Numerical Solution by Exponential Algorithm

By virtue of the replacement of the integral Eq. 5 in Part I with the system of differential equations in Eq. 1, the constitutive law is of a rate-type form, i.e., given entirely by differential equations (Eq. 1, and Eq. 7 of Part I). As shown before (Bažant 1971, 1975, 1982; RILEM 1986), effective numerical integration necessitates the so-called exponential algorithm that makes it possible to gradually increase the time steps to values greatly exceeding the shortest retardation time, while, at the same time, maintaining numerical stability and good accuracy. In this algorithm, it is assumed that within each

[1]Prof. of Civ. Engrg., Northwestern Univ., Evanston, IL 60208.
[2]Grad. Res. Asst., Northwestern Univ., Evanston, IL.

Note. Discussion open until January 1, 1990. Separate discussions should be submitted for the individual papers in this symposium. To extend the closing date one month, a written request must be filed with the ASCE Manager of Journals. The manuscript for this paper was submitted for review and possible publication on June 3, 1988. This paper is part of the ***Journal of Engineering Mechanics***, Vol. 115, No. 8, August, 1989. ©ASCE, ISSN 0733-9399/89/0008-1704/$1.00 + $.15 per page. Paper No. 23756.

step (t_i, t_{i+1}) the stress varies linearly, in which case exact solutions of the differential equations can be obtained. The linear variation of stress is $\sigma(t) = \sigma_i + (t - t_i)\Delta\sigma/\Delta t$ in which subscript i refers to time t_i, and Δ denotes changes over the time step. With this assumption, the general solution of Eq. 1 may be sought in the form $\gamma_\mu(t) = A + B(t - t_i) + Ce^{-(t-t_i)/\tau_\mu}$. Substitution into Eq. 1 yields the constants A and B, and subsequently the use of the initial condition $\gamma_\mu = \gamma_{\mu_i}$ at $t = t_i$ yields the value of C. In this manner one obtains for $t = t_{i+1}$

$$\gamma_{\mu_{i+1}} = \gamma_{\mu_i} e^{-\Delta y_\mu} + \frac{\sigma_i}{E_\mu}(1 - e^{-\Delta y_\mu}) + \frac{1 - \lambda_\mu}{E_\mu}\Delta\sigma \qquad (5)$$

with the notations

$$\Delta y_\mu = \frac{\Delta t}{\tau_\mu}, \qquad \lambda_\mu = \frac{1 - e^{-\Delta y_\mu}}{\Delta y_\mu} \qquad (6)$$

Note that the coefficient λ_μ is generally between zero and 1. For $\Delta t << \tau_\mu$, λ_μ approaches 1, and for $\Delta t >> \tau_\mu$, λ_μ approaches zero. This means that for time steps much shorter than the retardation time, the strain produced by the stress increment during the time step is very small, and for time steps much larger than the retardation time, the strain increment produced by the stress change during the time step is $\Delta\sigma_\mu/E_\mu$ since the stress in the dashpot has enough time to dissipate. The conventional numerical methods for ordinary differential equations do not satisfy these limiting conditions; they can be used only if the time step is much less than the shortest retardation time. This condition is unacceptably restrictive for such a broad retardation spectrum as that of concrete.

Calculating $\Delta\gamma = \Sigma(\gamma_{\mu_{i+1}} - \gamma_{\mu_i})$ from Eq. 6, we obtain for the solidified matter the following quasi-elastic stress-strain relation for each time step:

$$\Delta\gamma = \frac{\Delta\sigma}{D} + \Delta\gamma'' \qquad (8)$$

in which $$\frac{1}{D} = \sum_{\mu=1}^{N} \frac{1 - \lambda_\mu}{E_\mu} \qquad (9)$$

$$\Delta\gamma'' = \sum_{\mu=1}^{N} \Delta\gamma''_\mu = \sum_{\mu=1}^{N} \left(\frac{\sigma_i}{E_\mu} - \gamma_{\mu_i}\right)(1 - e^{-\Delta y_\mu}) \qquad (10)$$

According to Eq. 5 of Part I we further have

$$\Delta\epsilon^v = \frac{F(\sigma_{i+1/2})}{v_{i+1/2}}\Delta\gamma = \frac{F(\sigma_{i+1/2})}{v_{i+1/2}}\left(\frac{\Delta\sigma}{D} + \Delta\gamma''\right) \qquad (11)$$

in which subscript $i + 1/2$ refers to the middle of the time step in the logarithmic time scale, $t_{i+1/2} = t_0 + [(t_{i+1} - t_0)(t_i - t_0)]^{1/2}$. Finally, adding the increment of the flow strain, $\Delta\epsilon^f = F(\sigma_{i+1/2})q_4\Delta t/t_{i+1/2}$, and the increment of instantaneous elastic strain, $q_1\Delta\sigma$, we obtain the incremental quasi-elastic stress-strain relation

$$\Delta\epsilon = \frac{\Delta\sigma}{E''} + \Delta\epsilon'' \qquad (12)$$

in which $$\frac{1}{E''} = q_1 + \frac{F(\sigma_{i+1/2})}{v_{i+1/2}D} \qquad (13)$$

$$\Delta\epsilon'' = F(\sigma_{i+1/2})\left(\frac{\Delta\gamma''}{v_{i+1/2}} + \frac{q_4\Delta t}{t_{i+1/2}}\right) + \Delta\epsilon^0 \qquad (14)$$

Eqs. 13 and 14 make it possible to calculate the creep curves at constant stress (in which case $\Delta\sigma = 0$ for all the time steps except the first). One can also calculate from these equations the strain response for any stress history, as well as the stress response for any given strain history. Using Eq. 14 as a quasi-elastic stress-strain relation, one can calculate the creep response of any structure, e.g., by finite elements.

Eqs. 8–10 represent a special case of Bažant's (1971) exponential algorithm for aging concrete, and are essentially equivalent to the algorithms proposed previously for nonaging viscoelastic structures by Taylor et al. (1970), and Zienkiewicz and Watson (1968). The extension of these algorithms to aging creep is now accomplished by the transition from Eqs. 8–10 to Eqs. 12–14. It may be checked that the present algorithm is unconditionally stable, the same as the previous forms of exponential algorithm. The basic advantage, as already emphasized, is that the elastic moduli and viscosities used in this algorithm are constant rather than functions of time.

The algorithm which may be used to solve a structure on the basis of the present creep formulation may proceed in each time step (t_i, t_{i+1}) as follows:

1. The values of σ_i, ϵ_i, ϵ_i^v, ϵ_i^f, γ_i, γ_{μ_i} for the beginning of the time step are known for all the points of the structure. Estimate $\sigma_{i+1/2} = \sigma_i + \Delta\sigma/2$ for all the points of the structure, using $\Delta\sigma$ − values from the preceding iteration, or, for the first iteration, from the previous time step, and calculate $F(\sigma_{i+1/2})$. Then calculate D, $\Delta\gamma''$, $\Delta\gamma$, E'', $\Delta\epsilon''$ for all the points of the structure.
2. Solve the structure using the quasi-elastic stress-strain relation in Eq. 14 and the load or displacement increments prescribed for this time step. Then, if the condition for the termination of iterations is not met, return to 1 and start the next iteration of this step. Otherwise evaluate $\sigma_{i+1} = \sigma_i + \Delta\sigma$, $\epsilon_{i+1} = \epsilon_i + \Delta\epsilon_i$ for all the points of the structure, return to 1 and start the first iteration of the next time step.

The present formulation can be generalized to triaxial stress states on the basis of the assumption of isotropy, in the same manner as done with other formulations. Function $F(\sigma)$ must then be replaced by a function of the invariants of the stress tensor.

Dirichlet Series Expansion

It is useful to recall the series approximation of the logarithmic function, which is, according to Bažant and Wu (1973),

$$\log \xi = \sum_{\mu=1}^{N} (1 - e^{-\xi/\tau_\mu}), \qquad \tau_\mu = 10^{\mu-1}\tau_1 \qquad (15)$$

It is applicable in the range $0.3\tau_1 \le \xi \le 0.5\tau_N$. According to Bažant (1977),

TABLE 1. Coefficients of Dirichlet Series Approximation of ln $(1 + \xi)^n$

n (1)	b_1 (2)	z (3)
0.01	0.674	0
0.03	0.632	0
0.05	0.587	0
0.07	0.538	0
0.09	0.491	0.039
0.11	0.507	0.355
0.13	0.529	0.621
0.15	0.552	0.860
0.17	0.574	1.080
0.19	0.593	1.288
0.21	0.610	1.486
0.23	0.624	1.677
0.25	0.634	1.860
0.27	0.641	2.037
0.29	0.643	2.207
0.31	0.641	2.372
0.33	0.635	2.531
0.35	0.624	2.683
0.37	0.610	2.826
0.39	0.594	2.969
0.41	0.579	3.102
0.43	0.571	3.227
0.45	0.579	3.345
0.47	0.617	3.453
0.49	0.708	3.556
0.51	0.886	3.644
0.53	1.202	3.725

the Dirichlet series approximation of the power function for this range is

$$\xi^n = \sum_{\mu=1}^{N} b_\mu(n)\tau_\mu^n(1 - e^{-\xi/\tau_\mu}) \quad \text{(16)}$$

The coefficients $b_\mu(n)$ have been tabulated.

For the present formulation, we need the Dirichlet series approximation of the function ln $(1 + \xi^n)$ (Eq. 11 of Part I). Because ln $(1 + \xi^n) = \xi^n$ if $\xi << 1$, and ln $(1 + \xi^n) \approx n \ln \xi$ if $\xi >> 1$, Eqs. 15 and 16 must be the limiting cases of this approximation. For small ξ, the terms $1 - e^{-\xi/\tau_\mu}$, for which τ_μ is large, almost vanish, and for large ξ, the terms for which τ_μ is small are almost equal to 1. Thus, the limiting case for small ξ is dominated by the terms for small μ, and the limiting case for large ξ is dominated by the terms for large μ. At the same time we note that Eq. 16 changes to Eq. 15 (except for a factor) if n is replaced by zero. Based on these observations, Bažant suggested changing the constant exponent n from Eq. 16 to a function $m(\mu)$, such that the coefficients of Eq. 3 take the form

$$A_\mu = b_\mu(n)\tau_\mu^{m(\mu)}, \qquad m(\mu) = \frac{n}{1 + (c\mu)^z} \quad \text{(17)}$$

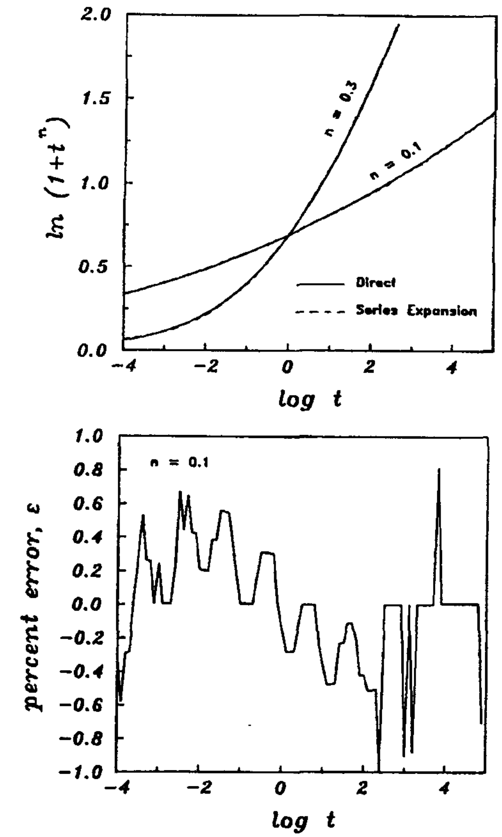

FIG. 1. Approximation by the Proposed Formula for Dirichlet Series and Errors of Approximation

Numerical studies confirmed that this approximation works very well provided that the retardation times are chosen as $\tau_\mu = 10^{\mu-2}\tau_2$ for $\mu = 2, \ldots N$, where τ_2 and N has to be chosen so as to cover the time range of interest, and $\tau_1 << \tau_2$, e.g., $\tau_1 \approx 10^{-5}\,\tau_2$. The values of $b_\mu(n)$, c, and z have been optimized by using a computer library subroutine for Levenberg-Marquardt algorithm. The results are $c = 0.146n^{-0.1}$, $b_\mu(n) = 1.1n(1 - n^3)$ for $\mu = 2, \ldots N - 1$, $b_N = 1.5n^{1.25}$, and b_1 and z are given by Table 1.

The curve ln $(1 + \xi^n)$, and its Dirichlet series approximation, are plotted in Fig. 1. The errors are almost undistinguishable graphically and, for the range $\tau_2 \le \xi \le 0.1\tau_N$, generally do not exceed $\pm 1\%$ for the values of n that can typically occur for concrete ($0.05 \le n \le 0.25$). For $n = 0.1$, the errors do not exceed $\pm 0.7\%$ of the value. If the range is increased to $0.25\tau_2 \le \xi \le 0.25\tau_N$, the maximum error increases from 0.7% to 1.15%.

The fact that A_μ is always nonnegative, according to Eq. 17, implies E_μ and η_μ to be nonnegative. This means that the thermodynamic restrictions are always satisfied.

Verification by Test Data

The present model describes the basic creep, and therefore only the creep data pertaining to concrete specimens that undergo no significant drying (at no heating or cooling) can be used to verify the present formulation. An abundance of basic creep data exist in the literature, and the best have been used (Figs. 2–13). They include the constant stress data for the Canyon Ferry Dam and Ross Dam, taken from Hanson (1953), Hanson and Harboe (1958), Rostasy et al. (1971), L'Hermite et al. (1965, 1968), and Wylfa Vessel data (Browne et al. 1975). Further, they include variable stress data from Kimishima and Kitahara (1964), Ross (1958), Polivka et al. (1964), and Mullick (1971), as well as the data of Komendant et al. (1976), and Mamillan (1959), which also include high stresses. The basic information on these data, as needed for their fitting, was previously summarized in Bažant and Panula (1978).

Optimum fits of the aforementioned data were first obtained for the present model considering not only $q_1, \ldots q_4$ but also m, n, and γ_0 as unknowns. In such a form the optimization problem is nonlinear, so a computer library subroutine for the Marquardt-Levenberg nonlinear optimization algorithm was used. After discovering that parameters m, n, and γ_0 can be fixed once for all, as indicated in Eq. 18 of Part I, the optimum fits have been obtained by linear regression with only four unknowns, $q_1, \ldots q_4$. The results are plotted in Figs. 2–6.

Fitting the present creep model to given test data requires a computer program that can calculate the response curves, such as the creep curve or the creep recovery curve, from any given values of the material parameters and the given loading. Such a program has been written using the present exponential algorithm. For the cases of constant stress, simple algebraic evaluation of strain, with the help of the approximate formula for function $Q(t,t')$, has been used in a linear regression program.

Figs. 7–12 show test data that revealed significant deviations from the principle of superposition in the service stress range and at medium stress levels. The optimum fits obtained with the present nonlinear model are shown in the figures as solid lines. The optimum values (multiplied by 10^9) of the material parameters $q_1, \ldots q_4$ for each data set, as well as the value of the coefficient of variation, ω, of the vertical deviations of the fits from the data points, are indicated in each figure. The coefficient of variation has been calculated in the same manner as in Bažant and Chern (1985), and Bažant and Panula (1979, 1980). This calculation does not use the actual data points because their unequal spacing represents a bias. Rather, the actual data points for each test are replaced by a hand-smoothed curve on which new data points are placed at regular intervals in the scale of log $(t - t')$. These smoothed uniformly spaced points are then used in the calculation of ω.

The comparisons with the constant stress data are shown in the scales of both log $(t - t')$ and log t. Each plot has the characteristic shape that is matched quite well by the present theory. It is seen that at increasing load duration, the creep curves asymptotically approach straight inclined lines, which all have the same slope, as is best apparent from the plots of strain versus log t. For short creep durations $t - t'$, the creep curves plotted in log $(t - t')$ have roughly the shape of exponentials.

The dashed curves in Figs. 7–12 for variable stress histories represent the

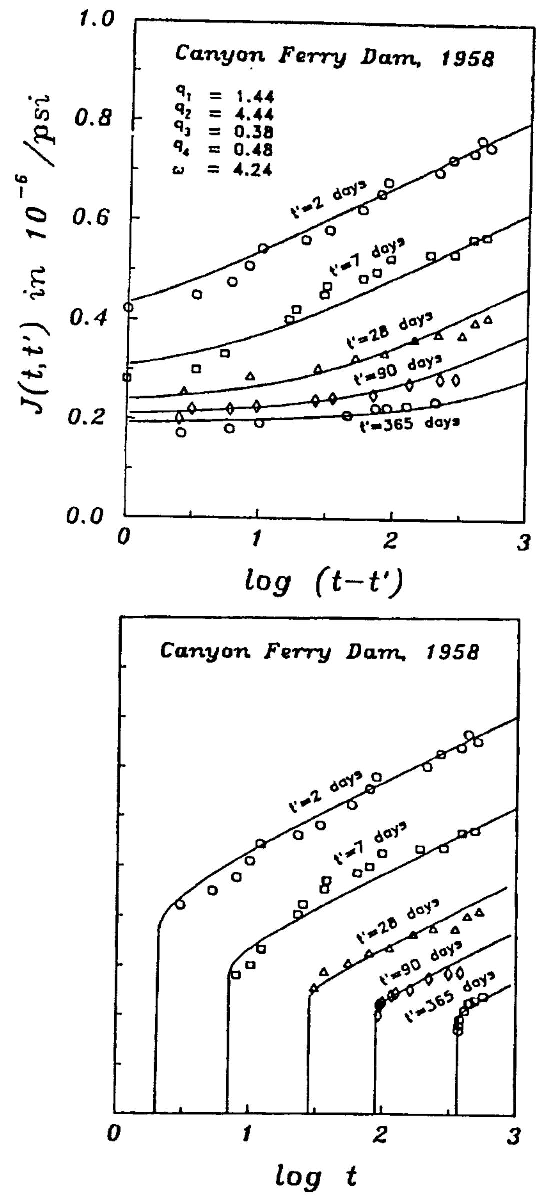

FIG. 2. Best Fit of Test Data by Hanson (1953), and Hanson and Harboe (1958) for Canyon Ferry Dam

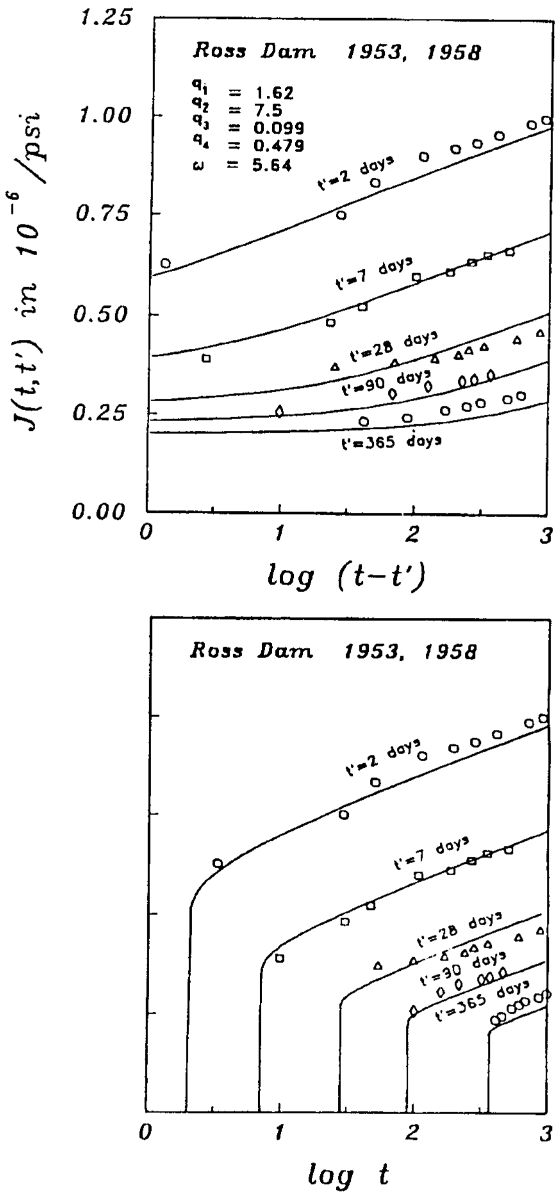

FIG. 3. Best Fit of Test Data by Hanson (1953), and Hanson and Harboe (1958) for Ross Dam

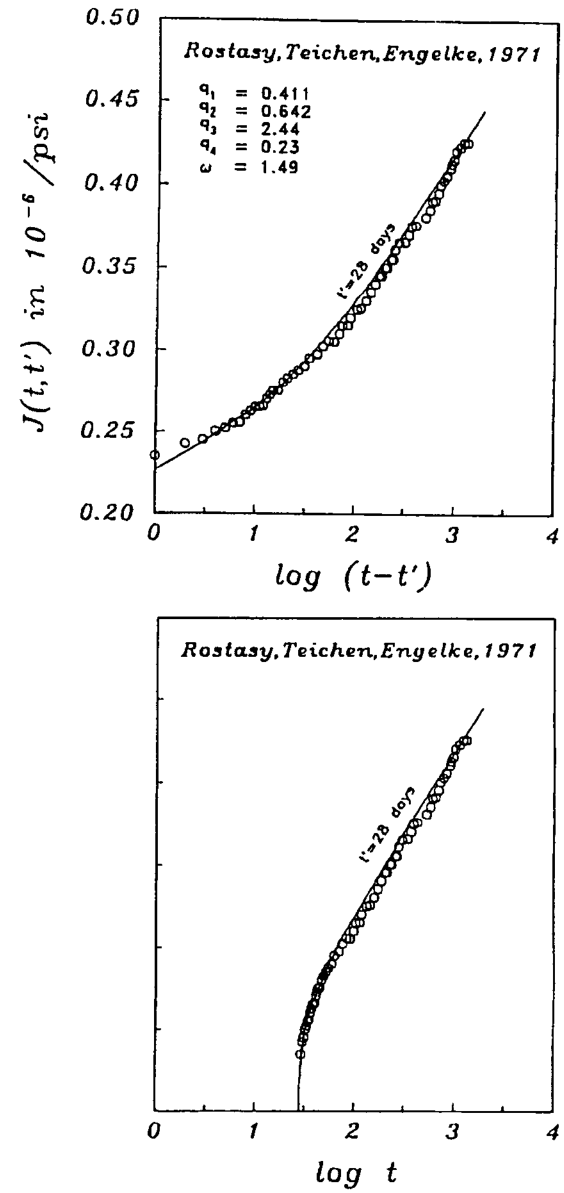

FIG. 4. Best Fit of Test Data by Rostasy et al. (1971)

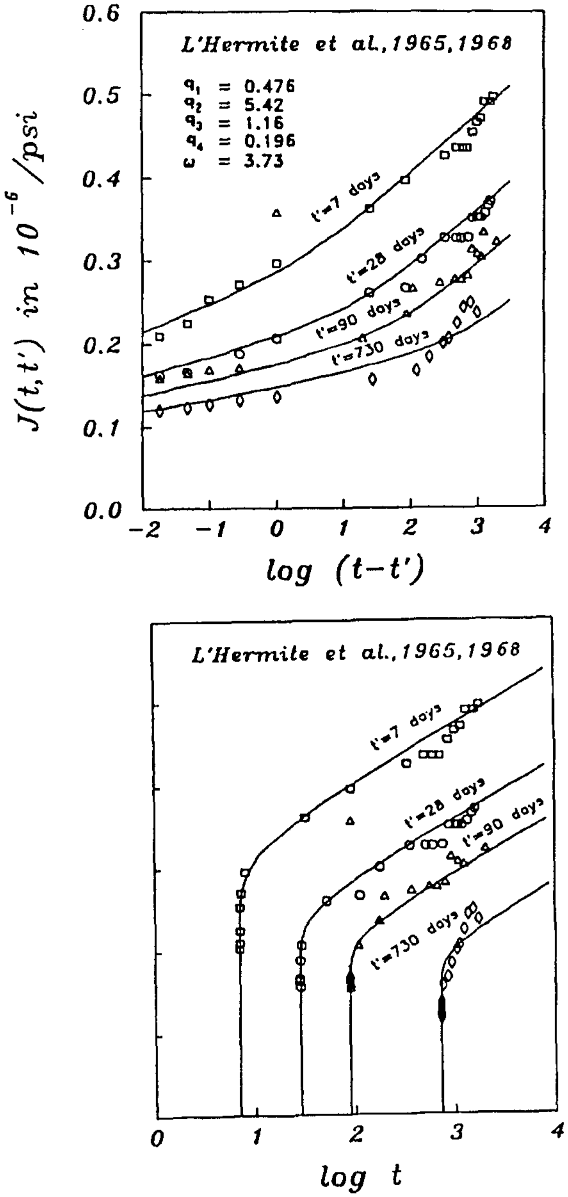

FIG. 5. Best Fit of Test Data by L'Hermite et al. (1965, 1968)

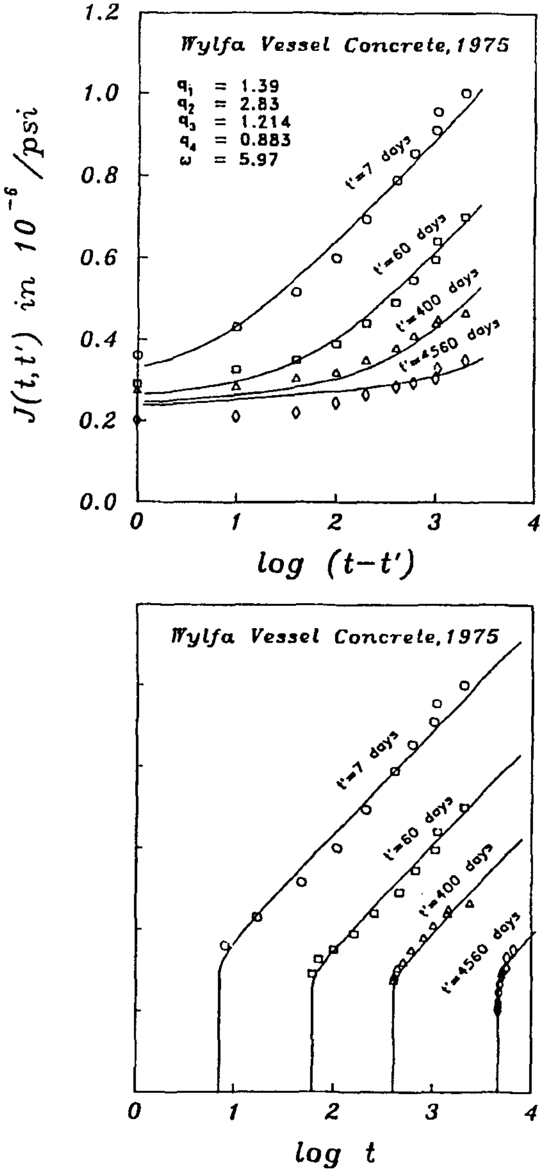

FIG. 6. Best Fit of Test Data by Browne and Bamforth (1975)

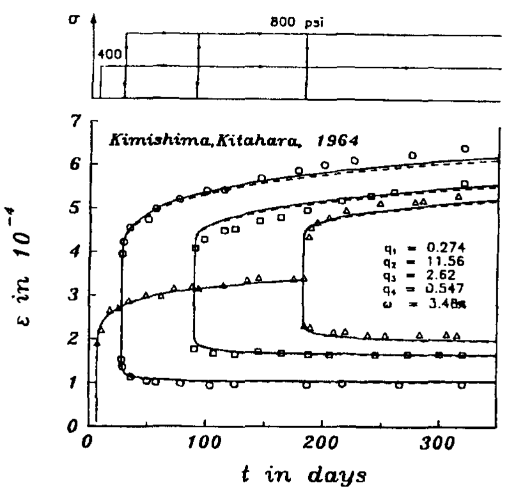

FIG. 7. Best Fit of Test Data by Kimishima and Kitahara (1964)

predictions, according to a linear version of the present theory, satisfying the principle of superposition. The formulation is linear if F is constant. Thus, the curves for the linear theory are obtained by keeping for the entire test period the value of F the same as for the first loading, all other parameters being the same as for the optimum nonlinear fit. For the first period of constant stress, the curve for the linear theory, of course, coincides with that for the nonlinear theory. A linear formulation with $F = 1$ is obtained by replacing Fq_2, Fq_3, and Fq_4 with q_2, q_3, q_4, where F is the value of $F(\sigma)$ for the initial stress.

From Figs. 7–12 we see that the present formulation agrees with test data, as closely as the previous nonlinear theories of Bažant and Asghari (1977), Bažant and Kim (1979), and Bažant, Tsubaki and Celep (1983), which were more complicated and did not represent the aging aspect as well. The nonlinear formulation generally gives a reduced creep recovery compared to the prediction from the principle of superposition, i.e., the test data and the nonlinear theory prediction lies above that for the linear theory (see Figs. 7, 8, 9, and 11). The principle of superposition overpredicts the magnitude of creep recovery. This is a generally recognized property of concrete, experimentally established already by Ross (1958). This property is clearly apparent from Figs. 8, 9, and 11. In Fig. 7, the linear and nonlinear predictions of recovery are very close, which means the response is almost linear. The reason is that in these tests the stress was very low.

For intuitive understanding of the reason for reduced recovery, consider Fig. 14, which shows the curves of creep strain (without the elastic term q_1). After the second stress change, the curves for $F = F_1$ represent the predictions of linear theory (superposition principle). For recovery, the stress is decreased ($F = 1 < F_1$), and therefore $b < a$. For the second stress increase, $F = F_2 > F_1$ and, therefore, $c > a$.

The reduced creep recovery is often regarded as one manifestation of the

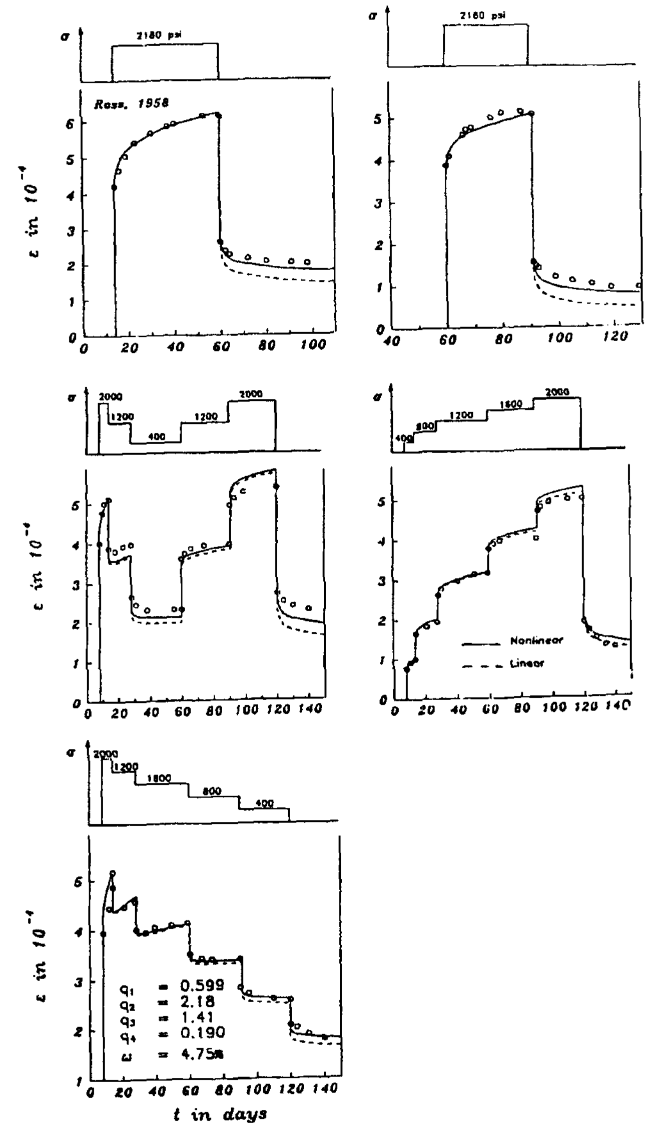

FIG. 8. Best Fit of Test Data by Ross (1958)

phenomenon of adaptation in concrete creep. In a broader sense, this phenomenon also means that after a longer period of creep under a relatively small compressive stress well within the linear range, the response to any second stress change [both a decrease (recovery) and an increase] is gen-

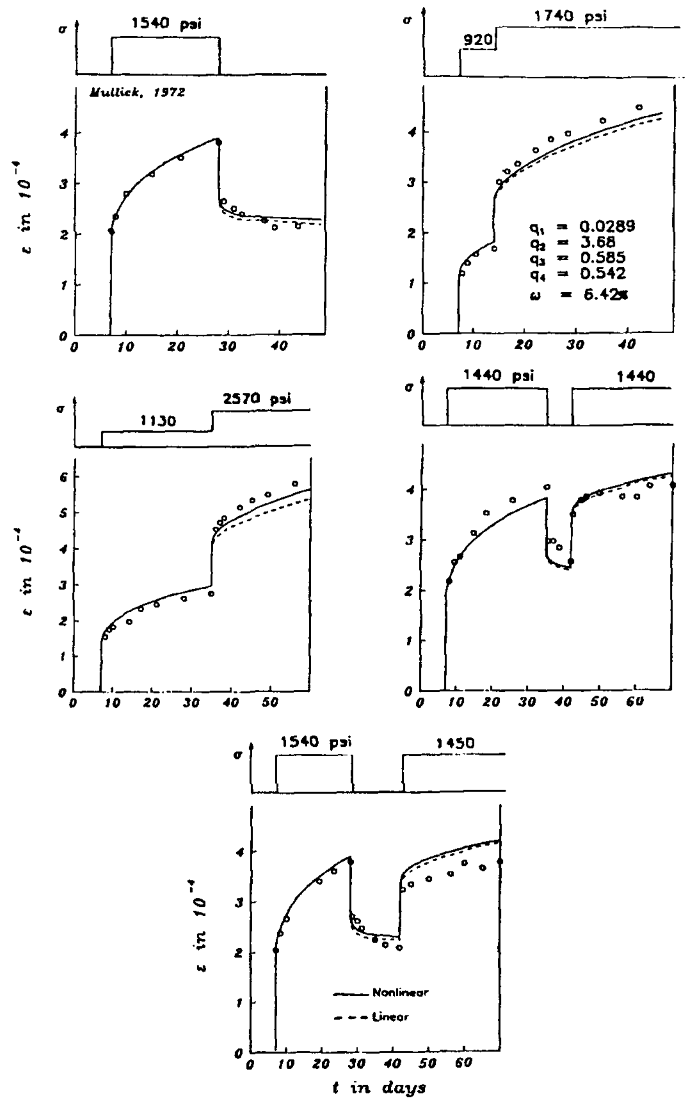

FIG. 9. Best Fit of Test Data by Mullick (1972)

erally stiffer than predicted from the principle of superposition, as if the material has "adapted" to the previous stress. The experimental support for this broader interpretation of the adaptation phenomenon is, however, ambiguous. The present theory predicts the additional creep resulting from a second stress increase will always be larger than that predicted by the prin-

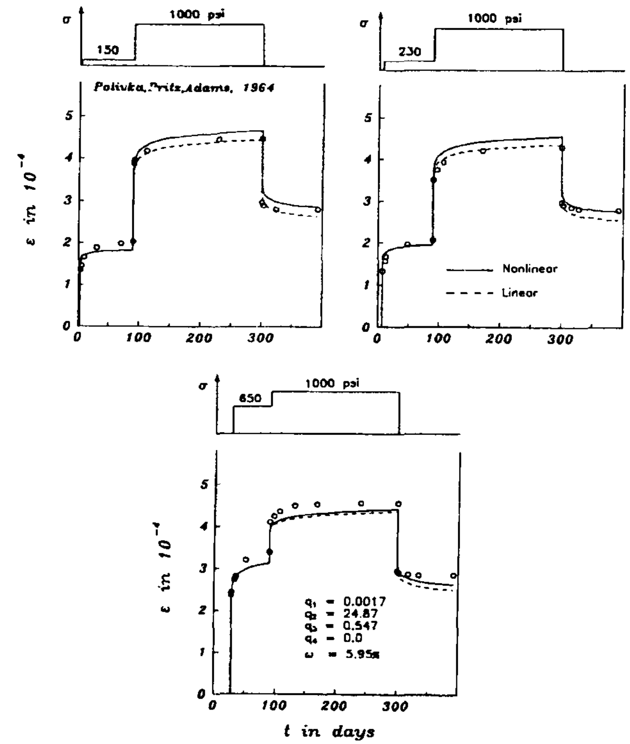

FIG. 10. Best Fit of Test Data by Polivka et al. (1964)

ciple of superposition (because the value of F increases). This is supported (or, at least, not contradicted) by the data analyzed here (see Figs. 7, 9, 10).

The relative magnitudes of various components of creep were already illustrated in Fig. 3 of Part I. It may be noticed that the flow term is insignificant for the early creep, especially when the concrete is old. It becomes dominant, however, for the long-time creep of concrete loaded at a young age.

To generalize the present formulation for creep at humidity or temperature changes, it is necessary to add not only shrinkage and thermal expansion, but also the stress-induced shrinkage and the stress-induced thermal expansion, which represent manifestations of the drying creep and the transitional thermal creep. Furthermore, η_μ must be multiplied by a factor depending on pore relative humidity. The mathematical description of these phenomena is a separate problem, and can be accomplished in the manner described before (Bažant and Chern 1985, 1987; Thelandersson 1983).

Prediction of Material Parameters and Sensitivity Analysis

By optimization of the fits of the data from Figs. 2–12, approximate empirical formulas (multiplied by 10^7) for the dependence of material param-

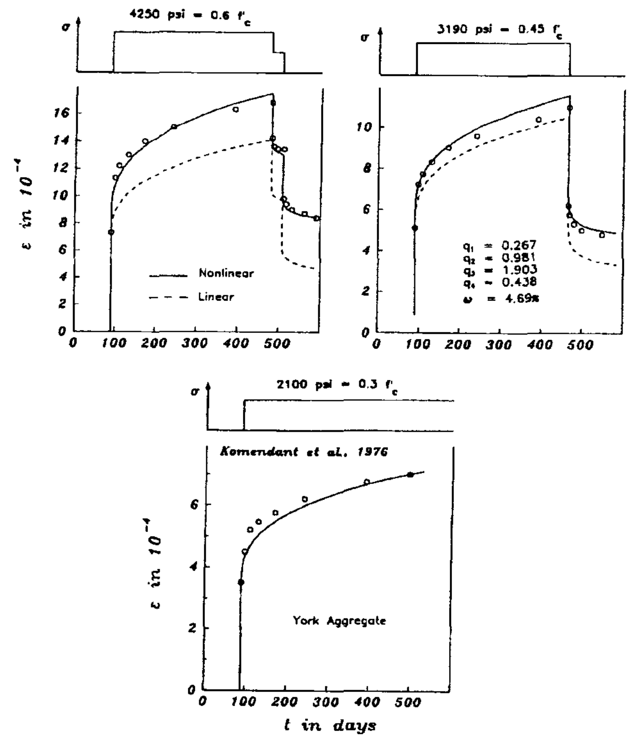

FIG. 11. Best Fit of Test Data by Kommandant (1976)

eters on the strength and composition of concrete have been found:

$$q_1 = 12.5\left(\frac{w}{c}\right)^{3.5} \quad (19)$$

$$q_2 = -22.8 + 2.5 \ln\left[\left(\frac{w}{c}\right)^5\left(\frac{a}{c}\right)f_c'^{1.5}\right] \quad (20)$$

$$q_3 = 16{,}000\left[\left(\frac{a}{c}\right)\left(\frac{w}{c}\right)^4 f_c'^{1.6}\right]^{-0.8} \quad (21)$$

$$q_4 = 0.000082\left[\left(\frac{w}{c}\right)\left(\frac{a}{s}\right)f_c'\right] \quad (22)$$

in which f_c' = 28-day cylindrical compression strength in psi (1 psi = 6,895 Pa); w/c = water-cement ratio of the mix; a/c = aggregate-cement ratio; a/s = aggregate-sand ratio; s/c = sand-cement ratio (all by weight). Sand is defined as the aggregate less than 4.7 mm in size (sieve No. 4).

The qualitative trends reflected in the preceding equations roughly agree with what is generally known or may be intuitively expected (BP Model, Bažant and Panula 1978, 1980). According to Eq. 19, the instantaneous

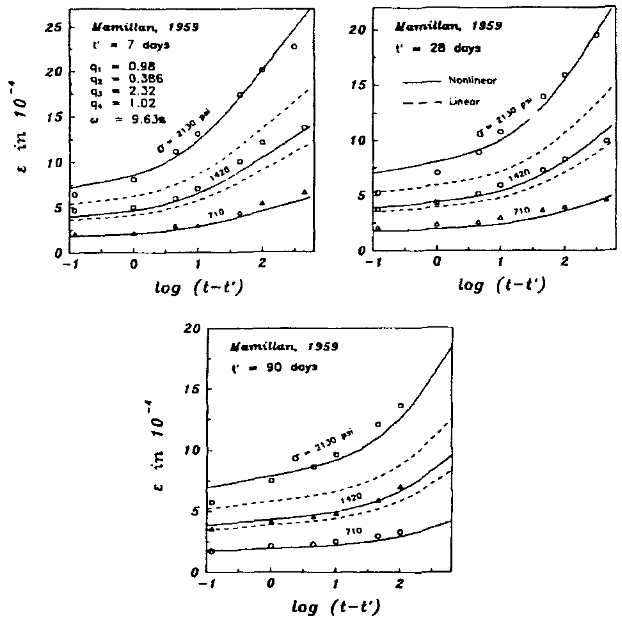

FIG. 12. Best Fit of Test Data by Mamillan (1959)

compliance parameter q_1 increases with increase in water–cement ratio. This is to be expected, however no dependence of q_1 on f_c' could be detected, contrary to the BP prediction model (Bažant and Panula 1978). Parameter q_2, which is associated with the aging viscoelastic creep and contributes more to the creep strain at small $(t - t')$, is seen to increase with an increase in water content. On the other hand, q_3, which is associated with the nonaging viscoelastic creep and affects mainly creep at large $(t - t')$, is seen to be inversely related to the water content and strength. Parameter q_4 of the viscous (flow) term, which controls mainly the long-term creep of concrete loaded at young age, increases when the water content and the aggregate-sand ratio are increased.

After obtaining the best fits and identifying the material parameters for

TABLE 2. Sensitivity Coefficients and Relative Importance of Various Parameters

Parameter (1)	ω_i^+ (2)	ω_i^- (3)	α_i (4)	β_i (5)
q_1	5.85	5.76	0.73	0.163
q_2	7.00	6.63	1.23	0.275
q_3	7.50	7.16	1.52	0.339
q_4	6.07	5.95	1.00	0.223

Note: $\bar{\omega}$ = 5.41.

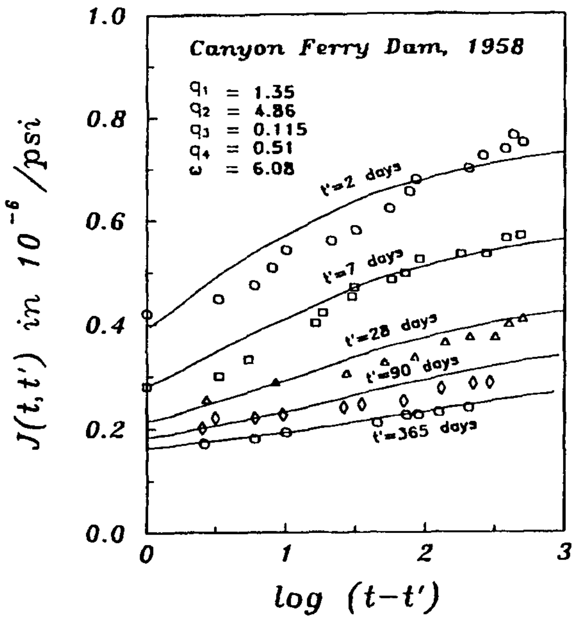

FIG. 13. **Best Possible Fit when Viscous Term (Flow) Is Omitted (q_4 = 0); Data of Hanson (1953), and Hanson and Harboe (1958)**

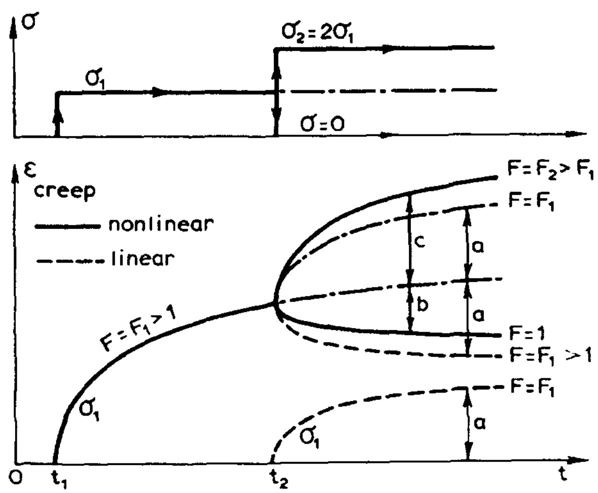

FIG. 14. Deviations from Principle of Superposition

each data set, sensitivity of creep to each parameter was analyzed, considering all the data collectively. Each parameter q_i was changed by $\pm 0.1q_i$, one by one, and for each case the changed overall coefficients of variation ω_i+ and ω_i^- (for all the data combined) were calculated. The sensitivity coefficients α_i and relative importance factors β_i were then calculated as

$$\alpha_i = \frac{1}{2}(\omega_i^+ + \omega_i^-) - \omega, \qquad \beta_i = \frac{\alpha_i}{\Sigma_j \alpha_j} \quad \text{(23)}$$

where ω = overall coefficient of variation of errors of data fits for the optimum values of $q_1, \ldots q_4$. The results are shown in Table 2. We see that all the four parameters have about equal importance, which is a desirable feature of a good model. Therefore, if any one of the terms q_2, q_3, q_4 is omitted, the best possible fits become distinctly worse. That this is true for the flow term is documented by the best possible fits without q_4 shown in Fig. 13.

Conclusions

1. Micromechanics of the solidification process make it possible to determine a rational form of the effect of aging on concrete creep. A relatively simple form is obtained under the hypothesis that aging is entirely due to the growth of the volume fraction of the load-bearing solidifying matter (hydrated cement), which itself is nonaging.
2. By separating aging in the form of volume growth from the viscoelastic behavior, it is possible to describe creep behavior by a Kelvin chain with age-independent properties. This brings about considerable simplification compared to the previous widely used models with age-dependent properties.
3. The creep strain may be regarded as a sum of aging and nonaging viscoelastic strains and aging viscous strain (flow).
4. For the present formulation, the elastic moduli and viscosities of the Kelvin chain are always nonnegative, which assures the thermodynamic restrictions to be always satisfied.
5. The age-independence of Kelvin chain properties leads to a simple numerical incremental algorithm for creep analysis of structures, which is of exponential type and is unconditionally stable.
6. The creep curves predicted by the present model exhibit no divergence. The consequence is that the principle of superposition always yields monotonic recovery curves.
7. Nonlinearity may be introduced by modifying the creep rate as a function of the current stress. This formulation causes deviations from the superposition principle to accumulate gradually during creep. Thus, the long-time response deviates from linearity more than the short-time response, as indicated by test results.
8. Contrary to various previous creep laws, the present formulation can be cast in a form in which identification of material parameters from test data can be accomplished by linear regression, while for previous formulations nonlinear optimization was necessary.
9. Despite being simpler, the present solidification theory for creep describes the existing test results better and over a broader range of conditions than the previous formulations.

Acknowledgment

Partial financial support of this study (both Parts I and II) was provided by NSF under Grant FED7400 to Northwestern University, and in the final phase by Center for Advanced Cement-Based Materials at Northwestern University (NSF grant DMR-8808432). Mary Hill deserves thanks for her superb secretarial assistance.

Appendix. References

Anderson, C. A. (1982a). "Numerical creep analysis of structures." *Creep and Shrinkage in Concrete Structures*, Chap. 8, Z. P. Bažant and F. H. Wittmann, eds., John Wiley & Sons, Inc., New York, N.Y., 297–301.

Anderson, C. A., Smith, P. D., and Carruthers, L. M. (1982b). "NONSAP-C: A nonlinear strain analysis program for concrete containments under static, dynamic and long-term loadings." *Report NUREG/CR0416, LA-7496-MS, Rev. 1, R7 and R8*, Los Alamos Nat. Lab., Los Alamos, N.M.

Bažant, Z. P. (1971). "Numerically stable algorithm with increasing time steps for integral-type aging concrete." *Proc., First Int. Conf. on Structural Mechanics in Reactor Technology*, 3, H2/3, Berlin, W. Germany.

Bažant, Z. P., and Wu, S. T. (1973). "Dirichlet series creep function for aging concrete." *J. Engrg. Mech.*, ASCE, 99, 367–387.

Bažant, Z. P. (1975). "Theory of creep and shrinkage in concrete structures: A precis of recent developments." *Mechanics Today*, 2, Pergamon Press, New York, N.Y., 1–93.

Bažant, Z. P. (1977). "Viscoelasticity of a solidifying porous material—Concrete." *J. Engrg. Mech.*, ASCE, 1049–1067.

Bažant, Z. P., and Asghari, A. A. (1977). "Constitutive law for nonlinear creep of concrete." *J. Engrg. Mech.*, ASCE, 103(1), 113–124.

Bažant, Z. P., and Panula, L. (1978). "Practical prediction of time-dependent deformations of concrete." *Materials and Structures*, RILEM, Paris, France, 11(65), 307–328; (66) 415–434; 12(69), 169–183.

Bažant, Z. P., and Kim, S. S. (1979). "Nonlinear creep of concrete—adaptation and flow." *J. Engrg. Mech.*, ASCE, 105, 429–446.

Bažant, Z. P., and Panula, L. (1980). "Creep and shrinkage characterization for analyzing prestressed concrete structures." *J. Prestressed Concr. Inst.*, 25(3), 86–122.

Bažant, Z. P. (1982). "Mathematical models for creep and shrinkage of concrete." *Creep and Shrinkage in Concrete Structures*, Chap. 7, Z. P. Bažant and F. H. Wittmann, eds., John Wiley & Sons, Inc., New York, N.Y., 163–256.

Bažant, Z. P., Tsubaki, T., and Celep, Z. (1983). "Singular history integral for creep rate of concrete." *J. Engrg. Mech.*, ASCE, 109, 866–884.

Bažant, Z. P., and Chern, J. C. (1985). "Concrete creep at variable humidity." *Materials and Structures*, RILEM, Paris, France, 18(103), 1–20.

Bažant, Z. P., and Chern, J. C. (1987). "Stress-induced thermal and shrinkage strains in concrete." *J. Engrg. Mech.*, ASCE, 113, 1493–1511.

Browne, R. D., and Bamforth, P. P. (1975). "The long-term creep of the Wylfa P. V. concrete for loading ages up to 12-1/2 years." *3rd Int. Conf. on Structural Mechanics in Reactor Technology*, H1/8, London, England.

Carol, I., and Murcia, J. (1986). "An incremental model for nonlinear time-dependent behavior of concrete in compression." *4th Int. Symp. on Creep and Shrinkage of Concrete: Mathematical Modeling*, RILEM, Paris, France, 797–805.

Chern, J. C., and Wu, Y. G. (1986). "Rheological model with strain-softening and exponential algorithm for structural analysis." *4th RILEM Int. Symp. on Creep and Shrinkage of Concrete: Mathematical Modeling*, Northwestern Univ., Evanston, Ill., Z. P. Bažant, ed., 591–600.

Espion, B. (1986). "Application of some time-dependent prediction models to the analysis of structural elements under sustained load." *4th RILEM Int. Symp. on Creep and Shrinkage of Concrete: Mathematical Modeling*, Northwestern Univ., Evanston, Ill., Z. P. Bažant, ed., 609–622.

Ha, H., Osman, M. A., and Huterer, J. (1984). "User's guide: Program for predicting shrinkage and creep in concrete structures." *File: NK38-26191P*, Ontario Hydro, Toronto, Ontario, Canada.

Hanson, J. (1953). "A ten-year study of creep properties of concrete." *Conc. Lab. Report No. SP-38*, U.S. Dept. of the Interior, Bureau of Reclamation, Denver, Colo.

Harboe, E. M., et al. (1958). "A comparison of the instantaneous and sustained modulus of elasticity of concrete." *Conc. Lab. Report No. C-854*, U.S. Dept. of the Interior, Bureau of Reclamation, Denver, Colorado.

Huterer, J., Brown, D. G., and Yanchula, S. (1985). "Arlington GS vacuum building." *Nucl. Eng. Des.*, 85, 119–140.

Jonasson, J. E. (1978). "Analysis of creep and shrinkage in concrete and its application to concrete top layers." *Cem. Concr. Res.*, 8, 397–418.

Kang, Y. J., and Scordelis, A. C. (1980). "Nonlinear analysis of prestressed concrete frames." *J. Struct. Engrg.*, ASCE, 106(2), 445–462.

Kimishima, H., and Kitahara, H. (1964). "Creep and creep recovery of mass concrete." *Tech. Report C-64001*, Central Research Institute of Electric Power Industry, Tokyo, Japan.

Komendant, G. J., Polivka, M., and Pirtz, D. (1976). "Study of concrete properties for prestressed concrete reactor vessels; part 2, creep and strength characteristics of concrete at elevated temperatures." *Report No. UC SESM 76-3*, Structures and Materials Research, Dept. of Civ. Engrg., Univ. of California, Berkeley, California.

L'Hermite, R., Mamillan, M., and Lefèvre, C. (1965). "Nouveaux resultats de recherches sur la deformation et al rupture du beton." *Annales de l'Institut Techn. du Batiment et des Travaux Publics*, 18(207-8), 325–260.

L'Hermite, R. G., and Mamillan, M. (1968a). "Further results of shrinkage and creep tests." *Proc., Int. Conf. on the Structure of Concrete*, Cement and Concrete Assoc., London, England, 423–433.

Mamillan, M. (1959). "A study of the creep of concrete." *Bull. No. 3*, RILEM, Paris, France, 15–33.

"Mathematical modeling of creep and shrinkage of concrete." (1986). "State-of-Art Report," *4th RILEM Int. Symp. on Creep and Shrinkage of Concrete: Mathematical Modeling*, Northwestern Univ., Evanston, Ill., Z. P. Bažant, ed., Technical Committee TC69, 39–455.

Mullick, A. K. (1972). "Effect of stress-history on the microstructure and creep properties of maturing concrete," thesis presented to the University of Calgary, Alberta, Canada, in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

Ožbolt, J. (1986). "Time-history analysis of reinforced concrete frames." RILEM Symp., 665–674.

Polivka, M., Pirtz, D., and Adams, R. F. (1964). "Studies of creep in mass concrete." *Proc., Symp. on Mass Concrete*, Am. Conc. Inst., 257–285.

Ross, A. D. (1958). "Creep of concrete under variable stress." *Proc., J. Am. Conc. Inst.*, 54, 739–758.

Rostasy, F. S., Teichen, K.-Th., Engelke, H. (1972). "Beitrag zur Klarung des Zusammenhanges von Kriechen und Relaxation bei Normal-beton. *Amtliche Forschungs-und Materialprufungsanstalt fur das Bauwesen.* Otto-Graf-Institut, Universitat Stuttgart, Strassenbau und Strassenverkehrstechnik, Heft 139, Stuttgart, Germany.

Taylor, R. L., Pister, K. S., and Goudreau, G. L. (1970). "Thermomechanical analysis of viscoelastic solids." *Int. J. Numer. Methods Engrg.*, 2, 45–60.

Thelandersson, S. (1983). "On the multiaxial behavior of concrete exposed to high temperature." *Nucl. Eng. Des.*, 75(2), 271–282.

Young, J. F. et al. (1986). "Physical mechanisms and their mathematical interpretations." *4th RILEM Int. Symp. on Creep and Shrinkage of Concrete: Mathematical Modeling*, Northwestern Univ., Evanston, Ill., Z. P. Bažant, ed., 41–80.

Zienkiewicz, O. C., Watson, M., and King, I. P. (1968). "A numerical method of viscoelastic stress analysis." *Int. J. Mech. Sci.*, 10, 807–837.