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Scaling of quasibrittle fracture: hypotheses of invasive and lacunar fractality, their critique and Weibull connection

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Abstract. Considerable progress has been achieved in fractal characterization of the properties of crack surfaces in quasibrittle materials such as concrete, rock, ice, ceramics and composites. Recently, fractality of cracks or microcracks was proposed as the explanation of the observed size effect on the nominal strength of structures. This explanation, though, has rested merely on intuitive analogy and geometric reasoning, and did not take into account the mechanics of crack propagation. In this paper, the energy-based asymptotic analysis of scaling presented in the preceding companion paper in this issue [1] is extended to the effect of fractality on scaling. First, attention is focused on the propagation of fractal crack curves (invasive fractals). The modifications of the scaling law caused by crack fractality are derived, both for quasibrittle failures after large stable crack growth and for failures at the initiation of a fractal crack in the boundary layer near the surface. Second, attention is focused on discrete fractal distribution of microcracks (lacunar fractals), which is shown to lead to an analogy with Weibull's statistical theory of size effect due to material strength randomness. The predictions ensuing from the fractal hypothesis, either invasive or lacunar, disagree with the experimentally confirmed asymptotic characteristics of the size effect in quasibrittle structures. It is also pointed out that considering the crack curve as a self-similar fractal conflicts with kinematics. This can be remedied by considering the crack to be an affine fractal. It is concluded that the fractal characteristics of either the fracture surface or the microcracking at the fracture front cannot have a significant influence on the law of scaling of failure loads, although they can affect the fracture characteristics.

Key words: Quasibrittle materials, scaling, size effect, fracture mechanics, fractal cracks, invasive fractals, lacunar fractals, fracture energy release, asymptotic analysis.

1. Introduction

Observations have shown that, within a certain range of scales, the fracture surfaces in many materials, especially brittle heterogeneous materials such as rock, concrete, ice, tough ceramics and various composites, exhibit partly fractal characteristics. Considerable advances in the study of the fractal aspects of crack morphology and energy dissipation by fractal cracks have already been made, and a correlation between the fractal dimension of the crack surface (observed over a limited range of scales) and the fracture energy or toughness of some brittle materials has been detected [2–31, 52].

Recently, it has been suggested that the fractal nature of crack surfaces might be the cause of the observed size effect on nominal strength of concrete structures [21, 24–30]. However, the connection between the fractal nature of cracks on the microscale and the scaling law on the macroscale has so far been based merely on intuitive analogy and geometric arguments. It has not been solidly established in terms of mechanics. This connection ought to be deduced by global energy release analysis and asymptotic matching [32, 33]. The general size effect of fractal fracture on the nominal strength of geometrically similar structures will be derived. An alternative size effect formulation based on the hypothesis that the microcrack array represents lacunar fractals will also be derived and its connection to Weibull-type statistical theory of size effect will be demonstrated. Comparisons of the predictions resulting from the fractal hypothesis to the typical experimentally observed size effect trends will allow an appraisal of the hypothesis.

The present analysis based on report [34], various aspects of which have been summarized at three recent conferences [35–37], will use the asymptotic and energetic approaches developed in the preceding companion paper [1]. The same notations will be used. For details of the method and the background, as well as for references to the test results and the nonfractal studies of the size effect, the preceding paper may be consulted.

A simplified fractal theory based on an approximate description of load sharing in a three-dimensional stack of cubes has been proposed for fracture comminution, with application to crushing of a sea ice plate pressing on an oil platform [38]. Another fractal theory for crushing of ice floes, based on a one-dimensional load-sharing model, was proposed by Bhat [39]. These simplified but practically useful theories are not based on fracture mechanics and are beyond the scope of the present paper.

2. Invasive and lacunar fractals

In two-dimensional fracture analysis, a crack in the Euclidean space is one-dimensional, i.e., it is a curve, which has fractal dimension $d_f = 1$ in a space of Hausdorff dimension $\mathcal{M} = 1$. In two-dimensional fracture analysis, the fractal generalization consists in considering the crack to be a (non-Euclidean) fractal curve having fractal dimension $d_f \neq 1$ in a space of Hausdorff dimension $\mathcal{M} \neq 1$. In three-dimensional fracture analysis, a smooth crack in the Euclidean space represents a two-dimensional surface, and its fractal generalization is a fractal surface of fractal dimension $d_f \neq 2$ in a space of Hausdorff dimension $\mathcal{M} \neq 2$. Our analysis will be confined to the two-dimensional treatment of fracture propagation, although generalization to three dimensions is possible without affecting the basic nature of the conclusions.

In previous studies, the fractal curve of a crack has been considered as a self-similar fractal curve, arising from disturbances of line segments that are self-similar when the scale is reduced. Such a curve has no finite length. The measured length of such a fractal curve, a_δ, depends on the length δ_0 of the ruler by which it is measured (Figure (1a)). For small enough δ_0 or large enough a_δ, the 'smooth' (or projected, Euclidean) length of the fractal curve is

$$a_\delta = \delta_0 (a/\delta_0)^{d_f}. \tag{1}$$

Here we introduced δ_0 as the lower limit of fractality of the crack curve, which is in practice always implied by the microstructure of the material and cannot be less than the atomic spacing. The value of δ_0 need not be known for the practical analysis of size effects in structures. Suffice to say, it may be a very small constant, much smaller then any crack length of practical interest.

There are two types of fractals which could conceivably influence fracture scaling:

(1) the invasive (densifying) fractals, for which $d_f > 1$ (Figure 1(a)), and (2) the lacunar (rarefying) fractals, for which $d_f < 1$ (Figure 1(d)).

The invasive fractals give a continuous curve (Figure 1(a)), whereas the lacunar fractals give a discrete set (Cantor set) of line segments corresponding to a row of microcracks (Figure 1(d)). When the so-called 'multifractal' scaling law (MFSL) was first proposed [21], the argument referred to the self-similar invasive fractals exemplified by the von Koch curve.

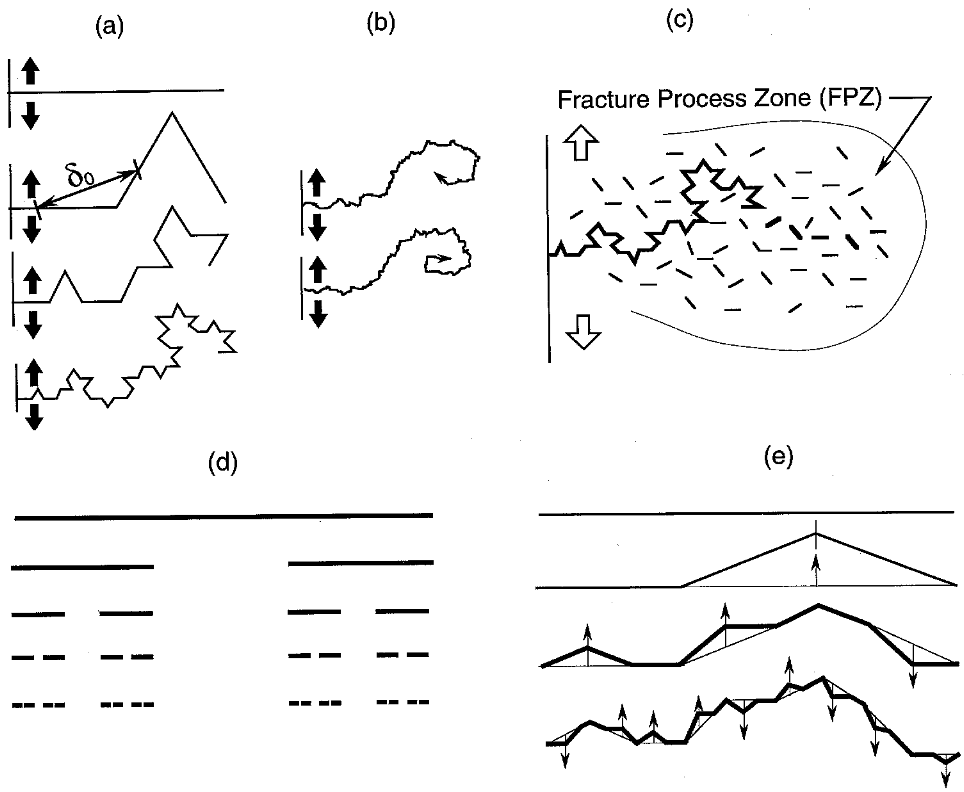

Figure 1. (a) Von Koch curves as examples of continuous fractal cracks (invasive or densifying fractals), at progressive refinement; (b) recession and spiraling which can be exhibited by fractal curves such as von Koch's; (c) continuous fractal crack forming in a fracture process zone; (d) row of discrete microcracks as lacunar (rarefying) fractals; (e) fractal curve created by displacements normal to overall fracture direction.

After the discussions at the 1994 IUTAM Symposium in Torino [35], the argument for MFSL was changed to the lacunar fractals [30], although, curiously, the concept of invasive fractals [29] has been retained for the explanation of the structure size effect on the apparent fracture energy. The consequences of the invasive fractal hypothesis will be studied first, and those of the lacunar fractal hypothesis second.

3. Dissipation of energy by a fractal crack

Due to progressive self-similar refinements or disturbances, the actual length of a fractal curve between two points a finite distance apart is infinite. This makes it impossible to use the classical definition of fracture energy, which consists of a finite energy dissipation per actual unit crack length and width and, therefore, would give an infinite energy dissipation by a fractal crack connecting two points a finite distance apart. To avoid this problem and make the fractal concept of crack propagation feasible, Mosolov and Borodich [17, 18] proposed a new unconventional definition of fractal fracture energy with a different physical dimension. They expressed the energy $\mathcal{W}_f$ dissipated by a fractal crack in a two-dimensional body of thickness b as follows: as

$$\mathcal{W}_f/b = G_{fl}a^{d_f}, \quad a < \delta_1, \tag{2}$$

where G_{fl} = fractal fracture energy, the metric dimension of which is $\mathrm{J\,m}^{-d_f-1}$ [17, 18], and constant δ_1 gives an upper limit of fractality. For $\delta_1 \to \infty$, the range of fractality would be unlimited. But this can never be the case in reality, i.e., δ_1 is finite and represents the crack length that is sufficiently larger than the maximum size d_a of material inhomogeneities (such as the maximum aggregate size in concrete or the maximum grain size in rock or ceramic). For larger scales, the energy dissipation is defined by the standard relation

$$\mathcal{W}_f/b = G_f a, \quad a \geqslant \delta_1, \tag{3}$$

where G_f = standard (or macroscopic, large-scale) fracture energy (the dimension of which is $\mathrm{J\,m}^{-2}$). Setting $G_{fl}a^{d_f} = G_f a$, we have $\delta_1 = (G_f/G_{fl})^{1/d_f}$.

In contrast to nonfractal cracks, we avoid introducing the stress intensity factor and the fracture toughness because the near-tip stress field of a fractal crack is unknown and nonexistent in the deterministic sense. The energy release rate and fracture energy, however, are concepts that a sound fractal model must accept.

4. Energy analysis

We adopt the following three hypotheses, of which the first two have been standard in the study of fractal cracks

(1) Within a certain range of sufficiently small scales, the failure is caused by the propagation of a single crack representing a single fractal curve.
(2) The fractal fracture energy, as defined by (2), is a material constant correctly describing energy dissipation [17, 18].
(3) The material may (although need not) possess a material length, c_f.

The rate of energy dissipation in the structure as a whole must be defined with respect to the 'smooth' (or projected) crack length a. Differentiating (2) and (3) with respect to a, we obtain the following dependence of the macroscopic energy dissipation rate $\mathcal{G}_{\mathrm{cr}}$ on the smooth crack length a

$$\mathcal{G}_{\mathrm{cr}} = \frac{1}{b}\frac{\partial \mathcal{W}_f}{\partial a} = \mathrm{Min}(G_{fl}d_f a^{d_f-1}, \quad G_f). \tag{4}$$

Setting both expressions in the parenthesis equal, we see that the transition from the first to the second expression occurs at

$$a_f = (G_f/G_{fl}d_f)^{1/(d_f-1)}, \tag{5}$$

which is different from δ_1. We accept this difference because $\mathcal{G}_{\mathrm{cr}}$ as a function of a would otherwise exhibit a jump at the transition from the fractal to the nonfractal regime (as shown for $a = \delta_1$ in Figure 2). Such a jump is physically inadmissible.

To characterize the size effect in geometrically similar structures of different sizes D (characteristic dimensions), we introduce, as in [1], the nominal stress $\sigma_N = P/bD$ where D = characteristic size (dimension) of the structure, P = applied load (or load parameter), and b = structure thickness in the third dimension (we restrict attention to two-dimensional similarity, although generalization of the present analysis to three-dimensional similarity would be easy). When $P = P_{\max}$ = maximum load, σ_N is called the nominal strength of the structure. The load is considered to be a dead load (i.e., independent of displacement).

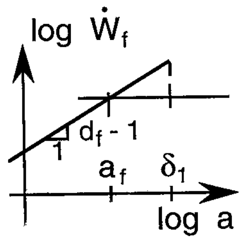

Figure 2. Variation of fractal and nonfractal energy dissipation rates at crack front.

According to Hypothesis 3 and similar to [1], we consider that in general there may be a material property having the dimension of length and called the material length c_f. This length governs the projected (smooth, Euclidean) size of the fractal fracture process zone. The special case when there is no material length will, of course, be included in our analysis as the limiting case for $c_f \to 0$. Based on the current understanding of the fracture process in quasibrittle materials, we must admit that the tip of the fractal crack may be preceded by microcracks and frictional slip planes, which may be fractal (Figure 1(c)). However, because of the way the energy dissipation is described in (2), the fractal fracture concept allows considering only fractal microcracks that form along the future path of the fractal crack. The fractal crack propagation concept cannot treat the case in which microcracks (whether fractal or nonfractal) would also form on the side of the future fractal crack path.

Same as [1], we have two basic variables, a and c_f, both having the dimension of Euclidean length. They must appear in the energy release expression nondimensionally. We again choose the dimensionless variables

$$\alpha = a/D \qquad \theta = c_f/D. \tag{6}$$

According to Buckingham's theorem of dimensional analysis, the complementary energy Π^* (representing, under isothermal conditions, the Gibbs free energy) must be a function of these dimensionless variables even for a fractal crack. Similar to [1]

$$\Pi^* = \frac{\sigma_N^2}{E} bD^2 f(\alpha, \theta), \tag{7}$$

in which f is a dimensionless continuous function of α and θ.

As pointed out in [1], the material length c_f need not be specified directly but is anyway implied if the material breakup in the fracture process zone is assumed to be governed by continuum damage mechanics. In that theory, the material failure is characterized in terms of a critical damage energy release rate W_d per unit volume of material. In fractal fracture mechanics, by contrast, the material failure is characterized by critical energy dissipation G_{fl} per unit fractal surface area. A quasibrittle material possesses both characteristics. So, instead of Hypothesis 3, one may assume that failure is governed by both G_{fl} and W_d. However, such a hypothesis is equivalent. Indeed, instead of (6), we may define

$$\theta = \frac{1}{D}\left(\frac{G_{fl}}{W_d}\right)^{1/(2-d_f)} \tag{8}$$

but this can again be written as $\theta = c_f/D$, where $c_f = (G_{fl}/W_d)^{1/(2-d_f)}$. Thus, a material length emerges in the formulation anyway.

Nonlinear fracture mechanics, too, implies material length c_f, as mentioned in [1]. In that theory, fracture propagation is governed not only by fracture energy, taken here as G_{fl}, but also by tensile strength f'_t. Instead of (6), the dimensionless parameter θ may then be defined as

$$\theta = \frac{1}{D}\left(\frac{EG_{fl}}{f_t'^2}\right)^{1/(2-d_f)}. \tag{9}$$

This can again be written as $\theta = c_f/D$ where $c_f = (EG_{fl}/f_t'^2)^{1/(2-d_f)}$.

Similar to [1], the first step is to impose the energy balance condition for crack propagation. In this regard, a point to note is that the energy release from the structure as a whole must be calculated on the basis of the 'smooth' (or projected, Euclidean) crack length rather than the fractal crack length a_δ. Indeed, this is the length that matters for the elastic stress field on the macroscale. Therefore,

$$\left[\frac{\partial \Pi^*}{\partial a}\right]_{\sigma_N} = \frac{\partial \mathcal{W}_f}{\partial a}. \tag{10}$$

Upon substitution of (7) and differentiation,

$$\frac{\sigma_N^2}{E} D g(\alpha, \theta) = \mathcal{G}_{\text{cr}}. \tag{11}$$

Here

$$g(\alpha, \theta) = \frac{\partial f(\alpha, \theta)}{\partial \alpha}, \tag{12}$$

where g is the dimensionless energy release rate function. For the special case $c_f \to 0$, the form of this function coincides with function $g(\alpha)$ of linear elastic fracture mechanics (LEFM). Same as f, function $g(\alpha, \theta)$ reflects the geometry of the structure, crack and load, but is independent of D.

Same as in [1], we now need to introduce the crack length $a = a_m$ at the maximum load, which represents the stability limit because we consider the loading under load control conditions ([40], Chapter 10). If the fractal crack curve has a perfectly sharp tip and if we restrict attention to the so-called positive structure geometries, which is the usual case for which the energy release curve of $\partial\Pi^*/\partial a$ versus a at constant load is rising, then $a_m = a_0 =$ initial traction-free crack length. For quasibrittle materials, however, $a_m > a_0$ because of the R-curve behavior associated with the existence of a large fracture process zone or material length, c_f. In (12) of the preceding companion paper [1], it was shown that the value of $\alpha_m = a_m/D$ is a function of α_0 and θ, i.e., $\alpha_m = \alpha_m(\alpha_0, \theta)$. This was deduced from the condition that, at maximum load, the energy release curve must be tangent to the R-curve, and alternatively in Appendix I also from the maximum load condition of the cohesive crack model. By analogy, one could now introduce a fractal generalization of the R-curve model or of the cohesive crack model to show that for fractal fracture, too, the maximum load occurs at

$$\alpha = \alpha_m(\alpha_0, \theta). \tag{13}$$

The corresponding crack length at maximum load, $a_m = \alpha_m D$, increases with D at constant α_0 and for $D \to \infty$ tends to a finite asymptotic value, $\lim a_m = a_0 + c_f$. The case of fractal generalization of linear elastic fracture mechanics (LEFM), which applies to fractal cracks with a perfectly sharp tip and no microcracks ahead of the tip, is obtained for $\alpha_m = \alpha_0$.

Substituting (13) in (11), and replacing also G_{fl} in (4) for $\mathcal{G}_{\text{cr}}$ with an R-curve, one may obtain from (11) a conclusion analogous to (14) of the preceding companion paper [1]

$$\sigma_N = \sqrt{\frac{E\mathcal{G}_{\text{cr}}}{D\hat{g}(\alpha_0, \theta)}}, \tag{14}$$

where α_0 denotes the value of the relative length of the initial traction-free crack, and $\hat{g}$ is a function of two variables. If the fractal crack displays no R-curve behavior, $\hat{g}(\alpha_0, \theta) = g[\alpha_m(\alpha_0, \theta), \theta]$. For $D \to \infty$, $\lim \hat{g}(\alpha_0, \theta) = g(\alpha_0, 0)$, which coincides with the usual dimensionless LEFM energy release rate $g(\alpha_0)$ as a function of one variable.

Because, same as in [1], $\hat{g}(\alpha_0, \theta)$ ought to be a smooth function, we may expand it into Taylor series about the point $(\alpha, \theta) \equiv (\alpha_0, 0)$. Equation (11) thus furnishes

$$\frac{\sigma_N^2}{E} D \left[\hat{g}(\alpha_0, 0) + g_1(\alpha_0, 0) \frac{c_f}{D} + \frac{1}{2!} g_2(\alpha_0, 0) \left(\frac{c_f}{D} \right)^2 + \cdots \right] = \mathcal{G}_{\text{cr}}, \tag{15}$$

where $g_1(\alpha_0, 0) = \partial\hat{g}(\alpha_0, \theta)/\partial\theta, g_2(\alpha_0, 0) = \partial^2\hat{g}(\alpha_0, \theta)/\partial\theta^2, \ldots$, all evaluated at $\theta = 0$. As in [1], we have thus acquired the large-size asymptotic series expansion of size effect. To obtain a simplified approximation, we may truncate the series after the linear term. Then, introducing the notations

$$k = \frac{\hat{g}(\alpha_0, 0)}{\alpha_0}, \qquad c_0 = c_f \alpha_0 \frac{g_1(\alpha_0, 0)}{\hat{g}(\alpha_0, 0)}, \tag{16}$$

we get the equation

$$(\sigma_N^2/E) k (a + c_0) = \mathcal{G}_{\text{cr}}. \tag{17}$$

Here we may substitute (4) and set $a = \alpha D$. Thus, solving this equation for σ_N, we conclude that the law of the size effect of fractal fracture is

$$\sigma_N = \text{Min} \left\{ \sigma_N^0 D^{(d_f - 1)/2} \left(1 + \frac{D}{D_0} \right)^{-1/2}, \quad B f_t' \left(1 + \frac{D}{D_0} \right)^{-1/2} \right\}, \tag{18}$$

with the notations

$$D_0 = \frac{c_0}{\alpha_0} = c_f \frac{g_1(\alpha_0, 0)}{\hat{g}(\alpha_0, 0)}, \tag{19}$$

$$\sigma_N^0 = \sqrt{\frac{E G_{fl} d_f \alpha_0^{d_f - 1}}{k c_0}} = \sqrt{\frac{E G_{fl} d_f \alpha_0^{d_f - 1}}{c_f g_1(\alpha_0, 0)}}, \tag{20}$$

$$B f_t' = \sqrt{\frac{E G_f}{k c_0}} = \sqrt{\frac{E G_f}{c_f g_1(\alpha_0, 0)}}. \tag{21}$$

It is obviously required that, for $d_f \to 1$, (18) must reduce to the nonfractal size effect law deduced by Bažant [41, 42], i.e., $\sigma_N = Bf'_t/\sqrt{1+\beta}$, $\beta = D/D_0$. In other words, the first expression in (18) must become identical to the second. Obviously, this requirement is satisfied.

For the simple specimen in Figure 2 of [1], Equation (17) can again be intuitively explained in terms of the stress relief zones, in the same manner as explained by Figure 2 of [1]. This provides a further justification of this equation.

5. Law of fractal size effect

To be able to apply (18), the α-value at maximum load must be known. Fracture test specimens (of positive geometry) reach maximum load for $c = a - a_0 \ll a_0$ where a_0 is the length of the notch, and if the notches are geometrically similar the value of α_0 is constant (independent of D). For brittle failures of geometrically similar reinforced concrete structures without notches, such as diagonal shear, punching of slabs, torsion, anchor pullout or bar pullout, extensive laboratory evidence as well as finite element solutions [43, 44] show that the failure modes are often approximately geometrically similar and $\alpha_0 \approx$ constant for a broad enough range of D. Then k, c_0, D_0, σ_N^0 and Bf'_t are also constant. In these typical cases, (18) describes the dependence of σ_N on size D only, that is, the size effect. (Note, however, that geometric similarity of failure mode is known to be violated for some cases, especially if a very broad size range is considered; e.g., for Brazilian split-cylinder tests of a size range exceeding 1:8).

Figure 3(a) shows the size effect plot of $\log \sigma_N$ versus $\log D$ at constant α_0, obtained according to the result in (18). Two size effect curves are shown

- the fractal one, which represents a transition from a rising asymptote, corresponding to a power law of exponent $(d_f - 1)/2$, to a descending asymptote corresponding to a power law of exponent $(d_f/2) - 1$, and
- the nonfractal one, which is the same as in Figure 3 of the preceding companion paper [1] and represents a transition from a horizontal asymptote, corresponding to a power law of exponent 0 (and to the strength theory), to a descending asymptote, corresponding to a power law of exponent $-1/2$ (characteristic of LEFM).

For both the fractal and the nonfractal curves, $D = D_0$ represents the point of intersection of the left-side and right-side asymptotes, that is, the center of the transition from one power law to another (Figure 3). On the microscale, i.e. for $D \ll D_0$, the energy release from the structure is negligible, and on the macroscale, i.e. for $D \gg D_0$, this energy release is dominant.

The special case when no material length exists is obtained as the limit for $c_f \to 0$. In that case, the fractal and nonfractal size effects consist of two power laws shown in the log-log plot in Figure 3(b) by lines of slopes $(d_f/2) - 1$ and $-1/2$. The latter slope corresponds to LEFM. The transition from the first to the second power law, which represents in the log-log plot a rotation, is called the renormalization group transformation.

For finite material length c_f, there is also a transition from one power law to another for the fractal regime alone, as well as for the nonfractal regime alone. These transitions have nothing to do with fractality and are not caused by a change of fractal scales, as in multifractal problems. Rather, they are a consequence of the effect that the existence of material length, c_f, has on the energy release – the fact that the energy release rate due to a unit σ_N-value is, for large structure sizes ($D \gg D_0$), nearly proportional to D but for small structure sizes

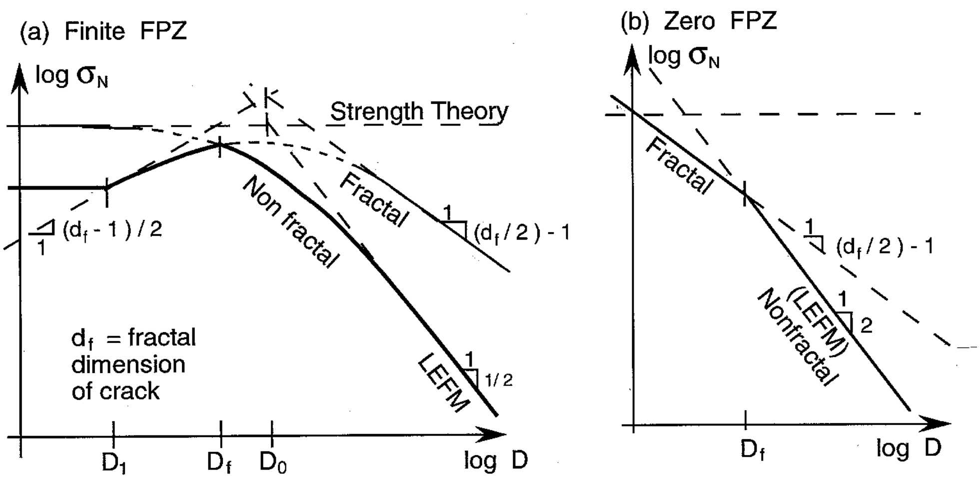

Figure 3. Size effect curves predicted for failures at geometrically similar continuous fractal cracks (invasive fractals) and nonfractal cracks, for (a) finite or (b) zero size of the fracture process zone (cohesive zone).

$(D \ll D_0)$ is approximately constant. This transition is centered, for both the fractal and nonfractal curves, at the value D_0 which we obtained by energy analysis.

When δ_1 is finite, we have a multifractal scaling problem, with two fractal scales, of which the second is nonfractal, having fractal dimension $d_f = 1$. In this case the fractal and nonfractal size effect curves shown in Figure 3 intersect. The intersection point D_f (whose value depends on structure geometry and on δ_1 or c_f) represents the transition from the fractal to the nonfractal size effect. Same as already mentioned for $c_f = 0$, this transition is in the fractal theory regarded as the renormalization group transformation. It is typical that this transformation represents a rotation of the log-log plot, as seen in Figure 3.

6. Can size effect be explained by crack fractality?

One salient feature of our result is that, for nonzero c_f, the first expression for σ_N in (18) is an increasing function of D when D is not too large. This is a strange feature, not supported by the available test results for quasibrittle materials. So it appears that, at very small scale, the fractal nature of the crack surfaces cannot be the major cause of size effect (except perhaps if $c_f = 0$).

To reconcile our result with this conclusion, we must assume that, for $D < D_1$, there ought to be a cut-off as indicated by the transition to a horizontal line at the left end of the size effect plot (Figure 3(a)). Constant D_1 corresponds to crack lengths $a < \delta_0$ where δ_0 is the supposed lower limit of crack surface fractality.

A second salient feature of our result is that the fractal size effect curve in the log-log plot approaches an asymptote of a slope much less that $-\frac{1}{2}$ (about -0.25 to -0.4, according to the d_f–values reported for concrete; Figure 3(a)). But there exist many test results for concrete and rock which clearly exhibit a close approach to an asymptote of slope $-\frac{1}{2}$; as two among many examples, see Figure 4(a) for data on diagonal shear failure of reinforced concrete beams [45], and Figure 4(b) on double-punch compression failure of concrete cylinders [46]. From these comparisons, we must conclude that the size effect in these test data cannot be caused by crack surface fractality.

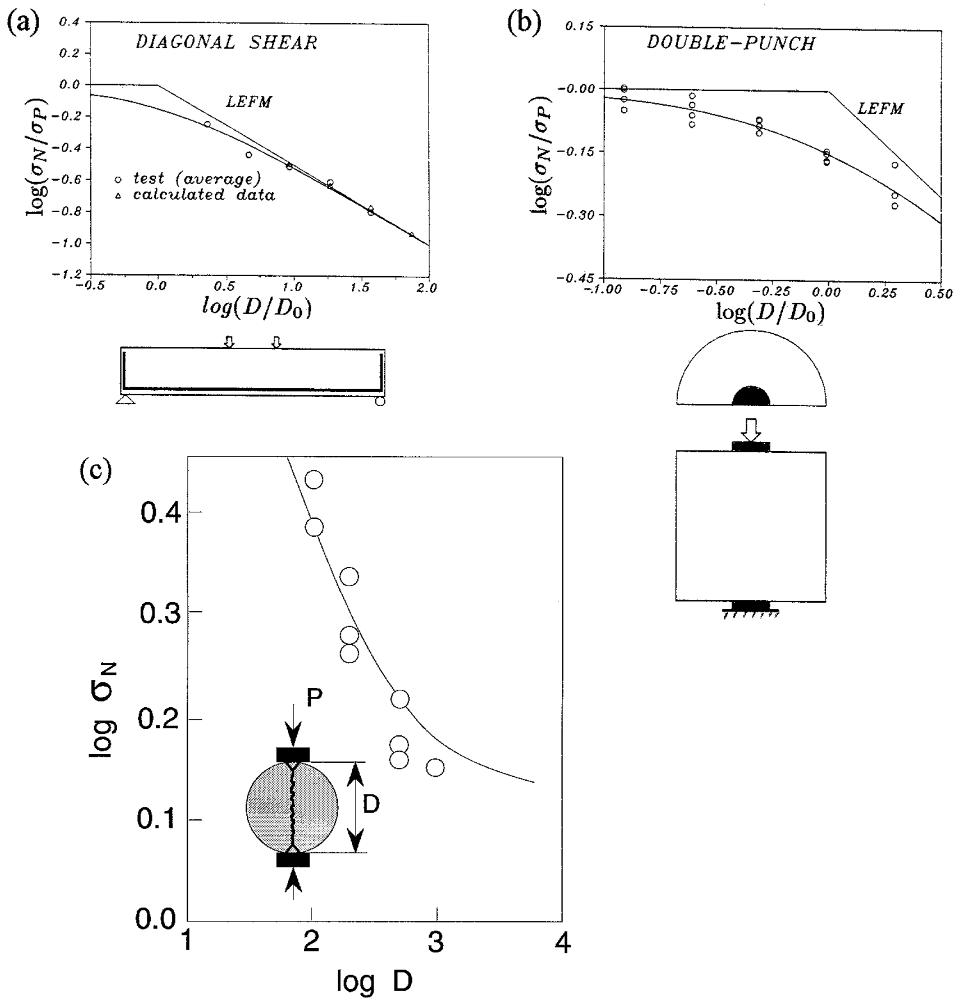

Figure 4. (a) Size effect test results for diagonal shear failure of longitudinally reinforced concrete beams without stirrups (Bažant and Kazemi, [45]) (top left) and (b) for double punch compression failure of concrete cylinders (Marti [46]) (top right), and (c) for Brazilian split-cylinder tests of Hasegawa et al. [50]. The first two are compared to Bažant's size effect law in Equation (21) of [1], and the third is compared to the extended size effect law in Equation (37) of [1].

There also exist size effect test data that do not approach an asymptote of slope $-\frac{1}{2}$ and exhibit a positive overall curvature in the logarithmic plot, as emphasized in [21, 26, 30]. The best examples are the Brazilian split-cylinder test data of a very broad size range (Hasegawa et al.'s data [51] of range 1:30, Figure 4(c); or Bažant et al.'s data of range 1:32, see [44]), These data suggest that, beyond a certain size range, the descending size effect curve of $\log \sigma_N$ versus $\log D$ might exhibit a transition to a horizontal line, i.e., the size effect might disappear for sufficiently large sizes.

However, in view of (18) it would be dubious to ascribe this phenomenon to crack fractality. There exist other, more plausible, explanations:

(1) a change of failure mechanism, or
(2) existence of a plastic mechanism providing residual strength.

When that is the case, the test data may be closely fitted by (37) or (38) of the preceding paper [1]. If the constant D_0 in that equation is sufficiently small, the plot of σ_N versus log D according to this equation exhibits a positive curvature. The fact that some test results show such a positive curvature has been invoked in [21, 26, 30] to support of the MFSL law (see Equation (39)). However, such test results can be equally well fitted by Equation (37) of [1].

In the Brazilian tests of very large cylinders, the nominal stress to cause splitting becomes so small that the failure is likely to be controlled by plastic-frictional slip on wedges that develop under the loading platens (this mechanism causes that the nominal strength is finite and not very small when the cylinder is cut by a saw along the axis and the two disconnected halves are then stacked together and loaded along the cut). Furthermore, for this or other reasons, the effective crack length at maximum load of very large cylinders apparently ceases being approximately proportional to the diameter D of the cylinder. A further cause of deviation in some Brazilian tests may be that the loading conditions violated geometric similarity because the loading strips under the platens were not scaled in proportion to D. Figure 4(c) documents that the Brazilian tests of Hasegawa et al. [51] can be reasonably well described by the extended size effect law with residual strength, given by (37) of the preceding paper [1]. So, crack fractality appears neither necessary nor logical for explaining these test results.

There is another objection to the fractal explanation of size effect. If the fracture of concrete developed in the form of a single smooth, continuous crack, the following relation would have to hold

$$G_f \approx 2\gamma, \tag{22}$$

where γ is the surface Gibbs free energy of the solid. The fractal tortuosity of a crack can conceivably cause the effective G_f to be several times larger than 2γ, but not much larger. However, the value of G_f for concrete (about 30 to 100 J/m^2) is several orders of magnitude larger than the γ value for the solids in concrete.

Therefore, the fracture front cannot consist of a single crack. Rather, the fracture process zone must consist of a band of many energy dissipating defects such as microcracks (including microcracks at mortar-aggregate interfaces) and plastic-frictional slips. They all must form before the fracture front can propagate. Indeed, this is what has been established experimentally, e.g. by locating the sound emissions from microcracks, or by unloading tests.

Some of the microcracks and slips eventually coalesce into a single continuous crack. But most of the microcracks, except those on the final crack surface, close. The final crack surface may be to a large extent fractal, but this is irrelevant for the fracture propagation criterion because the coalescence of microcracks and plastic-frictional slips occurs only at the end of the fracture process, in the wake of the fracture process zone. Most of the energy is dissipated in the fracture process zone by microcracks and plastic-frictional slip planes that do not become part of the final crack surface and thus have nothing to do with its fractality.

The difficulty of correlating G_f, or fracture toughness $K_c = \sqrt{EG_f}$, to γ is also supported by the observations of Cahn [6]. He found that while in chert and some ceramics K_c increases with d_f (approximately, $K_c \propto K_0^{d_f}$, where $K_0 =$ constant), in some steels it decreases with d_f. He pointed out that this may be caused by plastic phenomena in the fracture process zone, such as void growth. But this of course means that the crack front does not advance as a fractal curve. The fractal curve appears afterwards.

In view of the foregoing arguments, one must distinguish in the mechanics of fracture two types of continuous crack fractality:

(1) Fractality of the final crack surface, which is doubtless a reasonable morphological description of surface roughness for many materials (albeit applicable only for a limited range of scales); and
(2) fractality of the fracture process, that is, the process that controls energy dissipation at the fracture front.

Based on the present analysis, it is logical to conclude that the process of propagation of continuous cracks in quasibrittle materials such as concrete or rock cannot be predominantly fractal. The fractal nature of the crack surface is irrelevant for fracture propagation and for the scaling of failure loads. The recent intuitive suggestions [21, 30] that the size effect in concrete fracture may be caused by the fractal nature of crack surfaces are not defensible.

There is also a problem with the assumption that the crack is a self-similar fractal curve such as von Koch's, in which the fractality arises from locally transverse disturbances of a line. The crack curve (unlike the popular example of the shoreline of England) must be kinematically admissible, such that the zones of material adjacent to the crack can move apart as two rigid bodies. But a self-similar fractal curve such as von Koch's can have recessive segments (i.e. segments receding backward, against the overall direction of fracture propagation), and it can even develop spiral segments; see Figure 1(b). For such geometries, separation is kinematically impossible without additional fracturing around the fractal crack, but then the fracture would not grow as a single fractal crack. Although the probability of occurrence of recessive and spiral segments may be very low, they cannot be dismissed when the fundamental nature of fracture is studied.

For this reason, the self-similar (or locally transverse) fractality, i.e. fractality generated by displacements normal to individual local line segments, as is the case for the von Koch curve (Figure 1(a)), is unrealistic. It should be replaced (as J. Planas pointed out at FraMCoS-2, 1995) by the hypothesis of self-affine fractality, i.e., fractality generated only by crack line disturbances normal to the overall (global) crack direction, as illustrated in Figure 1(e).

7. Fractal size effect law in terms of material fracture parameters and identification of G_{fl}

Similar to the preceding companion paper [1], c_f can specifically be interpreted as the (Euclidean) effective length of the fracture process zone in a specimen extrapolated to infinite size (measured in the overall direction of propagation). In that case, $\theta = c_f/D = (a - a_0)/D = \alpha - \alpha_0$. This suggests that function $g(\alpha, \theta)$ can be approximated by a function of one variable, $g(\alpha)$, representing the usual dimensionless energy release rate function of LEFM. Also, $g(\alpha_0, 0)$ then reduces to $g(\alpha_0)$, $\partial/\partial\theta = d/d\alpha$, and $g_1(\alpha, 0)$ takes the meaning of $g'(\alpha) = dg(\alpha)/d\alpha$, with the prime denoting the derivatives. Instead of Equation (25) of [1], one obtains from these relations the fractal size effect law in terms of material fracture parameters

$$\sigma_N = \sqrt{\frac{E G_{fl} d_f \alpha_0^{d_f - 1}}{g'(\alpha_0) c_f + g(\alpha_0) D}}, \tag{23}$$

in which

$$\sigma_N^0 = \sqrt{\frac{E G_{fl} d_f \alpha_0^{d_f - 1}}{c_f g'(\alpha_0)}} \tag{24}$$

and, same as in [1], $D_0 = c_f g'(\alpha_0)/g(\alpha_0)$ and $Bf'_t = \sqrt{EG_f/c_f g'(\alpha_0)}$. If fracture propagation in some material were fractal, then G_{fl}, c_f and d_f could be identified by optimum least-square fitting of the last equation to maximum load data measured on notched specimens of various brittleness numbers.

8. General large-size asymptotic expansion

The same consideration as in [1] suggests that more general dimensionless variables and functions could be introduced as

$$\xi = \theta^r = (c_f/D)^r, \qquad h(\alpha_0, \xi) = [g(\alpha_0, \theta]^r, \tag{25}$$

with any $r > 0$. In the same manner as in [1], one can then obtain a more general fractal asymptotic expansion of size effect

$$\sigma_N = \sigma_P \left[\left(\frac{D}{D_0}\right)^r + 1 + \kappa_1 \left(\frac{D}{D_0}\right)^{-r} + \kappa_2 \left(\frac{D}{D_0}\right)^{-2r} + \kappa_3 \left(\frac{D}{D_0}\right)^{-3r} + \cdots \right]^{-1/2r}, \tag{26}$$

where $\kappa_1, \kappa_2, \ldots$ are certain constants, D_0 is given again by Equation (23) of [1], and

$$\sigma_P = \sqrt{d_f(\alpha_0 D)^{d_f - 1}} \sqrt{\frac{EG_{fl}}{c_f}} \left[\frac{\partial h(\alpha_0, 0)}{\partial \xi}\right]^{-1/2r}. \tag{27}$$

9. Small-size asymptotic expansion and asymptotic matching

Similar to [1], one can introduce into the expression for complementary energy Π^* instead of θ^r a new variable $\eta = \theta^{-r}$ and a new function, $\psi(\alpha_0, \eta) = [\hat{g}(\alpha_0, \theta)/\theta]^r$. Using the same procedure, one can thus obtain the small-size asymptotic expansion for fractal fracture

$$\sigma_N = \sigma_P \left[1 + \left(\frac{D}{D_0}\right)^r + b_2 \left(\frac{D}{D_0}\right)^{2r} + b_3 \left(\frac{D}{D_0}\right)^{3r} + \cdots \right]^{-1/2r}, \tag{28}$$

in which $b_2, b_3, \ldots$ are certain constants and

$$\sigma_P = \sqrt{\frac{EG_{fl} d_f (\alpha_0 D)^{d_f - 1}}{c_f [\psi(\alpha_0, 0)]^r}}; \qquad D_0 = c_f \left[\frac{1}{\psi(\alpha_0, 0)} \frac{\partial \psi(\alpha_0, 0)}{\partial \eta}\right]^{-1/r}. \tag{29}$$

Equation (29) shows that, for fractal fracture ($d_f > 1$) in small-size specimens, the nominal strength σ_P would have to increase with D proportionally to $D^{d_f - 1}$. Such a size effect contradicts experience. Therefore, the fractal hypothesis is untenable.

The large-size and small-size asymptotic series expansions in (26) and (28) have in common the first two terms. These two terms represent a simple asymptotic matching

$$\sigma_N = \sigma_P (1 + \beta^r)^{-1/2r} \quad (\beta = D/D_0). \tag{30}$$

This is the same expression as Equation (36) in [1], except that σ_P is different, depending on fractal characteristics. Same as before, the value $r \approx 1$ appears to be appropriate.

10. Fractal size effect for surface crack initiation and universal law

As already explained in [1], to treat the crack initiation from a smooth surface, as in the modulus of rupture (flexural strength) test, it is necessary to include the second (quadratic) term of the large-size asymptotic expansion because the first term is zero (as $g(0) = 0$). By the same procedure as in (15), the following law of the size effect for crack initiation may then be deduced

$$\sigma_N = \sigma_N^\infty D^{(d_f-1)/2} \left(1 + \frac{D_b}{D}\right), \tag{31}$$

with the notation

$$\sigma_N^\infty = \sqrt{\frac{EG_{fl} d_f \alpha_0^{d_f-1}}{g'(0)c_f}}, \qquad D_b = -\frac{g''(0)}{4g'(0)}. \tag{32}$$

Equation (31) is plotted in the logarithmic scales of D and σ_N in Figure 5, for both the fractal and nonfractal cases. For the nonfractal case, the plot has been discussed in [1]. For the fractal case, the size effect predicted from the fractal hypothesis represents a transition from a declining to a rising asymptote. Such behavior is contradicted by the experimentally observed trend (Figure 6). So we have another reason to conclude that the hypothesis of a fractal source of size effect is not defensible.

Note that, for $c_f = 0, \sigma_N^\infty$ can be finite only if the crack is fractal ($d_f > 1$), and $D_f \rightarrow= 0$ (which means the size effect disappears). This observation, of course, is not surprising because crack initiation from a smooth surface is, according to LEFM, impossible. Thus the size effect observed in modulus of rupture tests implies that $c_f > 0$.

Also note that (31) can be written as a linear regression equation $\sigma_N = A + CX$ where $A = \sigma_N^\infty D^{(d_f-1)/2}$, $C = AD_b$ and $X = 1/D$. Thus, for given d_f, one could identify the values of A and C by linear regression of test data on the modulus of rupture for various D. To identify d_f and G_{fl}, one could then repeat this analysis for various d_f and pick the case with the smallest sum of squared deviations from the regression line.

11. Does Weibull-type statistical theory apply to quasibrittle fracture?

Before discussing the possible effect of lacunar fractality, it is useful to make a digression to the statistical theory of size effect. Until about a decade ago, the size effect observed in concrete structures had been universally explained by randomness of strength and calculated according to Weibull statistical theory. Recently, however, it was shown [48, 49] that this theory cannot describe failure when large stable fractures can grow in a stable manner prior to maximum load, as is typical of quasibrittle fracture.

The basic hypotheses of Weibull-type theories are

(1) The structure behaves as a system of small material elements of random strength and fails as soon as the stress in one small element attains the strength limit.
(2) Stress redistribution and the associated release of stored energy caused by stable macroscopic crack growth before failure is negligible.

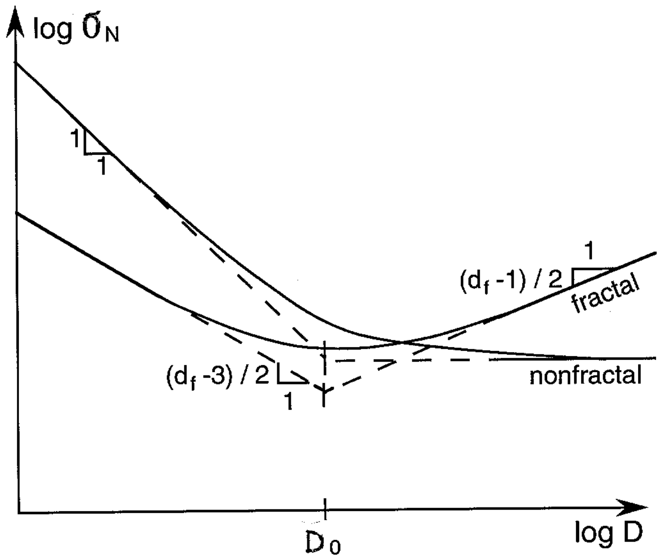

Figure 5. Size effect predicted for failures at initial of continuous (invasive) fractal crack or nonfractal crack in boundary layer.

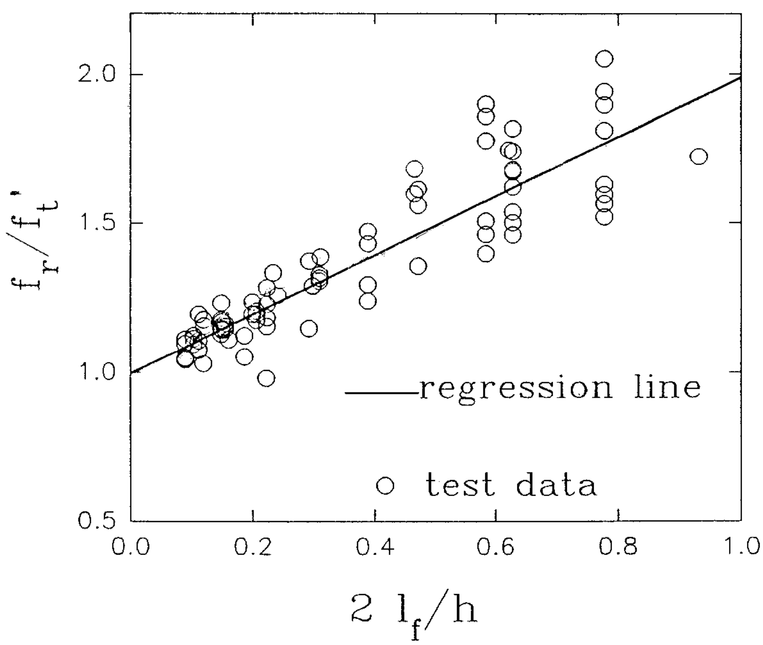

Figure 6. Test data for the dependence of modulus of rupture f_r of unreinforced concrete beams on beam depth D, and their fit by the nonfractal formula (31); $f_t' =$ direct tensile strength, $h = D =$ beam depth, $l_f =$ constant (after Bažant and Li, ref. [18] in [1]).

The proper tail distribution of failure probability P_1 of a small representative element of the material is the Weibull distribution [47], i.e.,

$$P_1 = \varphi(\sigma) = \left\langle \frac{\sigma - \sigma_u}{\sigma_0} \right\rangle^m, \tag{33}$$

in which φ = function giving P_1, m = Weibull modulus (approximately 12 for concrete), σ_0 = scale parameter (material constant), σ_u = threshold (a material constant which can normally be assumed as 0), and $\langle x \rangle = \max(x, 0)$ = positive part of x (Macauley bracket).

The failure condition that can be deduced from these hypotheses for the continuum limit of a structure consisting of many small elements having very small failure probability P_1 may be written (for $\sigma_u = 0$) in the form (e.g., [48, 49])

$$-\ln(1 - P_f) = \int_V \varphi(\sigma(\boldsymbol{x}))\, \mathrm{d}V(\boldsymbol{x})/V_r, \tag{34}$$

where P_f = failure probability of the structure, $\boldsymbol{x}$ = coordinate vectors, V = structure volume, V_r = representative volume of the material on which the strength probability is defined, which can be taken as $V_r = c_f^n$ where n = number of dimensions of the structure. One may set $\sigma(\boldsymbol{x}) = \sigma_N S(\boldsymbol{x})$ and introduce the scaling transformation of the coordinates (i.e., affinity) as $\boldsymbol{x}' = \boldsymbol{x}D/D'$, which transforms a structure of size D to a geometrically similar structure of size D'. One can then easily show [e.g., 48] that the size effect on nominal strength σ_N with any given probability cutoff P_f (as well as on the mean of σ_N) is a power law of the form

$$\sigma_N = k_v V^{-1/m} = k_0 D^{-n/m}, \tag{35}$$

where n = number of dimensions of the structure; $V \propto D^n$; and k_v, k_0 = constants.

The aforementioned hypotheses are strictly satisfied only for a long chain, as described by the weakest link model. But they also work well for brittle homogeneous structures that either fail or may be considered to fail as soon as a crack ceases to be microscopic compared to structure dimensions. This is the case for metallic structures embrittled by fatigue. In that case, the stress intensity factor of a microcrack in the small element in Figure 7(a) depends, at impending failure, only on the local stress σ calculated as if no cracks existed.

Even though (34) is an integral over the local stresses $\sigma(\boldsymbol{x})$, fracture mechanics has been introduced by evaluating the local failure probability from the statistical distribution of sizes of microcracks (or flaws). Validity of the formulation requires that the critical microcracks be so small that the remote stress field surrounding these microcracks be well approximated as a field of uniform stress that is equal to the local stress calculated for the same structure under the assumption that there are no cracks. The stress intensity factor of the microcrack is then simply approximated by the formula for a crack in an infinite solid subjected to a uniform stress at infinity. But it is important to note that this approach involves fracture mechanics only on the microscale (i.e. for a very small material element) rather than on the macroscale (i.e. for the structure as a whole). Obviously, with this approach, the energy release from the structure as a whole is negligible. Such a simplification is quite acceptable for steel structures but not for concrete or other quasibrittle structures.

Modest stress redistributions caused in structures in which a few material elements must fail before the structure can fail have been taken into account in early studies by means of various simple hypotheses about load sharing between the elements of the system. However, such phenomenological hypotheses ignore elasticity theory, and particularly the effect of the geometry of the structure and of the crack growth on the overall elastic stress field. They are normally insufficient for quasibrittle structures failing after large stable crack growth [48, 49], which typically occurs in reinforced concrete structures, as well as in penetration fracture or

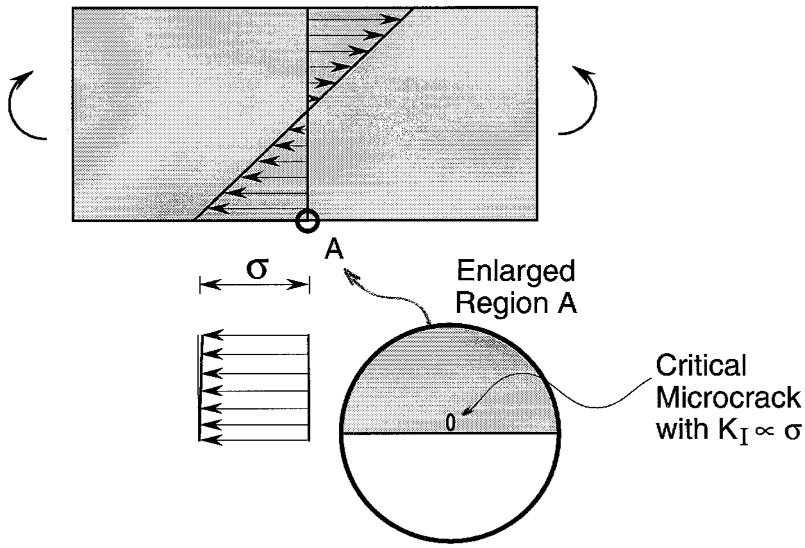

Figure 7. Basic hypothesis of Weibull theory: failure precipitated by a critical microcrack in microscopic zone A (marked by a circle), with K_{I} proportional to the local stress σ calculated for uncracked structure.

compression fracture of sea ice plate, or in fracture of a dam. The growth of large cracks before failure causes large stress redistributions with strain localization. This engenders a large release of stored energy and is, in itself, the cause of a major deterministic size effect. Because of ignoring it, the classical Weibull-type theory does not apply to quasibrittle structures [48, 49]. At present, extensive studies of the micromechanics of fracture are under way at several universities to overcome the aforementioned limitations of the classical Weibull theory. But they are based on large-scale numerical simulations which normally do not lead to easily interpretable analytical solutions.

It might seem that a simple way to take the stress redistribution due to a large crack into account would be to substitute the LEFM stress field with crack tip singularity into (34). However, this is incorrect for two reasons:

(1) Stresses approaching infinity are unacceptable for the Weibull distribution; and

(2) for the values of Weibull modulus m typical of most materials, the Weibull-type probability integral over the structure diverges.

A realistic approximation requires taking into account the blunting of sharp stress peak in the fracture process zone. This can be approximately done by assuming the failure probability P_1 of a small material element to depend on the nonlocal strain $\bar{\varepsilon}$ rather than the local stress σ at the same point, i.e., by replacing (34) with

$$-\ln(1-P_f) = k_1 \int_V \varphi[E\bar{\varepsilon}(\boldsymbol{x})]\,\mathrm{d}V(\boldsymbol{x})/V_r, \tag{36}$$

where $\bar{\varepsilon}$ can be most simply defined as the weighted average of strains ε within a certain characteristic neighborhood of a given point [48, 49]. Regarding the tensorial nature of strain,

ε is most simply interpreted as the maximum principal strain. Asymptotic matching based on (36), along with Weibull distribution, was shown to lead to the following law of size effect

$$\sigma_N = \frac{\sigma_P}{\sqrt{\beta^{2n/m} + \beta}}, \qquad \beta = \frac{D}{D_0}, \tag{37}$$

where σ_P and D_0 are constants. For $m \to \infty$, which gives the deterministic limit, this formula approaches the original Bažant's size effect law, $\sigma_N = \sigma_P(1+\beta)^{-1/2}$ (Equation (14) of [1]. For normal m-values, numerical studies [49] have shown that in practice (37) differs from this law only negligibly. For concrete, the difference was found to have no practical significance. The difference becomes appreciable only for mathematical extrapolation to structures smaller than the maximum aggregate size, which has no physical meaning.

For $D \to 0$, Equation (37) asymptotically approaches the classical Weibull size effect law in (35). The reason is that, for a very small structure, the nonlocal averaging operator in (36) represents averaging over the whole structure, and the average stress follows the Weibull-type size effect in (35).

An interesting feature is that, for $D \to \infty$, the statistical influence on the size effect disappears. Equation (37) asymptotically approaches the LEFM asymptote of slope $-\frac{1}{2}$ in the log-log plot. The reason is the redistribution of stresses caused by stable fracture growth prior to maximum load, which may be intuitively understood as follows.

If the Weibull probability integral is applied to the redistributed stress field, the dominant contribution to its value comes from the fracture process zone whose size is nearly independent of structure size D. The contribution from the rest of the structure nearly vanishes (reflecting the fact that the fracture cannot occur outside the process zone). Because, for specimens of very different sizes, this zone has about the same size, there can be no statistical size effect.

For quasibrittle materials, the Weibull-type size effect may operate only in very large structures that fail right at crack initiation – for example, in very deep unnotched plain concrete beams. In connection with the cohesive (fictitious) crack model, such a size effect has been studied under certain simplifying assumptions by Petersson [50]. He approached the problem numerically, by finite elements.

To obtain a formula, the deterministic size effect law for crack initiation at body surface, given by Equation (36) of [1], may be extended by asymptotic reasoning. The following generalization of the fractal formula (31), in which D_w, k_b are constants, matches the asymptotic properties

$$\sigma_N = k_b D^{(d_f-1)/2}\left[\left(\frac{D_w}{D}\right)^{n/m} + \frac{D_b}{D}\right]. \tag{38}$$

Even for beam depths such as $D = 10D_b$, the stress redistribution in the boundary layer, underlying (31), causes a significant deterministic size effect. Therefore, the beam depth beyond which the Weibull-type size effect could be dominant apparently needs to exceed $D = 100D_b$. Hardly any case satisfying this condition exists in concrete practice. Besides, the way to produce good, tough quasibrittle materials is to achieve that c_f be as large as possible. But this prevents the material from failing right at the crack initiation.

After discussing the Weibull-type statistical size effect, we are now ready to examine the possible role of lacunar fractals.

12. Lacunar fractality of microcracks

Another recently proposed law for size effect in quasibrittle fracture is [21]

$$\sigma_N = \sqrt{A + \frac{B}{D}}, \tag{39}$$

where A and B are empirical constants. This law has a similar plot as the nonfractal curve in Figure 5. It was called in [21] the 'multifractal scaling law', or MFSL. But it is unclear how it could be claimed to represent a consequence of fractality because the argument for Equation (39) stated in [21] completely lacked mechanics.[1]

Initially [21, 26], the argument for the law in (39) referred to fractal crack curves, for which the von Koch curve was used as an illustration. However, from the preceding analysis (presented at IUTAM Symposium, Torino 1994 [35]) it transpires that the hypothesis of fractality of the crack curves must lead to a different scaling law than the law in (39). Later [29] the fractal hypothesis for this law was modified to a lacunar fractality of microcrack distribution, but curiously the invasive fractality was retained [30] for explaining the variation of material fracture energy.

Microcracks with the geometry of lacunar fractals (Figure 1(e)) do not represent a continuous break of the material. So there can be no macroscopic stress redistribution and no global energy release causing size effect. Such microcracks can affect only the initiation of a macrocrack. Consequently, the argument for size effect must be made through some sort of Weibull theory.

This is what was proposed in [30]. The statistical distribution of microcrack sizes a was assumed to be the beta distribution with shape parameter α (Equation (4) in [30]), and the local stress σ required for microcrack propagation was assumed to follow the LEFM relation $\sigma/\sigma_u = \sqrt{a_M/a}$ where σ_u and a_M are material constants (Equation (6) in [30]). Then the failure probability P_f was assumed (in Equation (6) of [30]) to depend according to Weibull distribution on the local stress σ (calculated as if no crack existed).

Up to that point, the argument in [30] was the same as in the Weibull-type theory, and the same objections as before could be raised. After that, to justify the law in (39), a merely verbal argument was offered [30, p. 568], in which it was simply stated that the difference in physical fractal dimensions on the local scale of fracture and the global scale of the structure

[1] The argument for Equation (39) was stated on pages 196–197 of [21] as follows: 'the effect of microstructural disorder on the mechanical behavior becomes progressively less important at larger scales, whereas it represents a fundamental feature at small scales. . . . the disordered (damaged) microstructure is somehow *homogenized*, that is, it behaves macroscopically as an ordered microstructure. Therefore, the scale effect should vanish in the limit of structural size D tending to infinite, where an asymptotic finite strength can be determined. On the other hand, for small specimens (i.e. small compared to the microstructural characteristic size), the effect of the disordered microstructure becomes progressively less important, and the strength increases with decreasing size, ideally tending to infinite as the size tends to zero. In the bilogarithmic diagram [$\log \sigma_N$ versus $\log D$], the slope represents the fractal decrement of the ligament physical dimensions, which can be assumed as a measure of disorder on the mechanical behavior. Two limit conditions have to be satisfied: slope $\to$ 0 for large structures (homogenized microstructural effects), and slope $\to$ 0.5 for small structures. The last situation corresponds to the highest degree of disorder, which is a theoretical limit'.

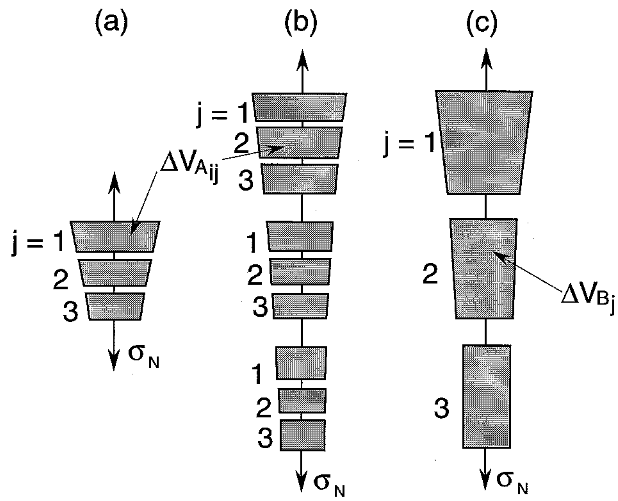

Figure 8. Large and small structures (a) and (b) subdivided into small material elements of same size and same failure characteristics, and large structures (b) and (c) subdivided into small and large material elements of different failure characteristics (considered for Weibull theory and for Carpinteri–Chiaia–Ferro MFSL law).

causes the nominal strength to depend on the structure size.[2]

In the consideration of 'two different fractal dimensions' (local and global) for the lacunar microcracks, i.e., different material characteristics, it was tacitly assumed that the size of the representative material element on which the material strength is defined increases with structure size D (Figure 8(a,b)). But this is not possible for structures made of the same material. If the size of the representative material element is not the same, the material is not the same. Let us now clarify these considerations by trying to formulate them mathematically.

For microcracks with lacunar fractal characteristics (see Figure 1(d)), Equation (1) is again valid but the fractal dimension $d_f < 1$. Because the microcracks are discontinuous, the continuum mechanics treatment must be based on damage mechanics rather than fracture mechanics. In the prototype one-dimensional form of continuum damage mechanics initiated by Kachanov, the effect of damage due to the microcracks is captured by the relations $\varepsilon = \tau/E$ and $\tau = \sigma/(1-\omega)$ where E is the Young's modulus, ε is the macroscopic strain, σ is the macroscopic stress (force per unit area of the continuum), τ is the true stress, and ω is the damage which is interpreted as the area fraction occupied by the microcracks. For a two-dimensional body, in which the fractal microcracks are one-dimensional, we have, according to Equation (1), $\omega = a_\delta/a = (\delta_0/a)^{1-d_f}$ where a_δ is the combined length of all the microcracks within a cross section of material of Euclidean length a, measured by a ruler of length δ_0. Because the ratio a/δ_0 is dimensionless, damage ω remains dimensionless. So we must conclude that, in contrast to cracks as invasive fractals, the dimensions of all the state variables and material constants remain the same. Hence, unlike fracture mechanics, the basic dimensional form of the continuum damage mechanics cannot be affected by fractality.

[2] On p. 568 of [30] the argument is stated as follows: The 'trend [of size effect] is correlated ... with the hypothesis of self-affinity for the material ligament. In other words, the physical dimension of the reacting [cross] section [of structure] at the peak load can be identified by two different values of fractal dimension: a local fractal dimension, in the limit of scales tending to zero, and a global fractal dimension, corresponding to the largest scales, strictly equal to the (integer) topological dimension. In consequence of this, the nominal tensile strength is constant for relatively large sizes, whereas it decreases with the size for relatively small sizes'.

A difference may nevertheless appear in the statistical properties of material strength, in the sense of Weibull, as proposed in [30].

The stresses in geometrically similar structures of different sizes may be written as $\sigma(\boldsymbol{x}) = \sigma_N S(\boldsymbol{\xi})$ where $\boldsymbol{\xi} = \boldsymbol{x}/D =$ dimensionless (relative) coordinates. In view of (1) for the fractal curve length a_δ, the Weibull-type failure probability of a small representative material element V_r needs to be redefined fractally, i.e.

$$P_1 = \left\langle \frac{\hat{\sigma} - \hat{\sigma}_u}{\hat{\sigma}_0} \right\rangle^m, \tag{40}$$

where the hats $\hat{}$ label lacunar fractal stress, $\hat{\sigma} = \sigma c_f^{1-d_f}$ (where $d_f < 1$ and $c_f \propto V_r^{1/n}$), and similar fractal generalizations of Weibull stress parameters.

In Weibull type theories, every structure can be made mathematically equivalent to a tensioned bar of variable cross section [48] (Figure 8). Since different lacunar fractal dimensions d_f associated with different structure scales are evoked in [30], one must consider subdivisions of the structure into sufficiently small elements of different sizes ΔV_A or ΔV_B associated with different dominant lacunar fractal dimensions d_{fA} and d_{fB}, as depicted in Figure 8 for geometrically similar structures of two sizes. The discussion in [30] implies different Weibull material strength distributions for different scales, which we label by subscripts $_A$ and $_B$

$$\varphi(\sigma(\boldsymbol{x}); d_{fA}) = \left\langle \frac{\sigma_N S(\boldsymbol{\xi}) c_f^{1-d_{fA}} - \hat{\sigma}_{uA}}{\hat{\sigma}_{0A}} \right\rangle^m, \tag{41}$$

$$\varphi(\sigma(\boldsymbol{x}); d_{fB}) = \left\langle \frac{\sigma_N S(\boldsymbol{\xi}) c_f^{1-d_{fB}} - \hat{\sigma}_{uB}}{\hat{\sigma}_{0B}} \right\rangle^m. \tag{42}$$

Now, if the material is the same, each material element of size $\Delta V_{Bj}, (j = 1, \ldots M)$ may further be subdivided into smaller material elements of sizes $\Delta V_{Aij}, (i = 1, \ldots N)$. According to the hypothesis in [30], the corresponding prevailing lacunar fractal dimensions, d_{fB} and d_{fA}, are different (Figure 8). So, the Weibull failure probabilities P_{fBj} of element V_B and the structural failure probability P_f are given by

$$-\ln(1 - P_f) = \sum_j \varphi(\sigma_N S_{Bj}; d_{fB}) \Delta V_{Bj}/V_r, \tag{43}$$

$$-\ln(1 - P_{fBj}) = \sum_i \varphi(\sigma_N S_{Aij}; d_{fA}) \Delta V_{Aij}/V_r. \tag{44}$$

Now, for the formulation to be objective, the structural failure probability calculated directly on the basis of the smaller elements ΔV_{Aij} must be the same, i.e.

$$-\ln(1 - P_f) = -\sum_j \ln(1 - P_{fBj}) = \sum_j \sum_i \varphi(\sigma_N S_{Aij}; d_{fA}) \Delta V_{Aij}/V_r. \tag{45}$$

Equating this to (43), we must conclude that, in order to meet the requirement of objectivity, the Weibull characteristics on very different scales, with different prevailing lacunar fractal dimensions, must be different and must be related as

$$\varphi(\sigma_N S_{Bj}; d_{fB}) = \Delta V_{Bj}^{-1} \sum_i \varphi(\sigma_N S_{Aij}; d_{fA}) \Delta V_{Aij}. \tag{46}$$

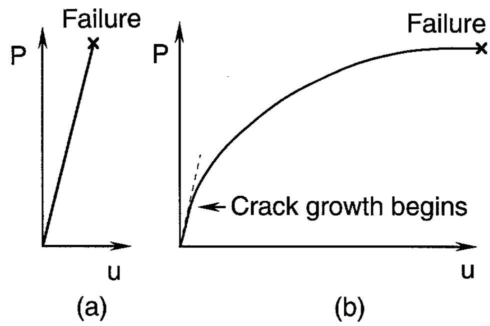

Figure 9. Load-deflection diagram of elastic structure in which a macroscopic crack (a) does not or (b) does grow in a stable manner before the maximum load.

Because of the transitive property of subsequent element subdivisions (Figure 8), expressed by (46) and (45), we must conclude that the consideration of different scales cannot yield different scaling laws, contrary to the hypothesis in [30]. In the case of a zero Weibull threshold, the same power law must result from the hypothesis of fractality of microcrack distribution, regardless of the scale considered. The scaling law is inevitably identical to the Weibull-type theory of failure. The only difference from the usual Weibull-type theory is that lacunar fractality can affect the values of Weibull material parameters.

Another objection against the argument advanced for the law in (39) is that, in quasibrittle structures, large cracks grow stably prior to reaching the maximum load. This is manifested by significant curvature of the load-deflection diagram before reaching the peak load (Figure 9(b)), which is typical of failures of reinforced concrete and other quasibrittle structures (if the diagram were straight up to the peak load, Figure 9(a), it would imply that the structure must be failing already when the critical crack is still microscopic). Due to the associated release of stored energy, the stable crack growth per se is a source of a major deterministic size effect, which may not be disregarded. Statistics of random crack roughness and of random microcrack distribution, possibly with partially fractal features, might of course have some additional effect. But that effect, even if it is not negligible, cannot stand alone. It would have to be appended to the size effect engendered by large stable crack growth. This contrasts with the classical applications of Weibull theory to brittle metals, in which the effect of energy release is on the macroscale negligible.

The property that the law in (39) approaches for large sizes a horizontal asymptote agrees with some size effect test data (e.g., for the Brazilian split-cylinder test). But there exist two other more realistic explanations:

(1) There is a transition to a residual plastic-frictional mechanism, as already embodied in Equations (32–33) of the preceding paper [1]; or
(2) the crack at maximum load ceases to increase in proportion to D but remains constant.

Such a situation can of course be described by the generalized size effect law in (19) of the preceding paper [1], with the shape effect embedded in function $g(\alpha)$; see Figure 4(c).

13. Conclusions

(1) Although the roughness of the crack surfaces in concrete, rock, ceramics or ice as well as the distribution of microcracks can be to a certain extent described as fractal, the fractal

characteristics do not cause a size effect on the nominal strength of structures. The arguments for such a size effect have not been based on mechanics. They have been purely geometric in nature, but this is insufficient. The stress redistribution and energy release caused by a fractal crack need to be taken into account.

(2) Asymptotic analysis shows that if the failure is caused by the propagation of a continuous fractal crack of fractal dimension d_f, then the scaling law for nominal strength of geometrically similar structures of sizes D containing large similar cracks can be described by a matched asymptotic that approximately describes a smooth transition from the power law $D^{(d_f-1)/2}$ to the power law $D^{(d_f/2)-1}$. The size D_0 at which the transition is centered is proportional to the material length c_f governing the size of the fractal fracture process zone and depends also on the geometry of the structure. The nonfractal size effect law derived by Bažant [41, 42] represents the limiting case for $d_f \to 1$. For quasibrittle structures that fail at crack initiation in the boundary layer (as in bending of unreinforced concrete beams), the fractal scaling law represents a smooth transition from power law $D^{(d_f-3)/2}$ to power law $D^{(d_f-1)/2}$.

(3) There are two objections to the hypothesis that the fractal nature of the crack surface is the cause of the observed size effect:

(a) If d_f is appreciably larger than 1, the large-size and small-size asymptotes of the fractal size effect for large cracks as well as for fracture initiation contradict the available experimental results on the size effect in concrete and other quasibrittle materials.

(b) The energy dissipated by the creation of the final crack surface, which may be fractal, is only a minute portion of the energy dissipated in the fracture process zone by the microcracks and frictional slips that lie away from the crack path.

(4) Self-similar fractal curves such as von Koch's are not acceptable for describing cracks because they exhibit recessive and spiraling segments which kinematically preclude separation of surfaces in a solid. The fractal crack curves can be considered only as affine fractals.

(5) The size effect law for the nominal strength of structures can be expressed in terms of the fractal fracture energy and other material parameters. If the fracture process of some material were fractal, these parameters as well as the fractal dimension of the propagating crack could be identified by the size effect method from the measured maximum loads of specimens of different sizes.

(6) The alternative hypothesis that the cause of the size effect is a lacunar (discrete) fractal distribution of microcracks in the fracture process zone rather than the fractal nature of the crack surface is also not a viable explanation of the size effect. Except for a possible influence of lacunar fractality on the Weibull material parameters, the predicted scaling law is identical to that in Weibull theory. Same as the Weibull theory, the hypothesis of lacunar fractal microcracks cannot explain the size effect in quasibrittle fracture because it ignores the stress redistribution and energy release due to large stable growth of cracks.

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