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Scaling of Failure of Beams, Frames and Plates with Softening Hinges

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Abstract. Between 1970 and 1986, Giulio Maier published a series of pioneering papers on equilibrium path bifurcations and instabilities in beam structures failing by inelastic hinges. His analysis is now extended to the size effect. First, the dependence of the bending strength on the postpeak softening slope of an inelastic hinge on the beam depth is analyzed based on the energy principles of fracture mechanics. Since exact analytical solutions for structures with many hinges softening simultaneously are very complicated, the present study focuses on the asymptotic case of sufficiently large structures, for which no two inelastic hinges are softening at the same time. Simple size effect trends are identified for this asymptotic case. For the opposite asymptotic case of very small structures, classical plasticity applies. For the intermediate situations, approximate formulae of asymptotic matching type are proposed. The size effect obtained is very different from two- or three-dimensional structures failing due to propagation of one dominant crack or damage band.

Sommario. Nel periodo tra il 1970 e il 1986, Giulio Maier ha pubblicato una serie di lavori pionieristici sulle biforcazioni nei percorsi di equilibrio e sui fenomeni di instabilità per sistemi di travi la cui crisi era dovuta a formazione di cerniere plastiche. I suoi studi vengono qui generalizzati ad includere effetti di scala. Dapprima si analizza, sulla base dei principi energetici della meccanica della frattura, la dipendenza della resistenza flessionale dalla pendenza del ramo “softening” successivo al picco per una cerniera plastica di trave. Dal momento che l’esatta soluzione analitica per strutture con molte cerniere plastiche contemporaneamente attive è piuttosto complicata, questo lavoro si concentra sul caso limite di strutture sufficientemente grandi in cui non più di una cerniera si attiva ad ogni istante, che consente di identificare in modo semplice l’influenza degli effetti di scala. Nel caso limite opposto, relativo ad una struttura molto piccola, la plasticità classica rimane valida. Per situazioni intermedie si propongono formule approssimate che sfruttano i risultati precedenti. Gli effetti di scala ottenuti risultano sensibilmente diversi da quelli relativi a strutture bio tri-dimensionali che collassano per la propagazione di una frattura dominante in una banda di danneggiamento.

Key words: Plastic hinges, Softening, Size effects.

1. Introduction

The classical approach to the calculation of load capacity of beams, frames and plates rests on the concept of inelastic hinge. With any set of kinematically admissible inelastic hinge locations, one obtains a lower bound on the load capacity, and if the hinge locations are correct, one has the exact solution. The evolution of the deformations of the structure and the progressive formation of the inelastic hinges need not be followed incrementally. It suffices to analyze only the collapse mechanism, writing a work balance equation for a small deflection increment in which all the inelastic hinges necessary to form a single-degree-of-freedom mechanism are rotating simultaneously. This classical approach, however, is valid only if the moment-rotation diagram of each hinge ends with a horizontal yield plateau.

It was Giulio Maier who recognized, with his co-workers, that this classical hypothesis is invalid in certain practically important situations and that it requires a very different type of analysis (Maier and Zavelani, 1970; Maier, 1971, 1976; Maier *et al.*, 1973). As shown by Maier and Zavelani (1970), postpeak softening of an inelastic hinge may arise in steel beams and frames because of large deflection buckling of the flanges.

In reinforced concrete beams, postpeak softening of inelastic hinges is encountered frequently. It occurs if a large enough axial compressive force is present (Bažant and Xiang, 1997), which is typical of columns and of beams in a frame with a large horizontal thrust, and also of prestressed beams. Overreinforcement of the cross section also causes postpeak softening, and although it is prohibited by the design codes it is often inevitable when damaged reinforced concrete beams are to be retrofitted with a fiber composite laminate.

The progressive growth of tensile bending cracks in plates, too, is a cause of postpeak softening of the moment-rotation diagram of a hinge line (Rice and Levy, 1972). This is, for instance, important for determining the load capacity of thick floating sea ice plates, as well as unreinforced concrete plates.

Maier and co-workers pioneered the studies of structures failing with softening hinges. They conducted incremental analysis and showed that various combinations of softening and unloading in different hinges cause bifurcations of the equilibrium path (Maier and Zavelani, 1970; Maier, 1971, 1976; Maier *et al.*, 1973). In a study that has not received deserved attention, Maier (1986) considered the example of a fixed-end beam to study the effect of the postpeak softening slope of the moment rotation diagram on the load-deflection diagram. He demonstrated that, as this slope is getting steeper, the load-deflection diagram of the beam develops multiple local maxima and the overall maximum (load capacity) decreases with an increasing post-peak slope. As transpired from later researches, the relative steepness of the postpeak moment-rotation diagram (Figure 1) increases with the beam size, and so Maier's (1986) analysis is now seen to have implied a size effect engendered by hinge softening.

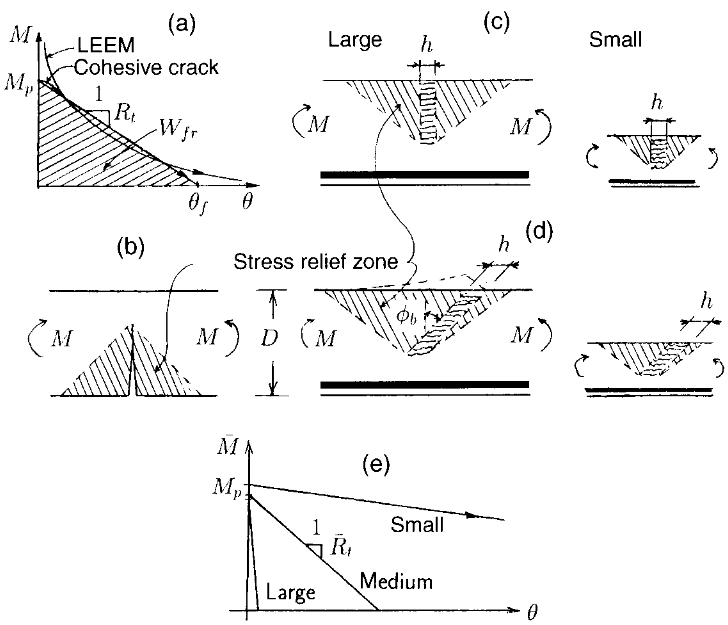

Figure 1. Softening inelastic hinge; (a) its moment rotation diagram, (b) hinge failing by tensile crack, (c) and (d) hinge failing by compression fracture band, and (e) size effect on moment-rotation diagram.

The objective of this terse paper is to briefly outline a simple method for extending Maier's innovative ideas and analyses to an approximate description of the size effect in beam structures with softening inelastic hinges as well as floating ice plates subjected to a line load. A detailed analysis with various rumifications will follow in two forthcoming articles (Bažant, 2001; Bažant and Guo, 2001).

In this paper, first the results of fracture mechanics of concrete are used to determine how the postpeak moment-rotation diagram depends on the size of the cross section. This diagram is then used to analyze the size effect in geometrically similar beam structures and ocean ice plates subjected to a line load.

The basic idea of the present approach is not to bother with a systematic analytical solution of the load-deflection diagrams of normal-size structures with softening hinges, which gets very complicated even for simple structures, but to focus instead on the asymptotic cases of very large and very small structures, for which analytical solutions are easy and amenable to simple formulae. An approximate solution for the entire size range is then be obtained by 'interpolation' between the asymptotic cases. This is an approach known in general as the asymptotic matching, which has for a long time been exploited in fluid mechanics (e.g. Barenblatt, 1979) and later used to deduce the size effect laws for fracture of quasibrittle structures (Bažant and Chen, 1997; Bažant and Planas, 1998; Bažant, 1999, 2001).

2. Causes of Moment-Rotation Softening

While the effective length of the plastic zone in inelastic hinges is proportional to the characteristic size (dimension) D, which may be taken as the beam depth D, the effective hinge length of fracturing hinges (Figure 1(b), (c), (d)) is, under certain plausible hypotheses, approximately constant, roughly-equal to the characteristic length of the material. The reason is the localization instability of strain softening (Bažant and Cedolin, 1991, Chapter 13).

Moment-rotation softening due to tensile bending fracture occurs for example in plain (unreinforced) concrete beams or plates or in floating ice plates (Figure 1(b)). The energy dissipated by the crack per unit area of the cross section is the fracture energy G_{f}. Moment-rotation softening due to compression fracture occurs for example in reinforced concrete beams if the bending moment M is accompanied by a sufficient axial compression force N (which is the case for prestressed concrete beams, for columns with a large enough axial force, and for frames or arches with a large enough horizontal thrust). The softening also occurs due to compression fracture if there is strong enough tensile reinforcement (Figure 1(c), (d)), for instance, if the beam is overreinforced because of a retrofit with a fiber laminate bonded to the tensile face.

The compression fracture typically consists of an inclined band of axial splitting microcracks (a crushing band). The plane of the band, inclined by angle ϕ_b with respect to the orthogonal cross section, can intersect the cross section in a line that is vertical (Figure 1(c)) or horizontal (Figure 1(d)). The band may be considered to dissipate, per unit area of the (normal) cross section, energy G_b that is approximately a material constant (Bažant and Chen, 1997; Bažant and Xiang, 1997).

3. Size Effect on Nominal Flexural Strength

The failure of a heterogeneous quasibrittle material is not decided by the local continuum stress but by the average stress within a representative volume of the material. Thus the maximum bending moment M_0 in a beam does not occur when the elastically calculated stress σ at the beam face equals the material strength. Rather it occurs when, approximately, the elastically calculated stress $\overline{\sigma}$ at the middle of a boundary layer of a certain thickness D_b, which is proportional to the maximum size of material inhomogeneities equals the material strength. From the bending stress formula, $\overline{\sigma} = (M/I)(D/2 - D_b/2)$ where D = beam depth, M = bending moment, and I = centroidal moment of inertia of cross section. Noting that the modulus of rupture f_r, representing the nominal strength σ_N, is defined as $f_r = \sigma_N = MD/2I$, we have

$$\sigma_N = f_{r\infty}\left(1 - \frac{D_b}{D}\right)^{-1}, \qquad (D \gg D_b). \tag{1}$$

It is exact only asymptotically, for $D \to 0$. According to the asymptotic expansion in terms of powers of $1/D$, one can make the replacement $(1 - D_b/D)^{-1} \approx (1 + rD_b/D)^r$ (r being any positive constant), which furnishes

$$\sigma_N = f_r = f_{r\infty} q(D), \quad \text{with} \quad q(D) = \left(1 + \frac{rD_b}{D}\right)^{1/r}, \tag{2}$$

where r is an empirical positive constant, and $q(D)$ is a positive dimensionless decreasing function having the limit $q = 1$ for $D \to \infty$. The size effect formula (2) becomes meaningless for $D \to 0$, but beams with $D < D_b$ cannot be produced. Therefore, this is not an important problem, although formulae with a finite σ_N value for $D \to 0$ can be devised. More logically, one can take into account the stress redistribution due to softening in the boundary layer (Bažant and Li, 1995), but the result is the same as formula (2) up to the first two terms of the asymptotic expansion. Even more logically, one can use fracture mechanics (Bažant and Li, 1996; Bažant, 1998; Bažant and Novák, 2000), but the result is again the same up to the first two terms.

The softening moment-rotation diagram will now be idealized, for the sake of simplicity, as linear (triangular, Figure 1(a), that is, $M = R_t(\theta_f - \theta)$ where R_t is the tangent stiffness of the hinge (representing the slope ot the $M - \theta$ diagram) and θ_f is the hinge rotation at complete break (Figure 1(a)). The energy W_{fr} dissipated by a total break of the cross section is given by the area under this diagram (Figure 1a); $W_{fr} = \frac{1}{2}M_p\theta_f$. Restricting now attention to rectangular cross sections, we may write the energy dissipated over the whole cross section upon reaching a complete break alternatively as $W_{fr} = G_b bD$ where b is the width of a rectangular cross section. Both foregoing expressions for W_{fr} must be equal, which yields

$$\theta_f = \frac{2G_b bD}{M_0 q(D)} = \frac{12G_b}{f_{r\infty}} \frac{1}{Dq(D)}. \tag{3}$$

Let us now introduce the dimensionless bending moment and dimensionless tangential softening stiffness of the softening inelastic hinge:

$$\bar{M} = \frac{M}{EbD^2}, \qquad \bar{R}_t = \frac{R_t}{EbD^2}, \tag{4}$$

where the tangent stiffness is defined as $R_t = M_p/\theta_f$. For non-softening elasto-plastic materials, the diagram of $\bar{M}$ versus θ is independent of the beam depth (e.g. Bažant and Jirásek, 2001), and so any change in this diagram as a function of the structure size reveals a size effect. Upon substituting (3) into (4), one finds that,

$$\bar{R}_t = \frac{q^2(D)D}{72l_0}, \qquad l_{fr} = \frac{EG_b}{f_{r\infty}^2}. \tag{5}$$

Here we may recognize l_{fr} to be an Irwin-type characteristic length of the material (e.g. Bažant and Planas 1998).

If $q(D)$ is almost constant (which occurs for large enough D), one finds from (5) and (3) that the dimensionless softening stiffness of the hinge increases in proportion to size D, and that the rotation at full break decreases at inverse proportion to D (Figure 1). As the structure size approaches infinity, $\bar{R}_t$ becomes infinite (a vertical drop). As the structure size tends to zero, $\bar{R}_t$ becomes zero (a horizontal line), and in that limiting case the softening hinge becomes equivalent to a plastic hinge.

4. Size Effect on Softening Beams and Frames

Consider a statically indeterminate (redundant) beam structure whose collapse necessitates N inelastic hinges to form ($N=4$ in the example of Figure 2(a)). Let the inelastic hinges be numbered as $j = 1, 2, \ldots N$ in the sequence in which they form during loading. Let K_i be the stiffness associated with P under the assumption that hinges $j = 1, 2, \ldots i-1$ have completely softened (i.e. $M=0$ and $\theta \geqslant \theta_f$ in these hinges) and hinges $j=i, i+1, \ldots N$ have not yet started to form (i.e. $M \leqslant M_p$ and $\theta=0$ in these hinges). Obviously, $K_1 > K_2 > K_3 > \ldots > K_N > 0$.

Consider now the case that all the hinges $j=1, 2, \ldots i-1$ have softened to $M=0$ and hinge i has not yet started to form. In this case, the load-point deflection w is decided solely by K_i and the start of softening in the next hinge i is decided by a critical stress, σ_i, representing the stress at the tensile face in the case of tensile fracture or at the compression face in the case of compression fracture; therefore

$$P = K_i w, \qquad \sigma_i = S_i w, \tag{6}$$

where S_i are constants. Limiting again attention to rectangular cross sections, one may introduce the dimensionless structure stiffness $\bar{K}_i$ and the dimensionless critical stress $\bar{S}_i$ in hinge i, such that

$$K_i = \bar{K}_i E b, \qquad S_i = \bar{S}_i E/D, \tag{7}$$

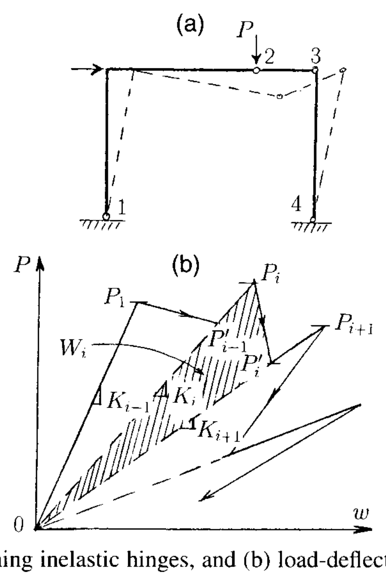

Figure 2. (a)Frame failing by softening inelastic hinges, and (b) load-deflection diagram when only one hinge is softening at a time.

where D = characteristic size of the structure, and b = characteristic thickness in the third dimension.

Assume now that the hinges form and fully soften one by one, that is, no two hinges are softening at the same time (it will be seen that a large enough structure always fails this way). Then the load-deflection diagram must look as shown in Figure 2(b), where the slope of each ray emanating from the origin is K_i, and the maximum load on each ray marked P_i ($i = 1, 2, \ldots$) corresponds to the start of softening of the next hinge. The trough P'_i on each ray is the load at which the softening of each hinge gets completed (i.e. $M = 0$ and $\theta = \theta_f$). Since the moment-rotation diagrams of the hinges are assumed to be linear, the load-deflection diagram from P_i to P'_i must be a straight line. The load peaks P_i are determined from the condition $\sigma_i = f_r = f_{r\infty} q(D)$. Noting that $\sigma_i = P S_i / K_i$, we find $P_i = f_{r\infty} b D q(D) \bar{K}_i / \bar{S}_i$. The nominal stresses at the load peaks are

$$\sigma_{Ni} = \frac{P_i}{bD} = \frac{\bar{K}_i}{\bar{S}_i} f_{r\infty} q(D). \tag{8}$$

Not surprisingly, the peaks σ_{Ni} exhibit only the size effect of fracture initiation (Figure 3(a)), the same as that for the modulus of rupture.

To determine the troughs P'_i, consider the area of the shaded triangle in Figure 2(b) between the rays of slopes K_i and K_{i+1}; the area is $W_i = \frac{1}{2} P_i^2 / K_i + \frac{1}{2}(P_i + P'_i)(P'_i / K_{i+1} - P_i / K_i) - \frac{1}{2} P'^2_i / K_{i+1}$ or

$$W_i = \frac{1}{2} P_i P'_i \left(K_{i+1}^{-1} - K_i^{-1}\right). \tag{9}$$

This area represents the work dissipated when hinge i softens from M_p to 0. Since it is assumed that no other hinge is softening simultaneously with hinge i, energy conservation requires W_i to be equal to the work dissipated by hinge i; $W_i = G_b b_i D_i = G_{fr} \beta_i \delta_i D^2$, with $\beta_i = b_i / D$, $\delta_i = D_i / D$. Here b_i and D_i are the width and depth of the cross section at hinge i, and β_i and δ_i are constants for structures geometrically similar in three dimensions. Setting both expressions for W_i equal, one can solve for P'_i and obtain:

$$\frac{\sigma'_{Ni}}{\sigma_{Ni}} = \frac{P'_i}{P_i} = \frac{2 E G_b}{\sigma_{Ni}^2} \left(\frac{1}{\bar{K}_{i+1}} - \frac{1}{\bar{K}_i}\right)^{-1} \frac{1}{D} = \frac{\text{const.}}{D} \quad (\text{for large} D). \tag{10}$$

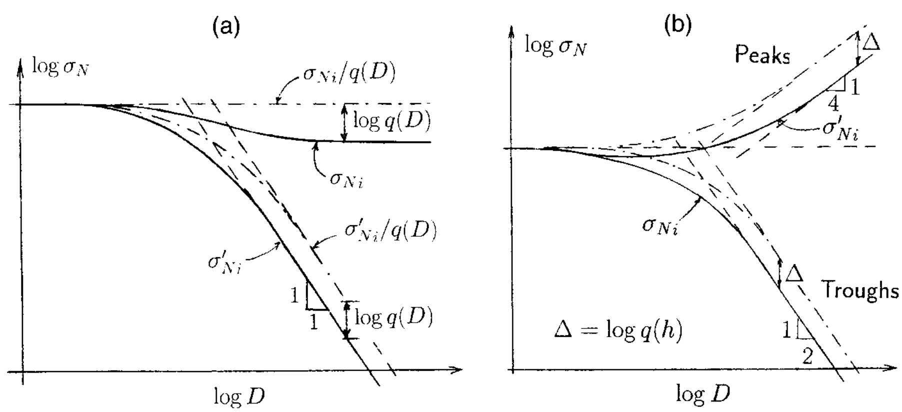

Figure 3. Size effects on nominal strengths at a peak and at a trough, matched to small-size plastic limit: (a) for beam (b) for floating ice plate under line load.

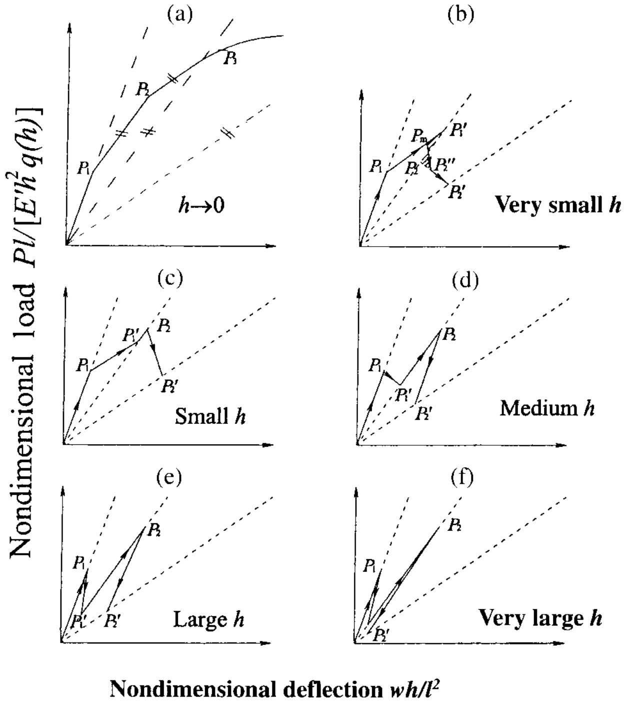

Figure 4. Evolution of load-displacement diagrams (in terms of dimensionless nominal strength σ_N/E and relative deflection w/D) with increasing size D of the structure failing by softening hinges (in such coordinates, the secant stiffness slopes, as well as the peak points for $q(D) = 1$, do not change with size). Parallel dashes in (a) mark parallel lines.

Factor $1/D$ represents the large-size asymptotic size effect on the troughs (Figure 3(a)). It is a very strong size effect, much stronger than the LEFM size effect of large cracks, which is of the type $1/\sqrt{D}$.

Figure 4 shows a sequence of typical load-deflection diagrams for geometrically similar structures of increasing sizes D. These diagrams are plotted in dimensionless coordinates w/D and σ_N/E, and the size effect on the peaks stemming from crack initiation is ignored, that is, diagrams are plotted for $q = 1$. Every change in these diagrams with the structure size signifies a size effect.

The scaling of the troughs P'_i, as described by (10), indicates that when the structure size increases, the response approaches a series of narrow spikes with softening slopes of progressively decreasing inclination (Figure 4). The descending part of each spike in Figure 4(e),(f) is unstable for any type of control of P and w, and is known as the snapback instability.

When the structure size decreases, the line from P_i to P'_i eventually changes its slope from negative to positive, that is, P_i ceases to be a peak and P'_i ceases to be a trough. With a further size increase, P'_i eventually becomes coincident with the peak P_i. For still smaller sizes, there exists a period of the loading process in which hinges i and $i + 1$ have both entered the softening regime. In that case, for reasons of bifurcation and response path stability (Bažant and Cedolin, 1991, Sections 10.2 and 13.4), one of the hinges must unload. After the other hinge softens completely, the first one reloads and then begins softening again. An analytical solution for simultaneous softening of several hinges is quite messy and will better be skipped.

For $D \to 0$, the slope of the diagram of dimensionless moment $\bar{M}$ versus rotation θ diminishes and a horizontal line is approached. Consequently, the plastic limit analysis applies in this asymptotic case. In Figure 4(a), the zero-size limiting response consists of line segments parallel to the rays emanating from the origin, which is easy to calculate by well-known methods (note the parallel double dashes marking parallel lines).

The behavior between this asymptotic case and the case of the smallest structure for which only one hinge is softening at a time (in which $P'_i = P_{i+1}$) can be approximately characterized by interpolation, better regarded as asymptotic matching. Matching of the large-size asymptotic size effect in (10) to the horizontal small-size asymptote may be achieved by the following simple approximate formulae for all sizes:

$$\sigma'_{Ni} \approx \frac{\sigma'_{0_i}}{1 + D/D'_{0_i}} \quad \text{or} \quad \frac{\sigma'_{0i}}{(1 + D_s/D'^s_{0i})^{1/s}}, \tag{11}$$

where σ'_{0i}, D'_{0i} and s are positive constants. For small sizes, however, the largest size effect is caused by the fact that the overall maximum load for different sizes occurs for different i; compare Figure 4(a) and (b). This aspect is not reflected in Figure 3.

Note that the peak σ_{Ni} exhibits no size effect when $q(h) = 1$. Yet there is an overall size effect because the overall maximum nominal stress occurs for different hinges (different i) as the size is increased.

When the structure is large enough for the load-deflection diagram to involve softening segments, interesting questions arise with respect to ductility and the effect of imperfections on stability. They are similar to those discussed in general for example in Bažant and Cedolin (1991).

Although a number of experimental studies dealt with softening of inelastic hinges in reinforced concrete beams (Barnard, 1965; Rosenblueth and Diaz de Cossio, 1965; Wood and Roberts, 1965; Barnard and Johnson, 1966; Wood, 1968), very little attention has been paid to size effect. No geometrically scaled tests of a sufficient size range have been reported, and the existing limited experimental evidence on size effect is quite ambiguous (Mattock, 1964; Corley 1966). Hillerborg (1989, 1990) was nevertheless able to show that some data were not inconsistent with his model for compression fracture, implying size effect. A detailed discussion of this literature is included in a forthcoming article (Bažant, 2001).

5. Size Effect in Floating Ice Plate Under Line Load

The foregoing analysis may be extended to a floating ice plate (Figure 5). The load capacity of an ice place under a vertical concentrated load was analyzed according to fracture mechanics by Bažant and Kim (1998a, b) and Bažant 2000. However, we consider here a line load, P, for which the failure mode is completely different, permitting a one-dimensional analysis.

In the case that the water does not flood the top of the plate (which can occur only if the plate is cracked and if the deflection exceeds 1/11 of ice thickness), the plate behaves exactly as a beam on elastic foundation. The failure occurs by parallel line cracks propagating vertically through the thickness of ice. As they propagate, the bending moment M in the cracked cross section decreases as a function of the additional rotation θ caused by the cracks. The diagram $M(\theta)$ is nonlinear, but can be simplified as linear (same as in Figure 1(a), in order to make an exact analytical solution feasible.

The only difference from the preceding section is that foundation (sea water) never reaches plastic response (unless water would flood the top). The response of the ice beam between the softening hinges is governed by the differential equation for beams on elastic foundation,

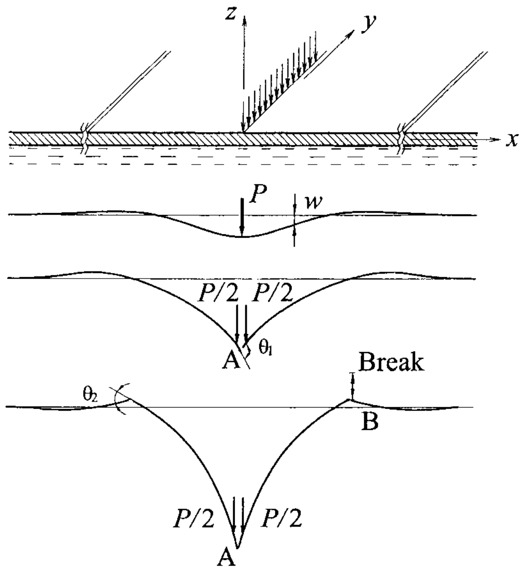

Figure 5. Top: Sequence of hinge formation in floating ice plate. Bottom: Evolution of load-displacement diagrams as ice thickness h increases (the size effect due to crack initiation is not shown).

whose solution is a sum of complex exponentials. Softening hinges initiate when $M = M_p$ at the location of the maximum bending moment in the ice beam.

The sequence of formation of softening hinges is shown in Figure 5 (top). In the last stage with three hinges, the beam alone (without the foundation) is a mechanism but, for small enough ice thickness h, the solution indicates that the load can increase further. It can be shown that further hinges cannot form; the two symmetric outer hinges spread, creating a continuous hinge segment of a finite length. Such a segment, however, cannot be assumed to transmit a shear force, which causes that the load P_2 at the moment of formation of the second hinge must in fact be the maximum load. Similar to the preceding section, the load-deflection diagrams evolve as shown in Figure 5 (bottom), and the nominal strengths $\sigma_{Ni} = P_i/h$ corresponding to the first and second peaks and troughs can be shown to be:

$$\sigma_{N1} = Aq(h)h^{1/4}, \qquad \sigma'_{Ni} = B\sqrt{12}q(h)h^{-3/4}, \tag{12}$$

$$\sigma_{N2} = (e^{\pi/4}/\sqrt{2})Aq(h)h^{1/4}, \qquad \sigma'_{N2} = 2.588Bq(h)h^{-1/2}, \tag{13}$$

where A and B are constants, $A = \left[3\rho(1-\nu^2)/E\right]^{1/4} 2\sigma_p/3$ and $B = \sqrt{12\rho E/(1-\nu^2)}\, G_f\sigma_p$, and function $q(h)$ is the same as $q(D)$ defined in (2) (since, in plate bending theory, D usually denotes the cylindrical stiffness, for plates we use h instead of D as the characteristic size). For a detailed solution and extensive discussion, see Bažant and Guo (2001).

The difference in scaling between the beams without and with elastic foundation can be easily explained as follows. In the former and latter cases, $K_i \propto 1$ and $K_i \propto h^{3/4}$, respectively. So, according to (9), $P_i P'_i \propto W_i K_i \propto hq(h)$ and $h^{7/8}q(h)$, respectively. Hence, $\sigma_{Ni}\sigma'_{Ni} \propto h^{-1}q^2(h)$ and $h^{-1/4}q^2(h)$, respectively. Since $\sigma_{Ni} \propto q(h)$ and $q(h)h^{1/4}$, respectively, we have (Figure 3b):

$$\sigma'_{Ni} \propto h^{-1}q^2(h)/\sigma_{Ni} \propto q(h)/h \quad \text{without elastic foundation}, \tag{14}$$

$$\sigma'_{Ni} \propto h^{-1/4}q^2(h)/\sigma_{Ni} \propto q(h)/\sqrt{h} \quad \text{with elastic foundation}. \tag{15}$$

6. Closing Remarks

Softening damage generally causes a size effect. The present results confirm this fact. However, it might be surprising that the size effect due to a mechanism of softening hinges is very different from that previously shown for various types of two- or three-dimensional structures (Bažant and Chen, 1997; Bažant and Planas, 1998). Nevertheless, on closer examination, the kind of size effect obtained here is not all that surprising.

The previous studies generally dealt with two- and three-dimensional structures failing due to propagation of one dominant crack or one dominant damage band. In those kinds of failure, the propagation of the crack or damage band generally releases stored strain energy which increases with the structure size *quadratically* (for the same nominal stress, and for geometrically similar failures), while the energy consumed and dissipated by the crack or damage band increases *linearly*. This energetic inbalance is the cause of size effect.

No such energetic inbalance exists for one-dimensional structures such as beams and, therefore, a similar kind of size effect cannot be expected in failures due to softening hinges.

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