Random Lattice-Particle Simulation of Statistical Size Effect in Quasi-Brittle Structures Failing at Crack Initiation

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Abstract: The modeling of statistical size effect of concrete structures that fail at crack initiation is studied, with special attention to the interaction between the autocorrelation length and the size of the failure zone. The mechanical failure of concrete is modeled by a network of axial springs with degrading stiffness. The heterogeneity of concrete is idealized by spatial variation of the tensile strength and fracture energy. As an example, the direct tensile test in plane stress, with the size range of 1:20, is simulated. Furthermore, simulations of a four-point bending test with varying bending span are compared to the experimental results reported in the literature. The interaction of the autocorrelation length and the size of the failure zone is identified as a key parameter for the modeling of statistical size effect.

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Introduction

Until the 1980s, any observed size effect on the nominal strength of structures was attributed to material randomness. This view is now known to be correct only for perfectly brittle materials, such as metals embrittled by fatigue, or fine-grained ceramics. Beginning in 1984 (Bažant 1984), it has gradually been established that quasi-brittle materials also exhibit a strong energetic size effect, which is not statistical.

Two basic types of energetic size effect may be distinguished, namely, Type 1 and Type 2 size effects. The latter one is caused by energy release engendered by stress redistribution due to large cracks or notches. It will not be considered here since its mean has a negligible statistical component. This study will deal only with the Type 1 size effect, which is both statistical and energetic. Same as in perfectly brittle structures, the statistical part is explained by Mariotte's idea (Mariotte 1686) that, the larger the structure, the lower is the minimum random strength value encountered in it, which was two and one-half centuries later formulated mathematically and verified experimentally by Weibull (1939). The energetic part of the Type 1 size effect is caused by the stress redistribution due to a large fracture process zone (FPZ), the size of which is not negligible compared to the cross section dimension of the structure.

The Type 1 size effect occurs for a broad class of (unnotched) structures of the so-called positive geometry. This is a geometry characterized by a positive derivative of the energy release rate with respect to the crack length (Bažant and Planas 1998; Bažant 2002). If the geometry is positive, the structure fails (under load control) at the initiation of macrocrack propagation from the FPZ.

The quasi-brittle heterogeneous materials include concrete (the archetypical example studied first), toughened ceramics, various types of fiber composites, wood, sea ice, rocks, stiff soils, snow slabs, rigid foams, paper, carton, bone, etc. They consist of brittle constituents and, due to their heterogeneity, develop a FPZ of nonnegligible size. Every brittle material becomes quasi-brittle on a sufficiently small scale, which means that the quasi-brittleness, with the associated size effect, becomes particularly important on the micro- and nanoscales. The FPZ width l_0 typically represents about two to three maximum inhomogeneity sizes and equals the size of the representative volume element (RVE) of quasi-brittle material.

A powerful approach to predicting the strength of structures made of concrete and similar materials is the discrete computer simulation of the microstructure, initiated in Kawai (1977) and Cundall and Strack (1979) and further developed in Bažant et al. (1990) and Jirásek and Bažant (1995a,b). The objective of the present paper is to extend this approach to the modeling of statistical-energetic size effect of Type 1.

Random Strength, Size Effect, and Quasi-Brittleness

If the local material strength is random, and if the structural geometry is positive, the strength of a perfectly brittle structure must follow the Weibull probability distribution function (pdf) (Weibull 1939) [mathematically first derived by Fisher and Tippett (1928) from the stability postulate of extreme value statistics (Gumbel 1958)], and the strength threshold must vanish. The perfect brittleness is closely approached only if the structure is much larger than one RVE or, more precisely, if the structure contains at least 10^4 equivalent RVEs [the equivalent number of RVEs is obtained from their actual number by scaling it according to the stress field, which is geometry dependent; see Bažant and Pang (2007)]. The Weibull pdf ensues mathematically as the asymptotic case of the weakest-link chain model when the number of links tends to infinity, under the hypothesis that the pdf of one RVE has a power-law tail when the stress tends to zero. The necessity of a

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power-law tail and of a vanishing strength threshold has recently been shown to be a mathematical consequence of three plausible hypotheses:

1. The frequency (or probability) of failure of interatomic bonds is controlled by the stress dependence of the activation energy barriers on the free energy potential of the atomic lattice;
2. The energy release due to random jumps of an atomic lattice crack causes a change in these activation energy barriers, but the change is only slight when the crack length jumps by one atomic spacing (which implies that the frequency, or probability, of forward and backward jumps differs little); and
3. The transition from the atomic scale to the RVE scale (spanning seven orders of scale magnitude in the case of concrete) can be statistically represented by a model consisting of a hierarchy of series and parallel couplings (or chains and bundles).

Hypotheses 1 and 3 were justified in Bažant and Pang (2005a, 2006) by a simplified consideration of stress dependence of activation energy of interatomic bonds. The more detailed, and more realistic, justification of Hypothesis 2 based on the probability of random crack jumps through an atomic lattice was provided in Bažant et al. (2008). Unlike other statistical strength theories, especially the Weibull theory with a finite threshold, these hypotheses lead to excellent agreement with the existing experimental data on strength histograms and size effect of quasi-brittle and brittle materials, and an extension of these hypotheses to structure lifetime at constant stress leads to agreement with the experimental histograms of lifetime (Bažant and Pang 2007; Bažant et al. 2008). The agreement with experiments verifies the mathematical hypotheses.

In the weakest-link chain model, each link is imagined to correspond to one RVE. If the structure is quasi-brittle, rather than perfectly brittle, the number of links must be considered as finite. It has been shown that the pdf of strength of one RVE, or one link, must be Gaussian (or normal), except for the far-left tail, which must be a power law with a zero threshold. The power-law tail of a quasi-brittle material must extend roughly to the cumulative failure probability P_f between 0.0001 and 0.01; typically $P_f \approx 0.001$ (Bažant and Pang 2006, 2007, 2005b,c). That the entire pdf of the strength of one RVE cannot be Weibullian is proven by impossibility of the opposite: if it were it would mean that the assumed RVE behaved as a chain of some sub-RVEs, but then the failure would have to localize into one sub-RVE, implying that the assumed RVE could not be the actual RVE. If, on the other hand, the power-law tail existed only for $P_f < 10^{-5}$, even small structures would exhibit no deviations from the classical Weibull theory (i.e., would not behave as quasi-brittle), and if the entire pdf of each RVE were Gaussian, then the structures consisting of such RVEs would never behave as Weibullian.

If the weakest-link chain-of-RVEs model has a finite number of links, the Gaussian distribution with a power-law tail for each RVE leads to a significant deviation from Weibull power-law size effect, increasing as the structure size decreases (Bažant and Pang 2006, 2007, 2005b,c). The Weibull size effect on the mean structural strength characterizes perfectly brittle behavior and is asymptotic, being approached closely for structures equivalent to a chain of more than about 10^4 RVEs.

It must be emphasized that, for the purpose of damage or failure analysis, the RVE cannot be defined by the homogenization theory, i.e., as the smallest control volume for which the first, or first few, statistical moments of random material properties do not change as the RVE is displaced through the material. Rather, the RVE must be understood as the smallest (size-independent) volume of material whose failure causes the whole structure of positive geometry to fail (under load control conditions) (Bažant and Pang 2006, 2007, Bažant and Pang 2005b,c). Briefly, the reason is that the first few statistical moments of RVE strength distribution are irrelevant for the strength of large structures of positive geometry and only the far-left tail of the pdf matters.

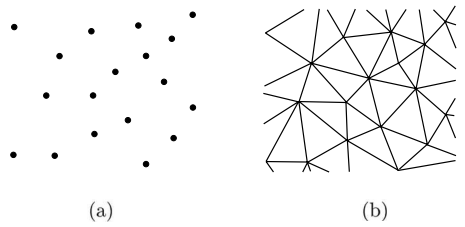

Fig. 1. Generation of the spring network: (a) random placement of vertices; (b) Delaunay triangulation

Because multiscale methods can determine only the low-order statistical moments [particularly the mean and coefficient of variation (CoV)] of the cumulative distribution function (cdf) of RVE strength, and not the far-out cdf tail, it further follows that these methods are not applicable to softening damage and failure of large structures, as already argued in Bažant (2004). Rather, inverse analysis of size effect and strength histograms are required for this purpose.

A number of investigators documented the statistical size effect experimentally. Among quasi-brittle materials, the most abundant data exist for concrete (Weibull 1939; Wright 1952; Koide et al. 1998, 2000), but Weibull distribution and Weibull size effect have never allowed close comprehensive fitting of test data. To improve the fit, a nonzero threshold was suggested (Weibull 1939), but even then the fit was less than satisfactory [Bažant and Pang (2007), Fig. 10(a)]. It now appears that a perfect fit can be attained with a finite chain of RVEs having Gaussian strength with a power-law tail of zero threshold (Bažant and Pang 2006, 2007, 2005b,c).

Random Particle Modeling Approach and Autocorrelation

In the lattice model, or random particle model, the properties of concrete are characterized by interaction of the large aggregate pieces in the coarse mesostructure. In this study, which is limited to the size effect in tensile failures caused by strength and fracture energy randomness, it will suffice to consider only the simplest planar form of particle model, which is adequate only for tensile distributed cracking and tensile (Mode I) fracture, and for the mean response only, see Fig. 1.

In this form, the movement of particles (lattice nodes or vertices), representing the centers of large aggregate pieces, is characterized only by displacements, and the connecting bars (or springs) resist only axial forces.

A certain degree of local statistical randomness is introduced by the random local geometry of the lattice connecting the particles. The geometrical randomness, however, will be seen to be unable to account for the size effect nor any of the observed statistical scatter of structural strength and fracture energy of concrete. To capture the size effect, it is inevitable to consider the strength of the lattice connections (springs) to be also random.

Clarifying the proper form of connection randomness is the main objective of this paper.

A continuum with identical but uncorrelated random strength distributions at all material points is undistinguishable from a homogeneous continuum of uniform strength. This elementary property is obvious if one notes that the number of material points in an arbitrarily small volume of a continuum is infinite, and so the strength of this volume must be equal to the average point strength. Therefore, only an autocorrelated field of local material strength makes physical sense. The basic question is the autocorrelation length l_0 of this field.

If each lattice bar (or spring) is assumed to have an uncorrelated random strength, independent of the strength of the neighboring bars, no statistical size effect is obtained. This is confirmed by numerical simulations. The explanation is that the strength values will roughly average out within a volume smaller than one RVE. In concrete, the RVE size is about two to three times the spacing of the largest aggregate pieces (or the length of the longest connection bars). So, the finite size of lattice bars does not suffice to introduce a meaningful autocorrelation of the strength field. A larger autocorrelation length must be imposed, at least as large as one RVE.

The objective of this study is to examine the role of the autocorrelation length in the statistical size effect at crack initiation, and, thereby, to clarify the interaction of the autocorrelation length with the size of the RVE (or the FPZ). Attention will be limited to tensile fracture, and in that case a simple two-dimensional lattice of bars resisting only axial forces will be adequate.

It must be emphasized, however, that a realistic model capable of simulation both of the cohesive tensile fracture and the general nonlinear triaxial behavior of concrete with postpeak softening damage in both tension and compression necessitates a three-dimensional lattice. The lattice nodes (particles, vertices) must be characterized by not only displacements but also rotations, and the bars must resist not only axial but also shear forces; see Cusatis et al. (2003, 2006) and Cusatis and Bažant (2006).

Would not a three-dimensional lattice with only axial forces suffice for general triaxial tensile-compressive modeling? In theory it would, but then each shear-resisting mesoscale lattice connection (a bar or spring) would have to be replaced by a shear-resisting sublattice with additional scale refinement of about one order of magnitude, capturing the underlying microstructure of fine sand grains in the mortar matrix. This would increase the number of degrees of freedom by about three orders of magnitude, and would, thus, be computationally prohibitive.

Similar to Jirásek and Bažant (1995a), the lattice bars (springs) are characterized by their strength and fracture energy, both of which are here considered to be random and to follow the grafted Gaussian-Weibullian distribution (Bažant and Pang 2006, 2007, 2005b,c). These material parameters must be considered as spatially autocorrelated.

The present study extends to statistical size effect various advances in particle models, which were presented in Zubelewicz and Bažant ((1987), Bolander and Saito (1998), Bolander et al. (2000, 1999), Roelfstra et al. (1985), Schlangen (1995), Schlangen and van Mier (1992), and Carol et al. (2001). It provides a discrete counterpart and mesostructural basis for the nonlocal Weibull theory (Bažant and Xi 1991), which is a continuum theory capable of capturing both the energetic (deterministic) and statistical parts of size effect on the mean and low-order statistical moments of nominal strength of structure (Bažant 2004). In this theory, in which the local failure probability is considered to depend on the nonlocal (or average) strain in the neighborhood of the continuum point rather than on the strength at that point, the nonlocal averaging has a similar effect as spatial autocorrelation, and leads to a similar deviation from the Weibull power-law size effect (Bažant 2004).

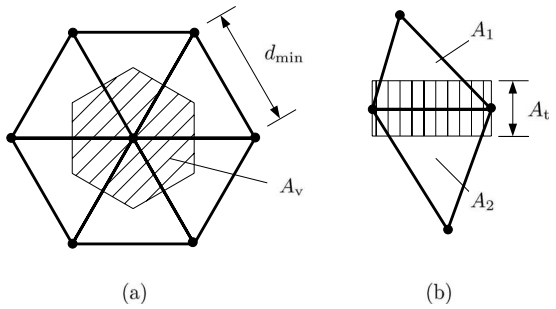

Fig. 2. (a) Definition of the mesh density; (b) approximation of cross-sectional area of struts

Formulation of Particle Model

It is well known that the tensile fracturing of concrete can be described quite well by a lattice (or truss) of bars (or springs) (Bažant et al. 1990; Jirásek and Bažant 1995a,b) in which the tensile failure of each bar is idealized by a uniaxial elastodamage constitutive law.

The lattice of bars is generated by placing vertices (or nodes) randomly into the domain of the given body [Fig. 1(a)] and subsequently performing Delaunay triangulation [see Fig. 1(b)]. The random placement of the vertices must be constrained by a specified minimum distance $d_{\min}$ between any two vertices (Bažant et al. 1990) and between a vertex and specimen boundary. The number of vertices N_v is determined by the specified mean density ρ of the lattice nodes as

$$N_v = \frac{\rho A_s}{A_v} \tag{1}$$

where A_s=specimen area, and $A_v=(d_{\min}^2\sqrt{3})/2$=area associated with one node (vertex) assuming a regular arrangement with a spacing of $d_{\min}$ [see Fig. 2(a)]. Eq. (1) is exact only for the limit of infinite specimen size for which the boundaries have no influence. In addition to the N_v nodes inside the specimen, further nodes, with the mean spacing of d_{ext}, are placed along the specimen boundaries.

The elastic force-displacement relation of the truss bars (springs) is

$$f = K\Delta u \tag{2}$$

where f=axial force; Δu=relative longitudinal displacement between the end nodes (vertices) of the bar; and K=elastic stiffness, which may be expressed as

$$K = \frac{E_t A_t}{L} \tag{3}$$

Here E_t=Young's modulus of the material of the bar; A_t=cross-sectional area; and L=length of bar=distance between the end nodes of the bar. The cross-sectional area A_t is determined so that the area content of all the bars is equal to the area of the specimen. This is achieved by

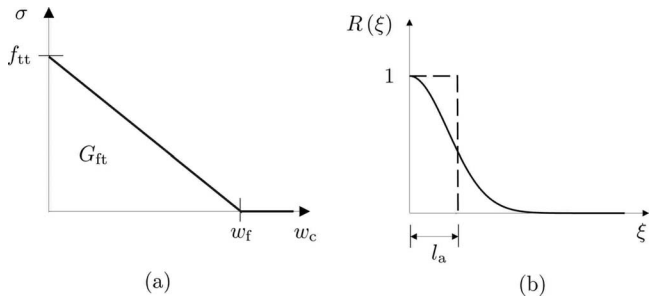

Fig. 3. (a) Stress-crack-opening relation of the struts; (b) autocorrelation function

$$A_t = \frac{A_1 + A_2}{3L} \tag{4}$$

where A_1, A_2=areas of the triangles adjacent to the bar.

The tensile failure is modeled by a uniaxial elastodamage law of the form

$$\sigma = (1 - \omega) E_t \varepsilon \tag{5}$$

where $\sigma = f/A_t$, and $\varepsilon = \Delta u / L$. The damage variable ω is related to the history variable κ as

$$\omega = \begin{cases} 0 & \text{if } \kappa \leqslant \varepsilon_0 \\ 1 - \dfrac{1 - \dfrac{\varepsilon_0}{\kappa}}{1 - \dfrac{\varepsilon_0}{w_f/L}} & \text{if } \varepsilon_0 < \kappa \leqslant w_f/L \\ 1 & \text{if } \varepsilon > w_f/L \end{cases} \tag{6}$$

where $\varepsilon_0 = f_{tt}/E_t$=strain at peak stress, and w_f=displacement at which the stress vanishes; here, f_{tt}=tensile strength of the bar. Eq. (6) describes a linear stress-crack opening curve [Fig. 3(a)]. The area under this curve is the local fracture energy

$$G_{ft} = \frac{1}{2} E_t \varepsilon_0 w_f \tag{7}$$

κ=history variable defined by the loading function

$$f(\varepsilon, \kappa) = \langle \varepsilon \rangle - \kappa \tag{8}$$

along with the loading-unloading conditions

$$f(\varepsilon, \kappa) \leqslant 0 \quad \dot{\kappa} \geqslant 0 \quad \dot{\kappa} f(\varepsilon, \kappa) = 0 \tag{9}$$

[often called incorrectly the Kuhn-Tucker conditions (Jirásek and Bažant 2002)]. The symbol $\langle \ldots \rangle$ in Eq. (8) is the positive-part operator (or Macauley brackets), defined as $\langle x \rangle = \max(x, 0)$.

The lattice exhibits two kinds of randomness of the microstructure: (1) the geometrical randomness, which is automatically introduced by random generation of the lattice nodes; and (2) the physical randomness, due to random distribution of the strength of lattice bars.

However, the effect of geometrical randomness on structural strength distribution is found to be negligible. Thus, the randomness observed in experiments must be attributed almost totally to physical randomness.

It is expected, and is verified by computations, that if the material strength of each lattice bar were considered as an independent random variable, its effect on the coefficient of variation of the material would be nil. Hence, it is essential to consider the strength of the lattice bars to be an autocorrelated random field.

If the autocorrelation length l_a were much smaller or much larger than the size of the representative volume element (RVE) of material, the physical randomness of lattice bars would have no effect on the coefficient of variation of the strength of specimens that are not much larger than the RVE. Therefore, l_a must be of the same order of magnitude as the RVE size (though not smaller than that size).

The randomness of microstructure of concrete is controlled by variable ε_0 in Eq. (6), which is here considered to be an autocorrelated random field. The random property of each lattice bar is given by the value of the field of ε_0 at the midpoint of the bar. The parameter ε_0 influences both the fracture energy G_{ft} and the tensile strength f_{tt}, and, thus, it seems realistic to assume that these two material parameters are fully correlated. The random field is characterized by an exponential autocorrelation function (independent of the spacing of the particles) and by a combined Gaussian-Weibull probability distribution function formulated in detail in Bažant and Pang (2007).

The generation of the random field is divided into two steps. First a Gaussian field is generated using the spectral representation according to Shinozuka and Jan (1972) and Shinozuka and Deodatis (1996). An exponential autocorrelation function of the form

$$R(\xi) = e^{-|\xi|^2/b^2} \tag{10}$$

is used, where ξ=separation distance. The autocorrelation length l_a is defined according to Fig. 3(b) and is related to the parameter b in Eq. (10) as

$$b = \frac{2 l_a}{\sqrt{\pi}} \tag{11}$$

In the second step of the generation of the random field, the Gaussian cdf is converted to the grafted Gaussian-Weibull cdf as

$$x_{\mathrm{GW}} = F_{\mathrm{GW}}^{-1}[F_G(x_G)] \tag{12}$$

where x_G and x_{GW}=values, and F_G and F_{GW}=cdfs of the Gaussian and the grafted Gaussian-Weibull field, respectively. The conversion leads to a change of the autocorrelation function defined in Eq. (10).

Yamazaki and Shinozuka (1988) developed an iterative approach, in which the autocorrelation function of the Gaussian field is modified so that the target random field has the desired cdf and autocorrelation function. In the present simulations, however, the random field was accepted after the first conversion, which means that the autocorrelation function was only approximate while the cdf was accurate. This approximation appears to suffice for the present purpose.

The grafted Gaussian-Weibull cdf is defined as

$$F_f = \begin{cases} r_f (1 - e^{-\zeta^m}) & \zeta \leqslant \zeta_{\mathrm{gr}} \\ r_f (1 - e^{-\zeta_{\mathrm{gr}}^m}) + \dfrac{r_f}{\delta_{\mathrm{Gn}} \sqrt{2\pi}} \displaystyle\int_{\zeta_{\mathrm{gr}}}^{\zeta} e^{-(\zeta' - \mu_{\mathrm{Gn}})^2/(2\delta_G^2)} d\zeta' & \zeta > \zeta_{\mathrm{gr}} \end{cases} \tag{13}$$

with the normalized mean $\mu_{\mathrm{Gn}} = \mu_G / s_1$ and the normalized standard deviation $\delta_{\mathrm{Gn}} = \delta_G / s_1$ of the Gaussian part, where s_1=scaling factor in $\zeta = x_{\mathrm{GW}} / s_1$; m=Weibull modulus (or tail exponent of cdf of RVE strength), and r_f in Eq. (13) = scaling factor required to normalize the grafted cdf; it has the form

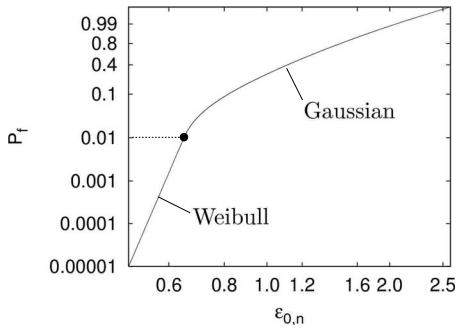

Fig. 4. Grafted Gaussian-Weibull cumulative distribution function (cdf)

$$r_f = [1 - F_G(\zeta_{gr}) + F_W(\zeta_{gr})]^{-1} \tag{14}$$

where $F_W(\zeta_{gr})$ and $F_G(\zeta_{gr})$=probabilities of the Weibull and the Gaussian distributions at the grafting point ζ_{gr}.

The overall mean μ and the overall coefficient of variation c_v are determined iteratively by keeping the Weibull modulus m and the grafting probability P_{Gr} constant and changing the scaling factor s_1 and the mean and standard deviation of the Gaussian part μ_g. For a detailed description of the formulation of the grafted cdf and its calibration, see Bažant and Pang (2005a,b,c, 2006, 2007).

The cdf for $m=12$, $P_{gr}=0.01$, and $c_v=0.2$ is shown in Fig. 4 on the Weibull probability paper, on which the Weibull cdf is a straight line of slope m. Furthermore, two representations of the random field for $l_a=0.02$ m and $l_a=0.04$ m are shown in Fig. 5.

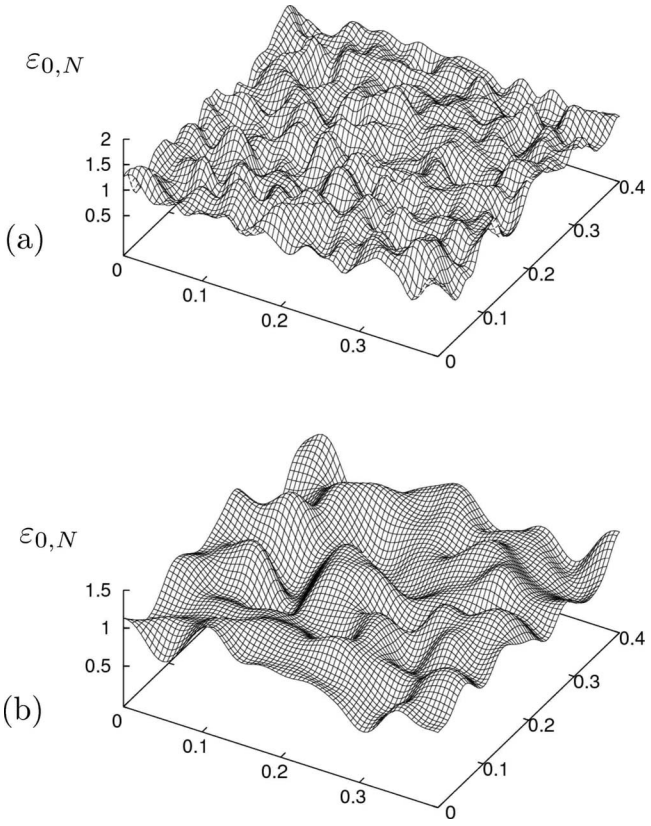

Fig. 5. Autocorrelation random fields for (a) $l_a=0.02$ m; (b) $l_a=0.04$ m

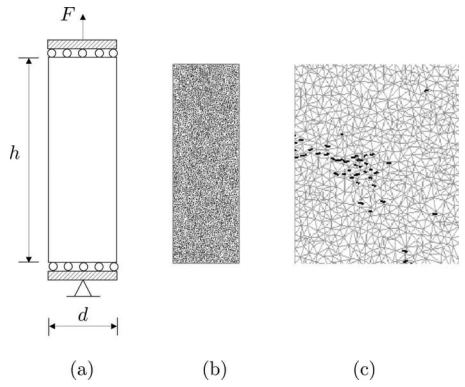

Fig. 6. (a) Geometry and loading setup of the direct tension test; (b) mesh of the large specimen ($d=0.4$ m); and (c) detail of the failure zone at peak load, damaged springs are marked with black lines

Determination of Model Parameters

The parameters of the lattice are not directly related to the macroscopic material properties of concrete. The macroscopic elastic Young's modulus E can be approximated by

$$E = \frac{1}{3}E_t \tag{15}$$

but this is exact only for a regular lattice (spring network) [Fig. 1(b)], which cannot simulate an arbitrary macroscopic Poisson's ratio ν but (in two dimensions) always gives $\nu \approx 1/3$ (Jirásek and Bažant 1995a; Griffiths and Mustoe 2001).

The macroscopic tensile strength f_t and the fracture energy G_F cannot be identified from a regular network, but must be determined by indirect analysis, e.g., by simulating a direct tension test of small size and a notched compact tension test, respectively. For the macroscopic material parameters $E \approx 35$ GPa, $f_t=2.7$ MPa, $G_F \approx 160$ N/m, inverse analysis provides $E_t=105$ GPa, $\bar{\varepsilon}_0=0.0011$, $w_f=0.0004$ m, where $\bar{\varepsilon}_0$ is the mean of the random field of parameter ε_0. Furthermore, the parameters of the random field are set to $m=24$, $c_v=0.2$, $P_{gr}=0.01$, and $l_a=0.04$ m.

The aforementioned inverse analysis gives only estimates of the proper values of the macroscopic material parameters, because the particle model described in the previous section yields only a relatively crude estimate of the postpeak softening response of concrete. A more accurate description could be achieved by considering also rotations of the particles and connecting them by normal and shear springs, as was done by Kawai (1977), Zubelewicz and Bažant (1987), Bolander and Saito (1998), Cusatis et al. (2003, 2006), and Cusatis and Bažant (2006). In the present study, we nevertheless choose connections with no shear resistance because our focus is not on accurate and general modeling of concrete, but on the understanding of the principles of computational simulation of statistical size effect.

Parametric Study

The interaction of the failure zone size with the autocorrelation length is studied by means of a direct tension test [Fig. 6(a)] with a plane stress state and a constant height-to-width ratio. Geometrically similar structures of five different sizes, with the characteristic dimension d ranging from 0.04 to 0.8 m, are simulated. The influences of the autocorrelation length l_a and the co-

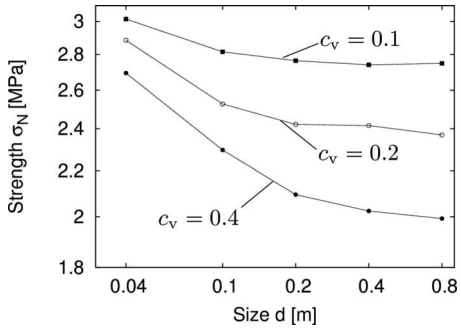

Fig. 7. Nominal strength σ_N versus size d for different material coefficient of variations

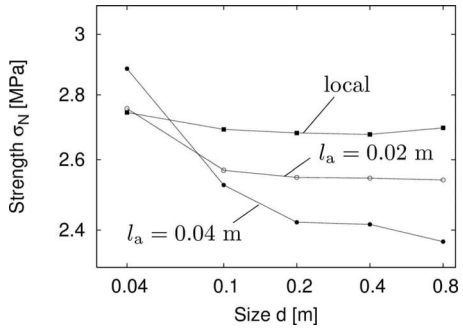

Fig. 9. Nominal strength σ_N versus size d for different autocorrelation lengths

efficient of variation c_v of the distribution of ε_0 are studied separately while keeping the other material parameters constant. The basic model parameters are: $\bar{\varepsilon}_0=0.00011$, $w_f=0.00004$ m, $E_t=105$ GPa, $m=12$, $P_{\mathrm{Gr}}=0.01$, $c_v=0.2$, and $l_a=0.04$ [m].

As mentioned in the section "Formulation of Particle Model," the lattice of bars [spring network, Fig. 6(b)] was, for all the sizes, generated on the basis of $d_{\min}=0.005$ m, $\rho=0.33$, and $d_{\mathrm{ext}}=0.01$ m, with $d_{\min}$ and ρ interpreted as material related parameters independent of the specimen size. For each size, 100 random nonlinear finite element simulations were performed. The peaks of the load-displacement curves were transformed into the nominal strength by dividing the mean of the peak loads $\bar{P}_{\max}=\Sigma_{i=1}^{n=100}P_{\max\, i}/n$ by the width d

$$\sigma_N=\bar{P}_{\max}/d \tag{16}$$

The typical predicted fracture process zone at failure (i.e., at the peak of the load-displacement curve) consists of several damaged elements, which are marked by black lines [see Fig. 6(c), showing the structural details]. Consequently, the size of the fracture process zone is an integer multiple of the lattice element length.

First, the coefficient of variation c_v is varied. The influence on the size effect curve is found to be strong, as shown in Figs. 7 and 8 in the form of the mean and coefficient of variation of the nominal strength versus size d. The greater the value of c_v, the stronger is the size effect on the nominal strength. Obviously, the reason is that, when c_v is increased, the strength of the weakest zone in the specimen is likely to get lower.

Second, the influence of the autocorrelation length l_a on the statistical size effect is studied (Figs. 9 and 10). The lower limit of the autocorrelation length l_a is represented as a random threshold considered as local (independent of any neighborhood, Figs. 9 and 10). For larger autocorrelation lengths, the approach described in the section "Formulation of Particle Model" is used.

As already mentioned, the nominal strength for failure at crack initiation is reached when one size-independent FPZ in the structure fails. For the present geometry and material parameters, the FPZ [Fig. 6(c)] is found to be a region whose size is two to three spacings of major aggregate pieces, and, thus, comprises all the lattice bars lying in this region. Consequently, the nominal strength of the structure is determined not by the failure of the lattice bar (spring) with the smallest strength, but by the average strength of all the bars within the weakest RVE region.

If the autocorrelation length l_a is less than the size of this region, the averaging leads to a considerable reduction of the coefficient of variation of strength of the failure region, i.e., the FPZ, with the consequence that the statistical part of the size effect becomes negligible, as seen in Fig. 7.

If, on the other hand, the autocorrelation length is significantly larger than the FPZ size, the averaging over the bars within the FPZ does not have much influence on the coefficient of variation and, therefore, does not diminish the statistical part of the size effect.

Size Effect in Flexural Strength Tests of Concrete

In addition to the preceding parametric study, four-point-bend tests (Fig. 11) with various bending spans are now simulated and compared to the extensive experimental results of Koide et al. (1998, 2000). The lattice mesh generation is based on $d_{\min}=0.005$ m, $\rho=0.23$, and $d_{\mathrm{ext}}=0.01$ m. Furthermore, the

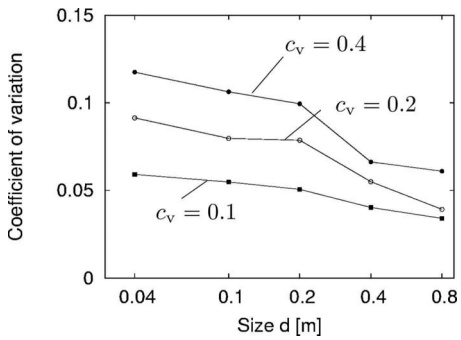

Fig. 8. Coefficient of variation versus size d for different material coefficient of variations

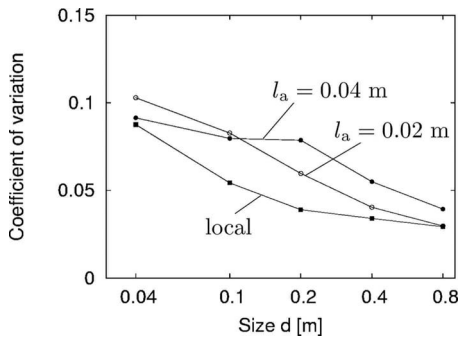

Fig. 10. Coefficient of variation versus size d for different autocorrelation lengths

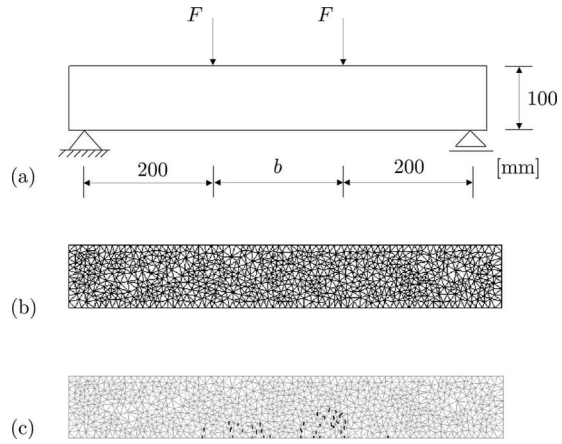

Fig. 11. (a) Geometry and loading setup of the four-point bending test; (b) mesh of the short specimen (b=0.2 m); and (c) active cracks at peak load

model parameters are chosen as E_t=105 GPa, ε_0=0.00012, w_f =0.00004 m, l_a=0.04, m=24, P_{Gr}=0.01, and c_v=0.25.

Earlier studies (Bažant and Novák 2003) indicated that a strangely small Weibull modulus (m=8) was needed to match the same data. However, by the time of those studies, the theoretical result that the cdf of local strength must have a broad Gaussian core and a remote Weibull tail had not yet been established. The present simulations, however, are performed with a combined Gaussian-Weibull cdf, where only the tail follows the Weibull distribution and the core is Gaussian. Note also that Koide's experiments had a relatively small size and narrow size range, which causes the Gaussian core to have a dominant effect on the simulation results (Fig. 12). It is for these reasons that Koide's data can be matched with a realistic Weibull modulus of m=24 for the tail of the grafted cdf, which is in agreement with the histograms and size effect for larger size tests (Fig. 13). As interesting discussion of some related questions of material length was presented in Vořechovský (2007).

Conclusions

1. The geometrical randomness of a lattice (with connecting bars of the same strength) causes only small and unrealistic scatter in structural strength and is unable to model the statistical size effect.

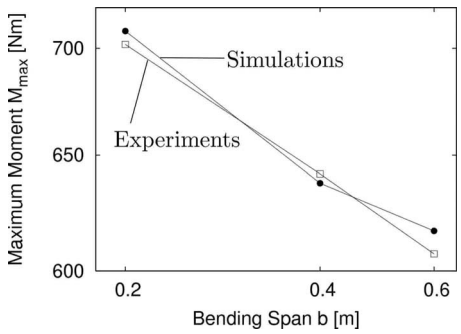

Fig. 12. Maximum moment $M_{\max}$ versus the bending span b for simulations and experiments by Koide et al. (2000)

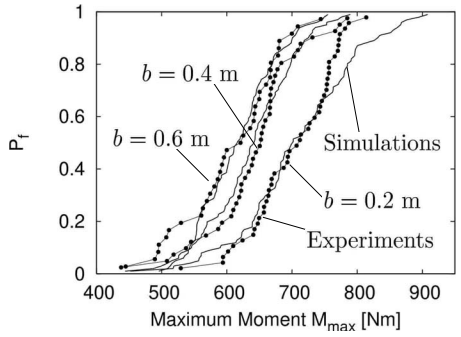

Fig. 13. Distribution function for the maximum moment $M_{\max}$ for three bending spans b for simulations and experiments

2. The ratio of the autocorrelation length l_a to the size of the representative volume element (RVE) [or fracture process zone (FPZ)] is the main parameter that controls the statistical size effect in structures that fail at crack initiation. The simulations document that the statistical part of size effect is significant if $l_a \geqslant$ RVE size and negligible if $l_a \ll$ RVE size. The reason is that what matters is the coefficient of variation (CoV) of the strength of one RVE. If $l_a \geqslant$ RVE size, this RVE has about the same CoV as the local material strength, while if $l_a \ll$ RVE size, it has a negligible CoV. So the experimental evidence of statistical size effect implies that the random material properties in the mesostructure must be autocorrelated.
3. The grafted Gaussian-Weibull cumulative distribution function of material strength, previously deduced theoretically, gives realistic results in qualitative comparison with test data. This provides additional support for its use in describing the statistical size effect over a wide range of sizes.
4. The simulated size effect verifies the fact that, for large specimens, the far-left tail of the cdf of strength of one RVE dominates the structural response, whereas for small and intermediate-size specimens, the cdf core does. This confirms the recent observation that the RVE properties relevant to the analysis of softening damage and failure of large positive-geometry structures can be obtained by neither the homogenization theory nor the multiscale methods, and that inverse analysis of size effect and strength histograms is the only way to obtain the RVE tail.

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