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FRACTURE ANALYSIS OF PENETRATION THROUGH FLOATING SEA ICE PLATE AND SIZE EFFECT

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Abstract. – Penetration of a small object through floating elastic-brittle plate from bottom up or from top down is studied as a two-dimensional fracture problem, in which the wedges formed by radial cracks are treated as thin plates floating on water. The growth of radial cracks is analyzed, with the fracture process zone being assumed to be a point in the plane of plate. Based on field observations, it is assumed that many symmetrical radial cracks grow radially from a small loaded area, and the maximum load is achieved at the initiation of circumferential cracks. The energy release rate due to the radial crack growth is calculated according to linear elastic fracture mechanics, which yields the dependence of the radial crack length on the applied load. Since the circumferential crack initiation is governed by a strength-type criterion, the size effect is not of the type (plate thickness)$^{-3/8}$ previously derived for fracture of floating ice plates. The maximum load is found to depend on the number of radial cracks. As a refinement and extension of a previous idea, a theory of initial crack spacing is proposed to deduce the dependence of the number of radial cracks formed during penetration on the geometry, dimensions and material properties. The size effect is enhanced by such a dependence of the maximum load on the number of radial cracks. The theory also indicates that for extreme sizes there is a change of failure mechanism, from failure by the formation of circumferential cracks in a radially cracked plate to failure by a conical crack.

1. Statement of Problem and Major Assumptions

The penetration problem of an ice plate floating on water has been studied by many investigators; see the excellent survey article by Kerr (1975). The main interest has been focused on determining the maximum load that is required to break the ice plate. The early studies have been confined to determining the critical load, which is the load corresponding to the first occurrence of cracks under the load. However, according to the field observation (Frankenstein, 1967), the critical load does not lead to immediate failure of the ice plate. Beyond the critical load, the ice plate normally accommodates an increase of load by developing various numbers of radial cracks or by forming a conical crack under the load, depending upon the stiffness of the ice plate and the size of the loaded area. To consider the effect of the radial cracks on the maximum load, an infinite wedge plate (with a certain given central angle), floating on water, was studied by some early investigators. Due to the analytical difficulty of finding a solution as well as unavailability of a powerful numerical technique in the early time, only a crude one-term semi-analytic approximation was used. But the accuracy of this solution is doubtful. When the wedge is very narrow, a wedge-beam assumption can be introduced to obtain an explicit solution expressible as an infinite series (Nevel, 1958).

In all these studies, however, three important problems remain unanswered. First, the evolution of the radial cracks is ignored in the infinite wedge models. In reality, there is a relation between the length of the radial cracks and the applied load. The knowledge of this relation is important for understanding the failure mechanism of the ice plate. Second, the number of the radial cracks formed during loading is not studied. According to the field observation, the number of the radial cracks is not a random quantity, but depends on the overall structural characteristics, such as the load size and the material properties of the ice plate. The relation between the overall structural properties and the number of radial cracks is essential for basic understanding of the penetration problem. Third, and most important - perhaps, because of the lack of the knowledge, the size effect on the nominal stress (the applied load divided by the square of the ice-plate thickness), has not been addressed. Knowledge of the size effect is indispensable when experimental results obtained in the laboratory only on relatively thin ice sheets are to be extrapolated to the field.

As the first step of the study, certain simplifying assumptions have to be introduced so that the problem could be solved with reasonable effort. Firstly, the ice sheet is assumed to be describable as a thin elastic plate, with uniform material properties and uniform thickness, resting on water which can be modeled as Winkler's foundation. The deflection is small and the flooding on top of the plate is not considered. The assumption of uniformity of plate thickness and of uniform material properties is rather crude. The reason is that Young's modulus, tensile strength and fracture toughness are functions of temperature. In reality, there is normally a large temperature gradient between the top and bottom surfaces of the ice sheet because the bottom temperature is always near 0°C; thus the material properties vary accordingly across the ice thickness.

Secondly, the applied load is assumed to be distributed uniformly in a circle with radius a_0. Under this assumption, the critical load at which the first crack occurs can be easily obtained using Bessel's functions. When radial cracks are introduced, then, in order to facilitate the calculation, the material within the loading circle is assumed to be removed, and the load is assumed to be distributed uniformly along the rim of the loading circle. Such an assumption is justifiable when the maximum stress that causes the rupture occurs far away from the loading circle. Thirdly, the radial cracks are assumed to be of equal length and uniformly distributed (that is, the angles between any two neighboring radial cracks are equal). This means we assume the problem to remain axisymmetric.

Finally, as for the radial crack propagation, it is assumed that linear elastic fracture mechanics (LEFM) is applicable and the cracks grow in a quasi-static manner, *i.e.*, no dynamic effect would be considered. The front of the radial cracks are assumed to be a straight line perpendicular to the surface of the plate so that it could be treated as a point in the plane of the plate. The radial cracks are assumed to remain totally open along their entire length during propagation. This assumption is actually rather crude, because in reality the radial crack is not totally open. Due to the very nature of the deformation, the upper surface of the ice sheet is always in compression, and the so called radial cracks are actually opened only near the lower surface of the ice plate. As noticed by Frankenstein (1963), the initial radial cracks are either invisible or whitish due to upper surface crushing; the radial cracks could be clearly seen only if the load were removed. This assumption is, nevertheless, employed because without it the frontal boundary of the opening crack would have to be determined, which would require a three-dimensional analysis.

There are further assumptions employed in this study. They will be mentioned or simply implied in the text. Some assumptions, such as the wedge-beam assumption, are well-known, and therefore will not be repeated.

In the following sections, first the mechanical modeling will be briefly described, and then the general deformation and moment distribution will be examined. The wedge-beam solution will be compared with the results of two-dimensional analysis. Fracture mechanics will be applied first to study the relation between the length of radial crack length and the applied load. Finally, the basic theory and main results of the radial crack initiation problem will be discussed.

It is known that radial cracking is not the only failure mode that happens in the penetration of an ice sheet. Under certain conditions, there are no radial cracks formed; rather, a conical crack forms under the load. Although a study of conical crack is beyond the scope of this paper, it would be interest to learn under what conditions the radial cracking becomes impossible. It seems that the present results can be used to estimate a possible conic failure.

The present paper gives only a succinct overview of the method and basic results. The detailed exposition is given in Bažant and Li (1992, 1993).

2. Mechanical Modeling and General Properties of the Solutions

Let an infinite ice plate of thickness h floating on water be subjected to a load from top down or from bottom up. Let $D = Eh^3/12(1-\nu^2)$ be the plate stiffness, and ϱ the specific weight of the water. The quantity $L = (D / \varrho)^{1/4}$ which is called the decay length of the plate-water system, is a basic characteristic of the system. The load is assumed to be uniformly distributed along the circumference of a circle with radius a_0. The material within the loading circle is removed, as is stated before. There are n radial cracks of equal length propagating radially from the center of the loading circle, as shown in Fig.1. In order to analyze this structure, it is further assumed that the deformation is symmetric with respect to each radial crack line and each central radial ray that bisects the angle between the two adjacent radial cracks. Under such an assumption, the infinite plate can be decomposed into n infinite wedges (see Fig. 2 for such a typical wedge), each of which has the following boundary conditions: (1) The circular loading rim is assumed to be free of bending and twisting moments, but subjected to uniformly distributed shear force which is one n-th of the total applied load. (2) Since the radial cracks are assumed to be totally open, the crack surfaces are assumed to be free of both moments and shear forces. (3) On the extensions of these radial cracks, the boundary conditions of symmetry are applied. (4) Theoretically, there is a zero deflection condition applied to the wedge on the infinitely far side, however, the wedge can be truncated at the distance of about $6L$, and the outer circular boundary can be considered free, since the deflection can hardly spread into the part of the wedge whose distance from the applied load exceeds $3L$. Furthermore, since the deformation is assumed to be symmetric with respect to the bisector ray, only one half of the wedge needs to be considered.

The numerical method employed in this study is the finite-difference method. A polar coordinate system is used, and finite-difference approximations of derivatives are introduced along the radial and angular directions. In contrast to the classical finite difference method, these finite difference approximation are not applied directly to the fundamental partial differential equation, but to the corresponding potential energy. In this way the inconvenient exterior (virtual) nodes, which are needed to impose boundary conditions in the classical method, can be avoided.

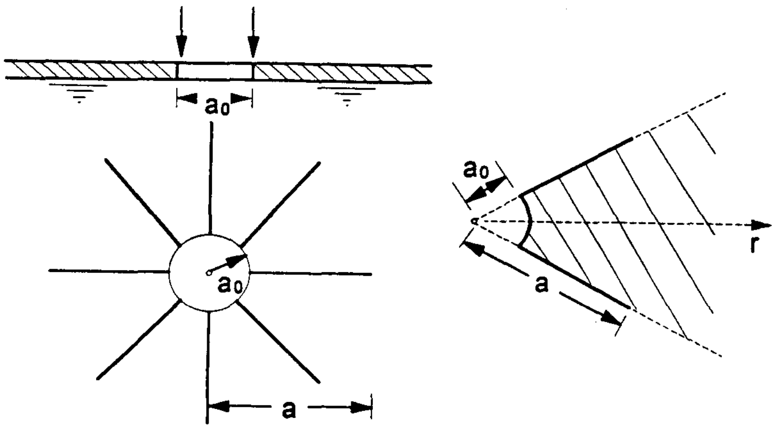

Fig.1(left) Floating plate subjected to vertical load; Fig.2 (right) wedge geometry

It is more convenient to use dimensionless quantities, such as the dimensionless radial crack tip coordinate $\alpha = a/L$ and load radius $\alpha_0 = a_0/L$. In the following discussion, all the mechanical quantities, such as the deflections and internal forces, are calculated for a unit value of $\bar{P} = PL^2/2\pi D$, unless specified otherwise. The average deflections (*i.e.* the average value of deflection along each arc) are plotted in Fig.3. It is seen that the deflection changes with n for a given loaded circle radius. For an infinitely narrow wedge, the wedge-beam solution is also presented in Fig.3 using a dashed line. It can be seen that the wedge-beam solution can be practically regarded as the limit of the wedge-plate solution for large n. The slight difference appears to be due to the fact that Poisson's ratio is not considered in the wedge-beam model. For very small n, the deformation pattern in a wedge is very different from that of a beam; Fig.4a which shows the deflection contour map for a wedge with $n = 2$. However, the deformation pattern becomes quite close to that of a beam even when n is as small as 4, as shown in Fig.4b with $\alpha = 3$, although the deformation close to the crack tip is complex.

It is natural to expect that the moment distribution is also complicated; see the angular profiles of moment components in Fig.5a for $n = 2$ and Fig.5b for $n = 6$. The stress singularity in the neighborhood of the radial crack tip (at dimensionless radial distance $x = r/L = 3$) is manifested by an abrupt increase in moment magnitude. The maximum principal moments always occur on the surface of radial cracks. As the length of radial crack increases, the strength of crack tip singularity decreases, as demonstrated in Fig.6; meanwhile a negative peak appears in the second principal moment. The singularity at the crack tip is the cause of radial crack propagation, and the negative moment is the cause of circumferential crack formation in the wedges. In this sense, Fig.6 explains why initially in the process of loading there is radial crack propagation, then the wedges are formed by radial cracks, and finally the wedges are broken by the circumferential cracks.

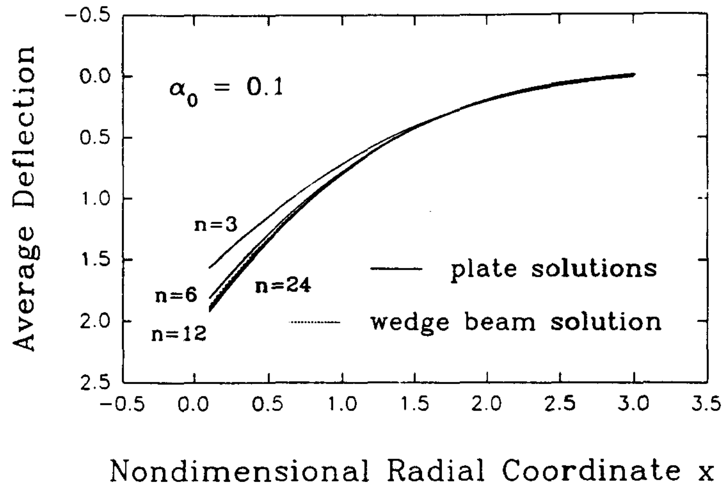

Fig.3 Average radial deflection profiles of wedge plates

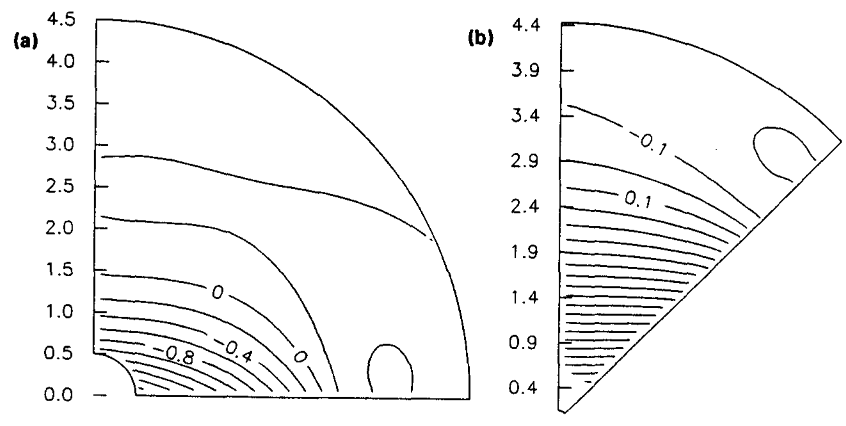

Fig.4 Deflection contours: (a) $n = 2$, $\alpha = 3$; (b) $n = 4$, $\alpha = 3$

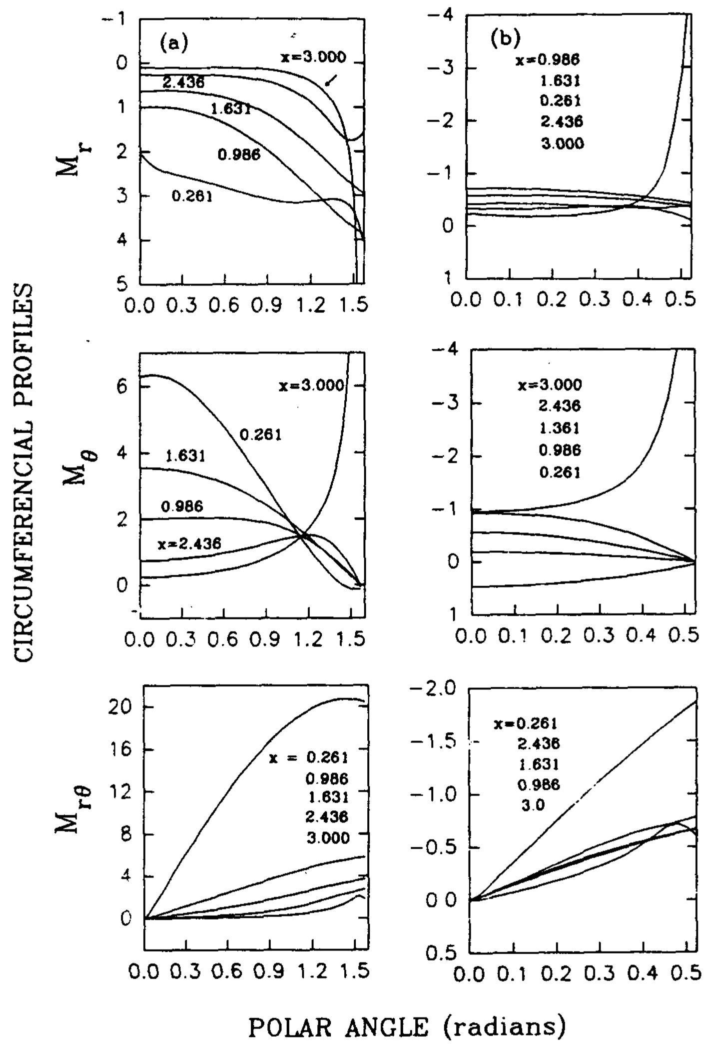

Fig.5 Circumferential profiles of moment components for (a) n = 2 and (b) n = 6

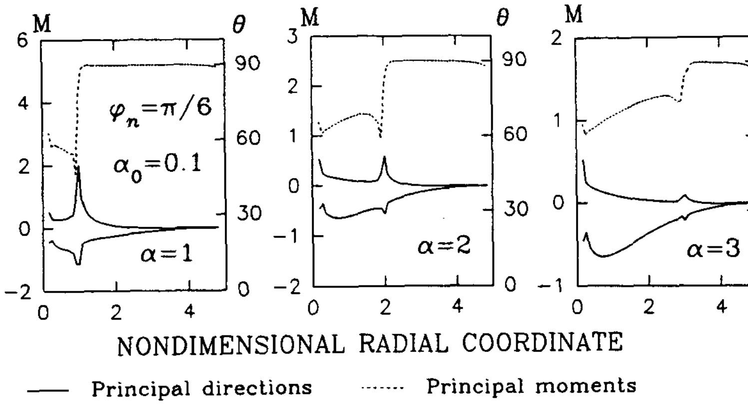

Fig.6 Distribution of the principal moments along radial cracks (n = 6)

3. Energy Release Rate and Crack Propagation

Denote the average deflection of the loading edge as $w_0 = \bar{P}\, f(\alpha,\alpha_0,n)$. The average compliance of the ice plate is $C = w_0/P$, which is proportional to the function $f(\alpha,\alpha_0,n)$ that is plotted in Fig.7a. As can be seen, when the radial cracks become sufficiently long, their compliance becomes almost constant (though still with a small oscillation which is common to all Bessel-type functions). The energy release rate $G = (P^2/2)(dC/d\alpha)$ is proportional to the function $df/d\alpha$. According to linear elastic fracture mechanics, $G = nhG_f$, which leads to the condition that the load required to propagate n radial cracks is proportional to $(df/d\alpha)^{-1/2}$. Thus, when $df/d\alpha$ approaches zero, the required load would approach infinity, which means that the radial cracks cannot propagate any more. The smallest value of α at which $df/d\alpha = 0$ is called the (non-dimensional) limiting radial crack length (Fig.7b), which is a function of α_0 and n.

When the length of the radial cracks is close to its limiting value, the applied load corresponds to a state in which the maximum stress at the bottom of the wedges is equal to its tensile strength f_t. As a result, circumferential cracks occur in the wedges, which leads to the final failure of the ice plate. The maximum tensile stress that causes circumferential cracks can be calculate as $3PM_{max}/\pi h^2$ (where M_{max} is the magnitude of the minimum negative moment under the unit value of $\bar{P}$). Since this stress must be equal to tensile strength f_t, the load level P_{max} can, therefore, be determined. Furthermore, since the energy balance equation should always be satisfied, one can also obtain an equation to solve for the length of radial cracks. For a typical ice plate, such a length is usually very close to the limiting crack length, and that is why the limiting length is a useful property.

Note that the negative M_{max} is a function of α_0, α and n. For large n, M_{max} is plotted in Fig.8a as function of α, which shows that M_{max} remains constant for large α and the constant value increases with decreasing α_0. What matters is the M_{max} value when α is sufficiently large. The location of M_{max} initially moves outward with α later but stays at a fixed position which depends on α_0, as shown in Fig.8b. When n is changed, the general behavior of M_{max} remains the same, although the actual values are changed. Fig.9a demonstrates the variation of M_{max} with respect to n for various α_0.

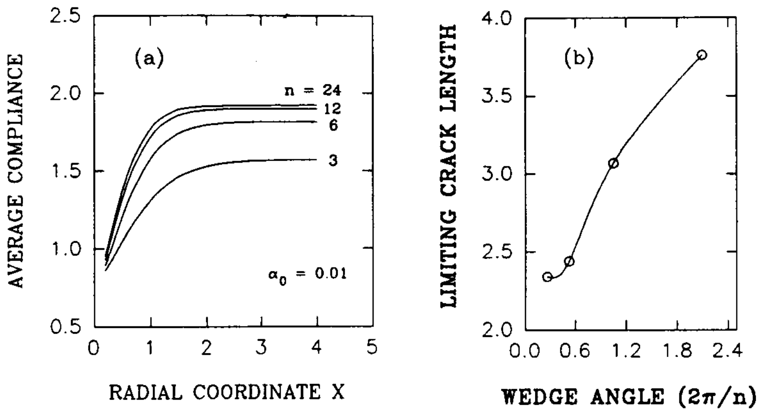

Fig.7 (a) Average compliance; (b) Limiting radial crack length

It is interesting to realize the benefits of using fracture mechanics in the ice penetration problem. Let P_f be the load level at which the first crack occurs, that is, when the maximum tensile stress reaches f_t of the ice plate. Assume that the applied load is uniformly distributed within a circle; then the solution is $P_f = f_t h^2/C(\alpha_0)$, with $C(\alpha_0) = 3(1+\upsilon)\mathrm{kei}'(\alpha_0)/\pi\alpha_0$. Here a prime denotes a derivative, and kei is one of the Kelvin functions (see, for example, Timoshenko and Woinowsky-Krieger, 1959). The ratio P_{max}/P_f is plotted in Fig.9b as a function of n for various given α_0. As it is seen, for a small α_0 value, the ratio can be significantly greater than 2. In other words, P_f would be a much too conservative estimate of the punch-through load.

4. Determination of the Number of Radial Cracks

So far we have not discussed how the number n of radial cracks can be determined. Field observations (Frankenstein, 1963) tell us n is not a fixed number but varies with structural dimensions, such as punch size a_0. Before n is determined, the penetration problem cannot be considered as solved. But, what is the law that governs the number of cracks?

We know that when the radial cracks are infinitely short, the energy release rate is zero and without cracks there can be no singularity. On the other hand, when the non-dimensional radial cracks approach their limiting lengths, the energy release rate will reduce to zero again. Because the energy release rate G is a smooth function of radial crack length, there must be a peak in this function, as is demonstrated in Fig.10. Under load control, the radial crack propagation is unstable before the energy release rate reaches its peak, but becomes stable after the peak. Therefore it is seen that once radial cracks have formed, the cracks will not stop until the peak is passed. Because after the peak the propagation is stable, the crack growth stalls unless the applied load is increased.

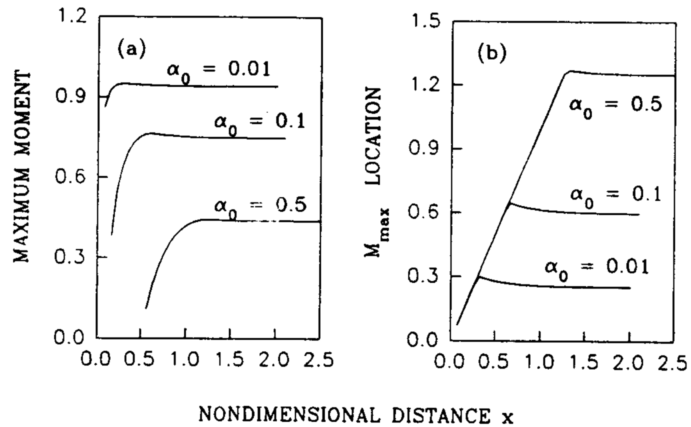

Fig.8 (a) Maximum moment as function of nondimensional radial crack length and (b) radial position of maximum moment

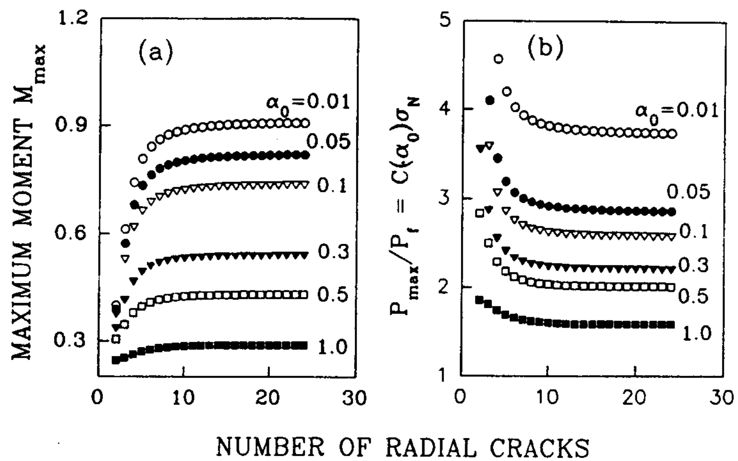

Fig.9 (a) Maximum moment and (b) the load factors *p*

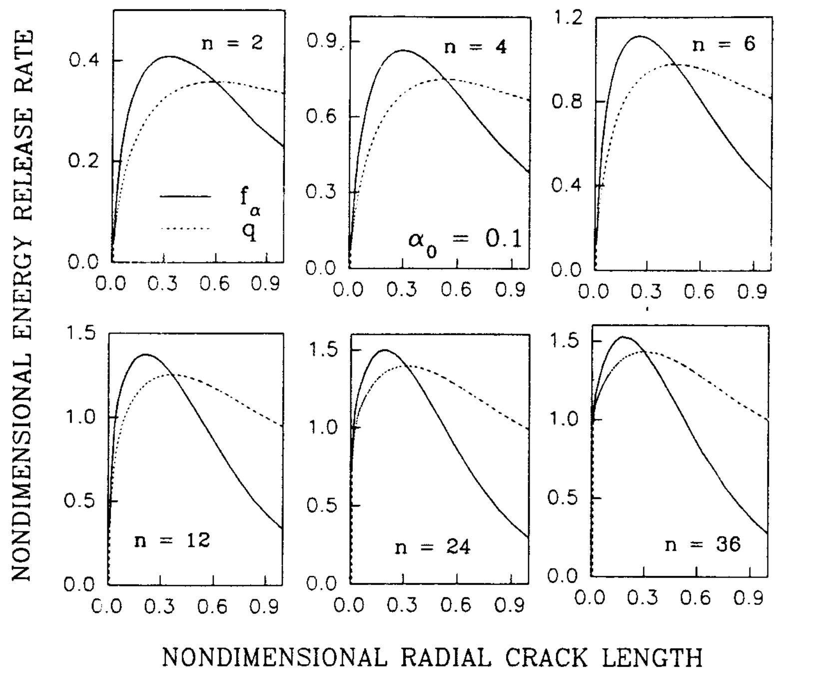

Fig.10 Tangent slope f_α and secant slope q of the compliance (α_0 = 0.1)

It seems reasonable, then, to assume that during the sudden crack jump, all the potential energy released from the ice plate is transformed into the surface energy to form new crack surfaces. Define an average energy release rate G_a during the finite crack jump, which is proportional to $[f(\alpha_0,\alpha,n)-f(\alpha_0,0,n)]/\alpha$. Geometrically, this expression is the secant slope, instead of the tangent slope in the ordinary energy release rate definition, of the function $f(\alpha_0,\alpha,n)$. According to our hypothesis, the average energy release rate should also be equal to nhG_f, which leads to a condition requiring the secant slope and the tangent slope of function $f(\alpha_0,\alpha,n)$ to be the same. It is an elementary mathematical fact that when such a condition is satisfied, the secant slope takes its maximum value, which will be henceforth denoted as $Q(\alpha_0,n)$. In Fig.10 the secant slope is dashed lines, and it is clearly demonstrated that each pair of the curves always intersects at the peak of the dashed lines. This condition is important because it establishes a relationship between n and the initial radial crack length.

With these notations, the average energy release rate can be written as $P^2L\, Q(\alpha_0,n)/4\pi D$ which is equal to nhG_f. Note that the crack jump would not happen if the applied load were less than P_f, which is the load level when the maximum stress in the ice plate first reaches the tensile strength f_t. When $P = P_f$, it is not clear whether unstable crack propagation would occur, and it is likely that some inelastic, yet stable, deformation may precede the unstable crack propagation. However, since the objective of this study is to formulate the basic concept of crack initiation, we assume, for the sake of simplicity, that unstable crack propagation occurs immediately when P reaches P_f. After substitution and some rearrangement, the above average energy release rate equation can be written as

$$n/Q(\alpha_0,n) = (L/l_o)3(1-\nu^2)/\pi C^2(\alpha_0)$$

where $l_o = EG_f/f_t^{\,2}$ — a material length characteristics often used in nonlinear fracture mechanics. From this equation, the number of radial cracks can be determined once L/l_o and α_0 are given. In one-dimensional analysis, in which Q depends only on α_0, the above equation can still be applied to determine n, although the solution becomes very inaccurate for small n.

Some important consequences of this theory now can be discussed. First note that the decay length L is proportional to $h^{3/4}$, and $n/Q(\alpha_0,n)$ is an increasing function of n. Therefore, the number of radial cracks increases with plate thickness h when the dimensionless load size α_0 is kept constant. Furthermore, the same trend remains the same even when the load size a_0 is kept constant. In this case, α_0 decreases with h, and $C(\alpha_0)$ increases with decreasing α_0. However, since $C(\alpha_0)$ approaches infinity as a logarithmic function when α_0 goes to zero, the rate of increase in $C^2(\alpha_0)$ is very slow, thus the overall qualitative behavior remains unchanged.

On the other hand, if the ice plate thickness is held constant and the size of the load is changed, the theory predicts that the number of radial cracks increases with increasing load size (which can be either a_0 or α_0, since L is constant). This is because $C(\alpha_0)$ is a decreasing function with increasing α_0, $1/C^2(\alpha_0)$ increases with increasing α_0. Field observations (Frankenstein, 1963) confirms the prediction for the case of constant h.

Second, it is known that the function $n/Q(\alpha_0,n)$ has a positive minimum value, which occurs at $n = 3$. This minimum places a lower bound for the plate thickness if either a_0 or α_0 is held constant, though different conditions lead to different lower bounds. When h is smaller than the lower bound, the radial cracks cannot be formed. This is because the energy release rate is proportional to the decay length L. When L is too small, the energy released by crack propagation may not be sufficient for the creation of new crack surfaces. Nevertheless, the ice plate would fail anyway. Thus, it is conjectured that below the lower bound, the ice plate would fail by a conical crack, since we do not know whether there is a third failure mode. It should be noted that the lower bound is a function of load size α_0. According to this theory, the lower bound increases with decreasing α_0. On the other hand, if the thickness h (therefore L) is kept constant, the same minimum value of $n/Q(\alpha_0,n)$ imposes a lower bound on α_0. Such a lower bound decreases with increasing thickness.

5. Size Effect on the Nominal Penetration Stress

Except for the initial crack formation, the radial cracks grow stably under load control, which means the applied load P needs to be increased as the radial cracks grow longer. P_{max} is reached when circumferential cracks initiate. The crack initiation is not governed by LEFM but by the strength theory, which exhibits no size effect (i.e. $\sigma_N \sim L^0$). But, as shown, the maximum bending stress that causes the initiation of circumferential cracks occurs when the radial cracks become sufficiently long; therefore, the size effect is a combination of LEFM size effect and strength size effect. The major factor that determines the actual size effect is the maximum tensile stress in the wedge. According to our analysis, the maximum tensile stress (which is in direct proportion to the maximum negative moment) is a function of load size α_0 as well as the number of radial cracks, as is shown in Fig.9a.

The size effect on non-dimensional nominal stress σ_N ($= P_{max}/h^2f_t$) depends on how the similarity is defined. If it is defined so that α_0 be constant for varying h, then there is no size effect if the influences of the modulus of rupture for bending and of changing crack number n are neglected, as shown in Fig.11a. In the calculation it is assumed that l_0= 0.2m and $L = 16h^{3/4}$. The size effect due to a change in n under constant α_0 is plotted in Fig. 11b. The left end of the curves corresponds to the threshold value L_0 as discussed in the previous section.

When the ratio a_0/h is kept constant for various h, a reversed size effect is superimposed on the size effect mentioned above. Since $\alpha_0 = (a_0/h)(h(1-\nu^2)/E)^{1/4}$, α_0 actually increases with h, and we know that when α_0 increases, the maximum value of the nominal stress increases, and thus the size effect gets reversed: the larger is h, the larger is σ_N. However, since the dependence of α_0 on h is very weak in the normal range of thickness, such a reversed size effect can often be neglected. Actually, keeping a_0 constant for all h is the case that is more relevant to the practical problem, in which an aircraft of a fixed and known contact area wants to land safely on the ice plate, or a submarine of a fixed and known contact area of its sail wants to penetrate upward through the ice. For this case, α_0 decreases with h, and so the maximum nominal stress decreases with h too, as is shown in Fig. 11c for several chosen n. This type of size effect has already been discussed by Bažant and Li (1992). The overall effect of changing n is to enhance the size effect under the condition of constant a_0, as can be seen from Fig.11d.

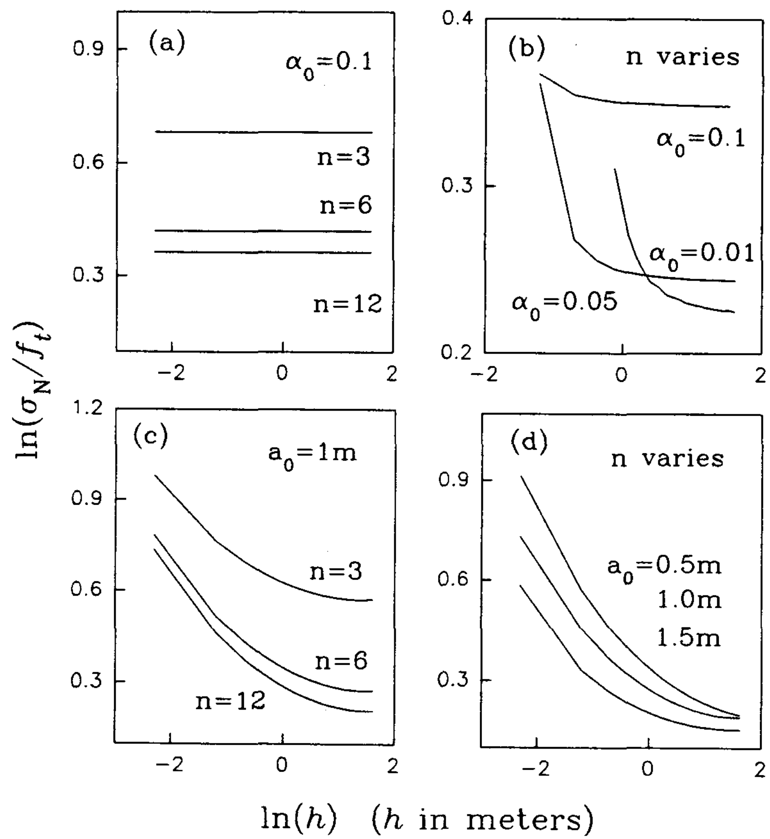

Fig.11 Size effect when (a) constatn α_0 and constant n; (b) constant α_0 and varying n; (c) constant a_0 and constant n; (d) constant a_0 and varing n.

Finally it may be noted that the basic underlying size effect of the type $\sigma_N \propto h^{3/8}$ is the same as in large-scale thermal bending fracture of sea ice plates (Bažant, 1992a, 1992b). This size effect, however, is modified by the fact that the initiation of the circumferential cracks is governed by the strength criterion, and the number of radial cracks varies with h.

6. Conclusions

1) Linear elastic fracture mechanics and Kirchhoff's thin elastic plate theory is used to simulate an ice sheet floating on water that is subjected to a load from a small object from upside down or from bottom up. The failure process observed in the field, in which extensive radial crack growth precedes circumferential cracking, is reasonably reproduced in this analysis.

2) It is demonstrated that fracture mechanics is indispensable in order to make a realistic estimate of the penetration load. Without using fracture mechanics, the obtained estimate is usually too low. The length of radial cracks depends on the applied load level, and failure occurs when the crack length is very close to its limiting value.

3) The number of radial cracks can be determined by considering three conditions: (a) the applied load must be such that tensile strength is reached, (b) the energy release rate equation must be satisfied, and (c) the *average* energy release rate in the process of crack jumping must be equal to the energy release rate. The last condition also provides an equation to determine the initial radial crack length. The theory further indicates that there must be a lower bound on the ice-plate thickness at constant size of loaded zone, or a lower bound on the size of loaded zone at constant ice-plate thickness, below which the ice must fail by another mechanism, likely to be a conical crack.

4) Aside from the size effect on the rupture modulus for bending, the size effect on the nominal penetration stress is basically determined by the maximum tensile stress in the wedge. With radial cracks long enough, the maximum tensile stress is a function of loaded zone size α_0 and the number of radial cracks n. Therefore, the size effect can vary depending upon how similarity is defined. The most useful form of similarity seems to be defined by given load size a_0. In this case, the nominal penetration stress decreases with increasing ice-plate thickness.

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