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Fracture of random quasibrittle materials: Markov process and Weibull-type models

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ABSTRACT: Quasibrittle materials, for example concretes, rocks, certain composites, toughened ceramics and ice, are materials that fail by fracture with a large fracture process zone. They have a heterogeneous random microstructure, which causes significant scatter in material strength. The classical statistical strength models of Weibull-type, as well as some recent random process models for crack growth, are not applicable because they ignore the stress redistributions and energy release caused by large stable crack growth before failure, and the variability of critical energy release rate (the R-curve). This paper presents, in the first part, a new random process model for crack growth which takes the R-curve into account. The second part focuses on another related problem — the Weibull-type model. It reviews a recent nonlocal generalization of Weibull theory which can take into account the existence of a large fracture process zone and stable crack growth prior to maximum load.

INTRODUCTION

Probabilistic fracture modeling of quasibrittle materials such as concrete is a problem of formidable complexity which must be simplified to be tractable. The simplifications required should obviously be patterned after the probablistic theories for the fracture of metals, although with certain salient special features which reflect the fact that the size of fracture process zone in quasibrittle materials is normally comparable to the cross section dimensions and that quasibrittle structures exhibit a significant stable fracture growth before the maximum load is reached. In contrast to metals, the fractures at maximum load of concrete structures typically occupy 50% to 90% of the cross section.

The present brief conference paper will expound two simplification, one describing the probabilistic nature of fracture from the static viewpoint and another from the evolutionary viewpoint. The former simplification represents an adaptation of Weibull theory in which the failure probability is estimated on the basis of the stress state of the structure before failure. This approach makes it possible to deduce a simple law for the effect of the structure size. The latter simplification treats fracture propagation as a random process, for which the Markov process is a natural choice. This approach can yield the probabilities of fracture growth at various stages. First we will consider the evolutionary aspect of random process and then will briefy outline the essential results in Weibull-type modeling.

1 MARKOV CHAIN MODEL FOR RANDOM CRACK GROWTH

The previous investigations of fracture in concrete structures concentrated mainly on the deterministic and statistical behaviors at the peak load (Bažant and Kazemi, 1990; Gettu et al., 1990; Bažant and Xi, 1991a, 1991b). A realistic theory is needed to consider the probabilistic nature of the steps in the crack growth process, especially, the question how the survival probability of one elementary volume is influenced by the preceding failure of an adjacent elementary volume. This requires following the incremental jumps of the fracture process in a probabilistic manner. To determine the probabilities of crack growth in each loading step, one must consider the probabilities of the lengths of the jumps of the crack tip for a given load increment when the major crack extends to a certain point. Furthermore, if a structure with a crack of a certain length is surviving at a given load level, one must decide what is the failure probability for a given load increment.

To answer these questions, a stochastic model for crack growth is required. We will now present a probabilistic model that can capture the randomness of progressive crack growth in a quasibrittle material such as concrete. The present method

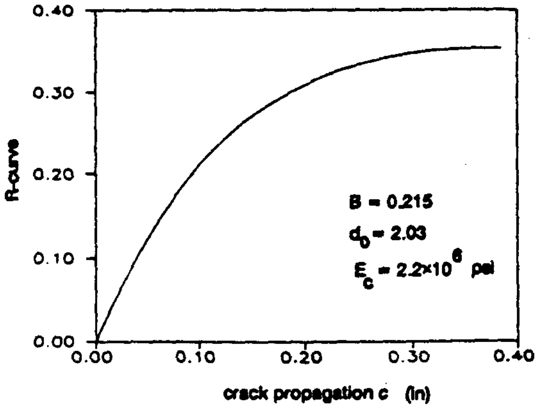

Fig. 1 R-curve

will be based on the Markov chain model and R-curve behavior which is derived from a new generalization of the size effect law combining Weibull statistical theory and nonlocal concept (Bažant and Xi, 1991a, b). The standard deviation of the peak load will be the only statistical information needed for the model. The parameter estimation method will be formulated, and some applications will be illustrated.

1.1 General Formulation

The Markov process (or Markov chain) is a general model commonly used to characterize and simulate many kinds of accumulative damage processes (Bogdanoff and Kozin, 1985). The well-known basic evolution equation for the Markov process is

$$\boldsymbol{p}_x = \boldsymbol{p}_0 \boldsymbol{P}^X \tag{1}$$

where X is the loading level, $\boldsymbol{p}_0$ is the initial state probability (a vector), $\boldsymbol{p}_0 = (\pi_1, \pi_2, \ldots \pi_{B-1}, 0)$, with $\sum \pi_j = 1$, in which $\pi_j =$ Prob(damage state j is initially occupied) and $\boldsymbol{P}$ is the probability transition matrix which characterizes the material properties. Eq. 1 means that the probability of crack advance depends only on the current state, i.e. is independent of the preceding states (the history). In the present study, we assume that always $\pi_1 = 1$, with all other $\pi_j = 0$, which means the crack or damage always starts from state 1; $\boldsymbol{p}_x$ is the damage state probability, $\boldsymbol{p}_x = \boldsymbol{p}_x(1), \boldsymbol{p}_x(2), \ldots \boldsymbol{p}_x(B)$, in which $\boldsymbol{p}_x(j) =$ Prob (damage state j is occupied at stress level X); and B denotes the failure state. Here we assume the crack can propagate only one unit at one loading level, which means a unit jump model is called for; then

$$\boldsymbol{P} = \begin{bmatrix} p_1 & q_1 & 0 & \ldots & 0 \\ 0 & p_2 & q_2 & \ldots & 0 \\ 0 & 0 & p_3 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \\ 0 & 0 & 0 & p_{B-1} & q_{B-1} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

where p_i = probability of remaining in the state i during one loading step, and q_i = probability that in one loading step the damage moves from state i to state $i+1$.

Let the random variable $X_{1,B}$ denote the load at failure reached by starting in damage state 1 at $X = 0$. Then the first two central moments of $X_{1,B}$ are found to be (Bogdanoff and Kozin, 1985):

$$E(X_{1,B}) = \sum_{j=1}^{B-1} (1 + r_j), \tag{3a}$$

$$\text{Var}\,(X_{1,B}) = \sum_{j=1}^{B-1} r_j (1 + r_j) \tag{3b}$$

where

$$r_j = \frac{p_j}{q_j}, \quad p_j = \frac{r_j}{1 + r_j}, \quad q_j = \frac{1}{1 + r_j} \tag{4}$$

We will focus on the statistical scatter of the failure load, which actually represents the macroscopic reflection of the microscopic randomness within the crack process zone. In other words, the failure load will be considered as a fictitious source of randomness, while the real source of randomness of course is the heterogeneity of material properties.

In the case of real engineering problems, the failure load and its standard deviation are the data most likely to be available, especially the failure loads obtained from small specimens. So, a suitable equation has to be introduced as a mean curve. One must realize that this mean equation is not just the equation obtained from fracture mechanics handbooks; it has to be calibrated from the peak load test results and averaged over all the specimens of different sizes. For quasi-brittle materials which show a very strong dependence of the nominal strength on the specimen size, the size effect law proposed by Bažant and Xi (1991a,b) might be one of the available method satisfying such a requirement.

1.2 Mean R-Curve

The deterministic equation for the nominal stress may generally be written in the form:

$$\bar{X} = \frac{\sqrt{R(a - a_0)\ E_c}}{\sqrt{\pi a}\ F(a/d)} \tag{5}$$

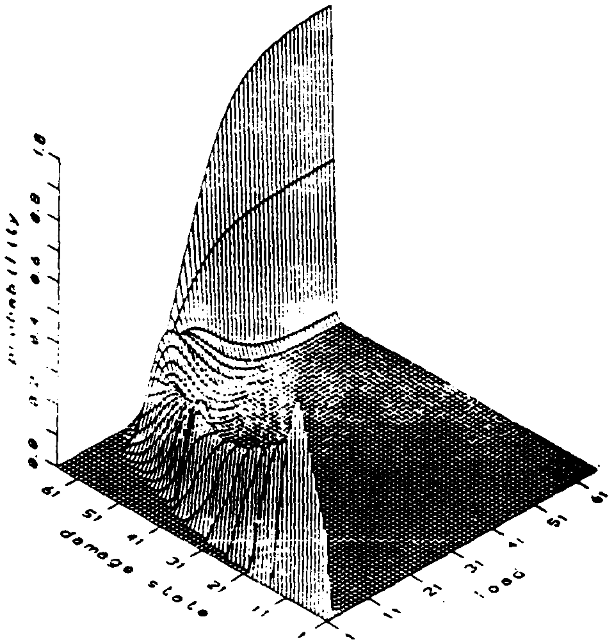

Fig. 2 Probability - loading level - damage state

where $\bar{X}$ represents the mean nominal stress (which is proportional to the applied load). E_c is the initial elastic modulus, $R(a - a_0)$ is the R-curve, $F(a/d)$ is a geometry-dependent function, available in fracture handbooks (e.g. Tada, 1983), a_0 is the current crack length, and a_0 is the initial crack (or notch) length.

The R-curve can be deduced from the statistical generalization of the size effect law proposed by Bažant and Xi (1991a,b). By fitting this law to the test results on maximum load values for geometrically similar fracture specimens of sufficiently different sizes, one can determine the fracture energy G_f of the material (defined as the energy required to propagate a crack in an infinitely large specimen), and the effective length of the process zone, c_f (defined for an infinitely large specimen). Then, according to the method of Bažant and Kazemi (1990), one can obtain the R-curve as follows:

$$R(a - a_0) = G_f \frac{g'(\alpha)c}{g'(\alpha_0)c_f} \frac{1}{1 - \frac{2n}{m}} \qquad (6a)$$

$$\frac{c}{c_f} = \frac{g'(\alpha_0)}{g(\alpha_0)} \left(\frac{g(\alpha)}{g'(\alpha)} - \alpha + \alpha_0 \right) \qquad (6b)$$

$$G_f = \frac{(bf_u)^2}{c_n^2 E} d_0 g(\alpha_0) \qquad (7a)$$

$$c_f = \frac{d_0 g(\alpha_0)}{g'(\alpha_0)} \left(\frac{d}{d_0} \right)^{2n/m} \qquad (7b)$$

in which $c = a - a_0, \alpha_0 = a_0/d, \alpha = a/d, g(\alpha)$ is the nondimensionalized energy release rate obtained from handbooks, and c_n is a factor in the size effect law chosen arbitrarily for convenience (see Bažant and Kazemi, 1990), b and d_0 are two constants (see Eq. 11). When m approaches infinity, Eqs. 6-7 degenerate to the same form as Bažant and Kazemi (1990). By choosing a series of α values, the corresponding c-values are obtained from Eq. 6b, and then, substituting each c into Eq. 6a, the corresponding R-curve values can be calculated. Then Eq. 5 will represent the mean curve of nominal stress as a function of the crack length.

1.3 Variance of Nominal Stress before Failure

As a simplifying approximation, we may assume the variance σ_j^2, at state j, to be proportional to the length of the fracture process zone. This assumption seems reasonable in view of the experience from testing. The length of the process zone at the initial state is 0, and the standard deviation of the load is also 0. At the failure state, the process zone is fully developed and the standard deviation of the failure load must, therefore, also reach its maximum value at the same time. Since the fracture process zone size depends upon the crack length a, the variance at state j may be expressed approximately as a linear function of the crack length a:

$$\sigma_j^2 = \frac{(a_j - a_0)}{(a_{max} - a_0)} \sigma_{max}^2 \tag{8}$$

where a_{max} can be obtained from Eq. 5. σ_{max}^2, representing the variance of peak load, may be considered to be size independent, since, as we already explained, the random scatter is mainly related the size of the fracture process zone during the loading process, and at ultimate state the fracture process zone size is almost independent of the structure size. This has been shown by test results of laser speckle interferometry (Ansari, 1989) and by size effect analysis (Bažant and Kazemi, 1991).

1.4 Markov Chain Model Combined with R-Curve

Based on Eqs. 5 and 8, we can derive the expression for the parameters in Eq. 2 and damage state B_j. First, we divide the damage states $j = 1, \ldots, B-1$ into j groups as follows: $1, \ldots B_1 - 1; B_1, \ldots B_2 - 1; \ldots$; and $B_j, \ldots B - 1$. Then we assume r_1 for $1, \ldots B_1 - 1$; r_2 for $B_1, \ldots B_2 - 1; \ldots$, and r_j for $B_j, \ldots B - 1$. The following recursive equations are obtained

$$B_j = \frac{(\bar{X}_j - \bar{X}_{j-1})^2}{(\bar{X}_j - \bar{X}_{j-1}) + (\sigma_j^2 - \sigma_{j-1}^2)} + B_{j-1}. \tag{9a}$$

$$r_j = \frac{\bar{X}_j - \bar{X}_{j-1}}{B_j - B_{j-1}} - 1 \tag{9b}$$

1.5 Numerical Example

Consider a notched three-point-bend beam specimen of high strength concrete (Gettu et al. 1990) as an example. The R-curve obtained from the peak loads by the size effect method is shown in Fig. 1. Fig. 2 shows the probability of each damage state and nominal stress. One can see that, for example, at loading level 61 (almost the peak load, which is 66) the probability for the occurrence of the damage state 61 (almost the failure state, which is 66) is very high, more than 90%. On the other hand, the probabilities for the occurrence of the lower damage states, 1 - 50, at the same loading level are almost 0; this is true in reality, because at such a high loading level the probabilities for a very low damage state should be very small. The ridge of the probability surface represents the mean path of damage evolution.

An advantage of the present model is that the sample curve can be easily simulated by the computer. Fig. 3 shows the relation between crack extension and loading level. One can clearly see that the generated sample curves represent the observed test curves quite well. This means that the present model can characterize the probabilistic structure for the entire loading history from the initial state up to the failure load.

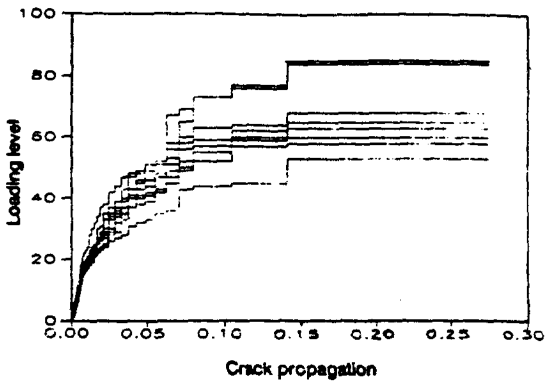

Fig. 3 Sample of load - crack propagation

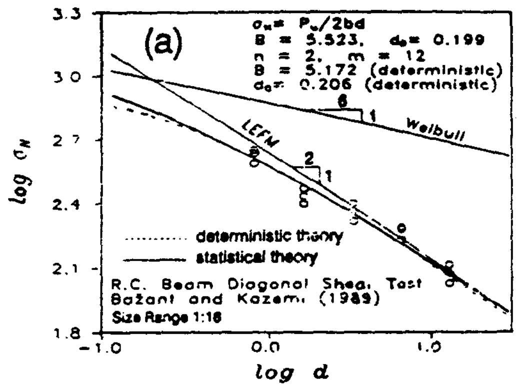

Fig. 4 Test result on diagonal shear failure of reinforced concrete beam without stirrups, and optimum fit by (11)

1.6 Conclusions on Markov Chain Model

1. The present discrete Markov chain model can be used to determine the probabilistic structure of progressive cracking under monotonic loading conditions. The crack propagation probability for an existing crack at any loading step before the peak load and the failure probability from any damage state can be calculated.

2. The determination of model parameters requires a large number of sample curves. When only the peak load data are available and the number of samples for the peak load does not suffice to obtain the statistical parameters required in the model, the mean curve based on fracture mechanics may be used as a substitute.

3. The standard deviation of the mean curve in the entire process of crack propagation may

be predicted from the standard deviation of the peak loads of specimens. The standard deviation of the peak load is affected by the crack process zone. In turn, the crack process zone is affected by the length of crack propagation. Hence, the peak-load deviation can be assumed as a function of the crack propagation length.

4. The R-curve obtained by size effect analysis of a series of experimental results on peak loads for different sizes is employed as the mean curve. The peak load deviation is assumed to be a linear function of the length of crack propagation. The obtained probabilistic structure for the progressive damage process agrees with the general observation of experimental results. (Note: In detail, the Markov process model will be presented in a journal article; Bažant and Xi, 1993).

2 NONLOCAL GENERALIZATION OF WEIBULL STATISTICAL THEORY OF RANDOM STRENGTH

The second part of the conference presentation briefly reviews a recent development of nonlocal Weibull-type theory for concrete and other quasibrittle materials (Bažant and Xi, 1991a,b) and discusses some new consequences of this model. The classical Weibull theory applies only to those case where the maximum load (failure load) is attained at the initiation of the macroscopic crack propagation. Applications have been made to concrete structures, in which large crack growth with large stress redistributions that occur prior to the maximum load are ignored. But such application *fs* are not very realistic. They do not yield the correct size effect.

One might think of remedying the problem by substituting the stress distribution according to linear elastic fracture mechanics into the Weibull-type probability integral, but the integral then diverges. The root of the problem is that the probability of material failure is, in the classical Weibull approach, assumed to depend on the local stress at any given point of the material. This is not realistic, as clarified by the recent deterministic nonlocal continuum models for strain softening damage. In similarity to those models, it is proposed that the material failure probability be considered as a function of the average deformation of a certain neighborhood of a given point. Thus, generalizing the basic results of the classical Weibull theory, the basic hypothesis is that

$$\ln(1 - P_f) = \int_v \sum_{i=1}^{n} \left(\frac{E\bar{\epsilon}_i(x)}{\sigma_0} \right)^m \frac{dV(x)}{V_r} \qquad (10)$$

where P_f is failure probability, x is coordinate vector, E is elastic modulus, σ_0 and V_r are constants, m is Weibull modulus, n is the number of dimension, and $\bar{\epsilon}_i(x)$ are the averaged principal strains.

The most important consequence of the nonlocality is a change in the size effect on the nominal strength, σ_N, of geometrically similar structures of different sizes. It can be shown that (Bažant and Xi, 1991b):

$$\sigma_N = \frac{bf_u}{\left[\left(\frac{d}{d_0} \right)^{2n/m} + \left(\frac{d}{d_0} \right) \right]^{1/2}} \qquad (11)$$

where f_u is the tensile strength, d is the characteristic dimension, and b and d_0 are two constants which can be identified by linear regression of the test results. This law has as the deterministic limit ($m \to \infty$) the size effect law proposed by Bažant and Kazemi (1990).

Eq. 11 has been considered in dealing with the test results by Bažant and Kazemi (1991) on the diagonal shear strength of longitudinally reinforced concrete beans without stirrups, which had a particularly broad size range, 1:16. These results are shown in Fig. 4, in comparison to the optimum fits by Eq. 11, as well as by the deterministic size effect law. Both fits are good, but the difference between them is minor. It has been concluded that, in this type of failure, the deterministic size effect due to energy release dominates, and the statistical contribution to the size effect is unimportant. This may be explained by the fact that, due to localization of cracking, major contribution to the probability integral in Eq. 10 comes only from the fracture process zone, the size of which is almost the same even for specimens of different sizes. However, in other quasibrittle structures, in which such localization of damage prior to the maximum load does not take place, the nonlocal generalization of Weibull statistical theory could be more important.

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