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SEISMIC LOCALIZATION OF SOFTENING CRACKING DAMAGE IN CONCRETE FRAMES

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Abstract

The bending failure in reinforced concrete beams and frames under static as well as dynamic loading is typically caused by localization of cracking into inelastic hinges. This phenomenon, which is of greatest concern for high-strength concrete because of its high brittleness, has a profound destabilizing effect under both static and dynamic conditions. The paper analyzes the behavior at typical static bifurcations with localization for a single column, a portal frame, and a multibay frame. Under dynamic loading near the state of static bifurcation or near the static limit point, the primary (symmetric) path of dynamic response as well as the periodic response becomes dynamically unstable in the sense of Lyapunov. The implication for seismic loading is that the kinetic energy of the structure must be absorbed by fewer hinges, which means faster collapse. The dynamic localizations are demonstrated by semi-analytical solution of horizontal shear excitation of a building column. Regarding the current practice, it is important to note that the usual simplification of a structure as a single-degree-of-freedom oscillator becomes inapplicable after the static bifurcation state is passed.

1 Introduction

Many structures possess some type of symmetry. Usually, symmetry of the structure is reflected in symmetry of the response path in the space of generalized displacements. However, geometrically nonlinear effects or softening-induced localization due to fracture or damage can cause breakdown of the symmetry of response, i.e., a bifurcation of the response path. Generally, the solution corresponding to the secondary, symmetry-breaking path is the solution that actually occurs.

In building frames, softening damage tends to localize into short beam segments called inelastic hinges. When there is more than one inelastic hinge, the damage can further localize into some of the hinges. For example, for a symmetric portal frame subjected to horizontal excitation, there always exists a symmetric solution in which the moments in the corners have at all times equal magnitudes and opposite signs. But if inelastic hinges develop and exhibit softening (i.e., decreasing moment at increasing rotation), any imperfection can cause a large asymmetry in the magnitudes of hinge moments because the softening damage localizes into only one of the two hinges, causing the failure to progress faster.

The localizations due to softening in inelastic hinges have been demonstrated for static loading of reinforced concrete and steel frames (Maier, 1971; Bažant, 1976; Bažant et al., 1987; Hunt and Baker, 1993; etc.). Currently, a problem of great interest is the behavior of building or bridge frames under dynamic loading, e.g., during an earthquake. In reinforced concrete frames, the large deformations that occur in earthquake cause strain-softening damage in the form of distributed cracking of the material and multiple fractures. When such damage occurs only on the tensile size of the cross section, along with tensile yielding of reinforcing bars, there is no significant softening and the hinge is a plastic one. But when such damage occurs on the compressed side of the cross section, the inelastic hinge ceases to be plastic (unless there is a very strong lateral confinement, as for concrete encased in a steel tube). Rather, development of compression splitting cracks and crushing of concrete before the yielding of steel in tension causes gradual flexural softening in which the bending moment decreases at increasing curvature. Such behavior can be especially marked in prestressed beams and in reinforced concrete beams under high axial force, as in columns of tall buildings or wide-span frames with high horizontal thrust. The phenomenon can be particularly pronounced for high-strength concrete, due to its extreme brittleness. Under seismic loading, the softening damage is intensified by load repetitions.

This paper will first present a few typical examples of static equilibrium bifurcations due to softening. Next, the dynamic response of structural models exhibiting static bifurcation or limit points due to softening will be discussed in general terms of nonlinear dynamic systems, and a simple dynamic model of a column will be analyzed. The paper will conclude by comments on practical implications of the present theory. For details, see Bažant and Jirásek (1995).

2 Static instabilities of frames with softening hinges

As a simplifying idealization, the inelastic response can be modeled by lumping it into a single cross section regarded as an inelastic hinge, while the rest of the beam is assumed to respond elastically. The relative rotation θ in the inelastic hinge represents the rotation difference between the ends of the hinging segment caused by the inelastic part of the curvature.

As a first approximation, the moment-rotation diagram may be idealized as linear (Fig. 1a). It is characterized by peak moment M_p and fracture rotation θ_f at which M is reduced to zero.

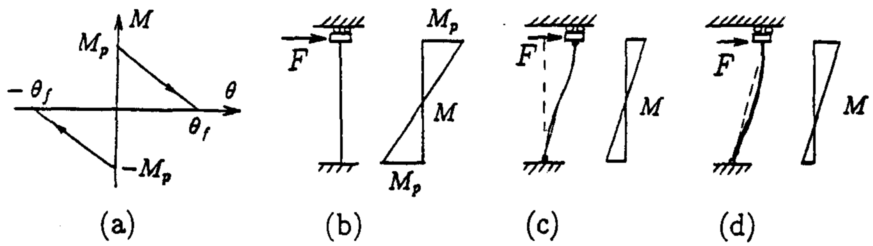

Figure 1: (a) Idealized moment-rotation diagram, (b-d) static model of a column

The simplest model of a typical column in a building frame responding principally by shear is shown in Fig. 1b. The bottom of the column is assumed to be fixed and the horizontal displacement u on top to be prescribed. The column fails by softening hinges which form at the ends. The peak value of the horizontal load F is $F_p = 2M_p/H$ where H is the height of the column.

One possible post-peak response is symmetric, with equal magnitudes of bending moments and hinge rotations at the ends (Fig. 1c). There exists another possible response, representing the bifurcating secondary branch (Fig. 1d). It is nonsymmetric, such that the bottom hinge first rotates while the top cross section (a potential hinge) unloads elastically from the maximum moment state. It can be shown by incrementally linear analysis that the secondary response exists and is stable under displacement control if $0.25 < \beta < 0.5$ where $\beta = \theta_f EI_c/M_pH$ = nondimensional ductility parameter; EI_c

= bending cross-sectional stiffness of the column. For $\beta > 0.5$ the secondary response does not exist because the condition that the top moment cannot increase would be violated. For $\beta < 0.25$ the load-displacement diagram exhibits snapback.

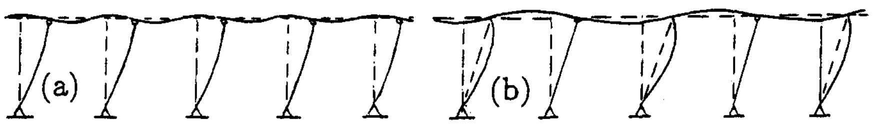

Figure 2: One-story multi-bay frame: (a) symmetric mode, (b) alternating mode

A similar situation arises in a one-story frame with a large number of identical bays. For columns remote from the ends of the row, one may idealize the row of columns as infinite and assume the response to be spatially periodic. One possible mode of collapse is symmetric, with all the hinges softening simultaneously (Fig. 2a). Another possible mode is that in which softening hinges alternate with unloading ones (Fig. 2b). Again, it can be shown by elastic analysis that the symmetry-breaking solution exists and is stable if

$$\frac{(4+3\alpha)(4+\alpha)}{24(2+\alpha)} < \beta < \frac{4+3\alpha}{12} \tag{1}$$

where $\alpha = LI_c/HI_b$ is a nondimensional parameter characterizing the ratio between the bending stiffness of the column and the beam; L= span of the frame, and EI_b = bending cross-sectional stiffness of the beam.

In real frames, even if the number of bays is very large, the moment distribution always differs from the ideally periodic one in the first and last few bays. It might be expected that, in some situations, damage can localize into a single column, and the second column is the best candidate. Analysis of this assumed collapse mode indicates that snapback occurs if

$$\beta \leq \frac{1}{3} + \frac{\alpha(4+3\alpha)}{28+24\alpha+8\gamma+6\alpha\gamma} \tag{2}$$

where $\gamma = \sqrt{3+3\alpha+9\alpha^2/16} - 3\alpha/4 - 2$. A stable solution with damage localized into one hinge can be expected for θ_f slightly larger than the right-hand side of the above inequality.

A similar localization of damage into fewer softening hinges can occur in multi-story frames at any floor. The symmetric mode of collapse, in which hinges form simultaneously on top and bottom of each column and rotate equally, is similar to elastic deformation and is exhibited in plastic (nonsoftening) response. If the hinges are softening, however, various collapse patterns with damage localized into fewer hinges are possible.

3 Numerical analysis of static response

Numerical analysis of softening damage in frames can be based on special beam elements with embedded softening hinges. A detailed development of such elements is given by Jirásek (1995).

The general conditions (1) and (2) valid for an idealized frame with an infinite number of bays can be confirmed by numerical analysis of frames with a large but finite number of bays. As an example, consider a frame with 20 bays (21 columns), $L = H$ and $I_b = I_c$, from which $\alpha = 1$ and $\gamma = -0.1883$. According to (1), a solution with alternating softening hinges is predicted for $0.486 < \beta < 0.583$, and according to (2), a solution with one softening hinge is predicted for β slightly above 0.475. Indeed, the numerical results show that: for $\beta \leq 0.4751$ snapback occurs after the simultaneous occurrence of the first two softening hinges in the second and penultimate column; for $0.4752 \leq \beta \leq 0.487$ damage localizes in the two most critical hinges under increasing floor displacement; for $0.488 \leq \beta \leq 0.583$ damage localizes in every other hinge (starting from the second and penultimate one and progressing from both sides into the interior of the frame); for $0.584 \leq \beta \leq 0.589$ the final failure pattern shows 17 softening hinges; for $0.59 \leq \beta \leq 0.69$ the final failure pattern shows 19 softening hinges (all except the first and last one); and finally for $0.7 \leq \beta$ all the hinges undergo softening. We observe a very close agreement between the analytical prediction and the numerical results.

4 Dynamic instability near static bifurcation or limit point

Consider now what happens near the static bifurcation state under dynamic loading. In dynamics, bifurcation is impossible because it would imply a sudden change of some components of velocity, which is impossible provided that the applied forces remain finite.

Let $\boldsymbol{q}^{(1)}(t)$ (called the primary response) describe the evolution of the displacement vector for given initial conditions and given load vector history. Imagine that the state of the system at the moment when it is passing the static bifurcation point is slightly perturbed, which leads to a different solution $\boldsymbol{q}^{(2)}(t)$, called the perturbed (or secondary) response.

Subtracting the equations of motion for the primary response from those for the secondary response, we obtain for the difference $\boldsymbol{w}(t) = \boldsymbol{q}^{(2)}(t) - \boldsymbol{q}^{(1)}(t)$ the equation of motion

$$\boldsymbol{M}\ddot{\boldsymbol{w}}(t) + \boldsymbol{K}^L \boldsymbol{w}(t) = \boldsymbol{0} \tag{3}$$

where $\boldsymbol{K}^L$ is the tangential stiffness matrix of the structure for loading only at the state of static bifurcation (or limit point). Of course, this equation is valid only for a short enough time interval after the static bifurcation state.

In the vicinity of the static bifurcation state, $\boldsymbol{K}^L$ can be considered constant and the displacement difference can be sought in the form $\boldsymbol{w}(t) = \boldsymbol{a}e^{i\omega t}$ in which i = imaginary unit, ω = constant, and $\boldsymbol{a}$ = amplitude vector (column matrix). Substituting this into (3), we obtain the generalized eigenvalue problem $(\boldsymbol{K}^L - \omega^2 \boldsymbol{M})\boldsymbol{a} = \boldsymbol{0}$.

Because matrix $\boldsymbol{M}$ is always positive definite, all the eigenvalues $\lambda = \omega^2$ of this matrix equation are positive as long as $\boldsymbol{K}^L$ is positive definite. At the first static bifurcation state, the first eigenvalue λ_1 is zero if matrix $\boldsymbol{K}^L$ evolves continuously, and is nonpositive if $\boldsymbol{K}^L$ evolves discontinuously (which is for example the case for softening materials with a bilinear constitutive law). After the state of static bifurcation, we may set $\omega = \pm i\lambda$ in which $\lambda = \sqrt{-\lambda_1} > 0$, and so the displacement difference after the static bifurcation state must contain a growing exponential term $\exp(\lambda t)$.

Imperfections in structures are never zero. No matter how small $\dot{\boldsymbol{w}}^0$ or $\boldsymbol{w}^0$ at the state of bifurcation, the difference $\boldsymbol{w}(t)$ will subsequently grow and deviate from the primary response. This indicates a dynamic instability of the primary solution, which is normally of limited duration (temporary). Of course, the stiffness matrix might later become positive definite again, in which case dynamic stability will be restored. But even in that case, the initial imperfections will have been magnified by the time the lowest eigenvalue of $\boldsymbol{K}^L$ might change its sign again.

The foregoing solution applies to motion near the static bifurcation state. The solution of motion near the static limit point (maximum load) is similar but its character is deduced more simply. It suffices to say that, because of the existence of a negative eigenvalue of $\boldsymbol{K}^L$ after the static limit point, the general solution of the equations of motion must contain growing exponential terms. Therefore, small deviations from the primary solution for the symmetric structure can grow rapidly. This represents again dynamic (Lyapunov type) instability.

5 Semi-analytical solution of dynamic response

Fig. 3 shows a simple dynamic model of a column. The inertial properties of the column are represented by three lumped masses with no rotational inertia so that the rotational degree of freedom ϕ_2 can be eliminated from the equations of motion by static condensation. The remaining degrees of freedom u_1 and u_2 represent the displacement of the top and center, respectively, while the displacement history $u_3(t)$ at the base is assumed to be prescribed. The resulting equations of motion read

$$m_1\ddot{u}_1 + k(30u_1 - 48u_2 + 18u_3 - 9L\theta_1 + 3L\theta_3) = 0 \tag{4}$$
$$m_2\ddot{u}_2 + k(-48u_1 + 96u_2 - 48u_3 + 12L\theta_1 - 12L\theta_3) = 0 \tag{5}$$

where $k = 2EI_c/L^3$. The inelastic rotations in softening hinges θ_1 and θ_3 must be determined by substituting the end moments

$$M_1 = (18u_1 - 24u_2 + 6u_3 - 7L\theta_1 + L\theta_3)\,kL/2 \tag{6}$$
$$M_3 = (-6u_1 + 24u_2 - 18u_3 + L\theta_1 - 7L\theta_3)\,kL/2 \tag{7}$$

into the moment-rotation law.

Depending on which branch of the moment-rotation diagram is currently active, three basic situations can be distinguished: (i) both hinges "locked" (elastic loading/unloading); (ii) one hinge softening, the other one "locked"; (iii) both hinges softening. If a hinge is "locked", the corresponding rotation remains constant. If a hinge is softening, the moment-rotation law combined with (6) or (7) provides an additional equation, so that the corresponding rotation can be expressed in terms of the displacements and eliminated from (4) and (5). Thus, the problem is always reduced to a system of two linear second-order differential equations with constant coefficients and with unknown functions $u_1(t)$ and $u_2(t)$. Given a prescribed harmonic horizontal displacement history $u_3(t)$ at the column base and prescribed initial conditions, we can construct the analytical solution of this system, which remains valid until the status of one of the hinges changes (e.g., from virgin loading to softening, or from softening to unloading). The time at which the status changes must be solved numerically. The displacements and velocities calculated for that time are then used again as the initial values for the next interval. In this manner, the entire solution can be calculated exactly.

Fig. 4a shows the calculated exact histories of the bending moments at the base and at the top of the column for the case of elastic response. The column is excited by a prescribed sinusoidal history of the horizontal displacement at the base. This must lead to a periodic dynamic solution. The question is whether an instability

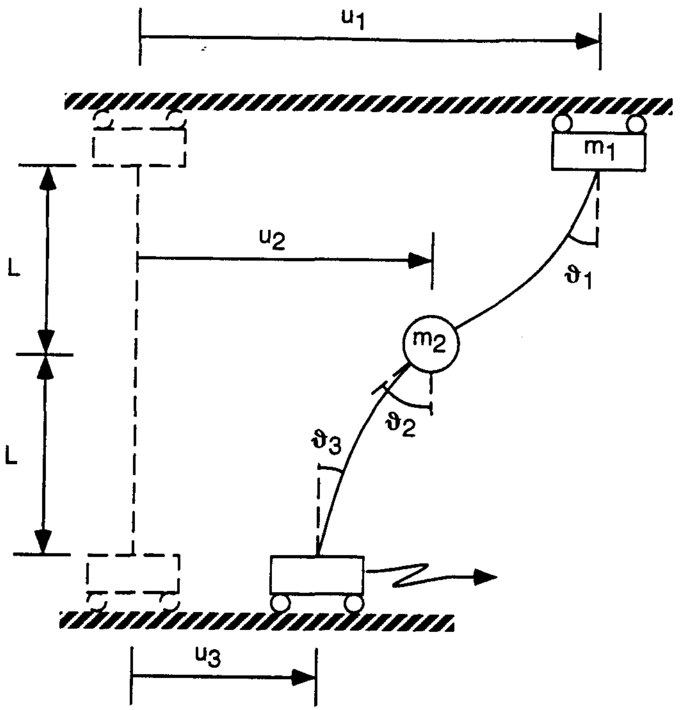

Figure 3: Dynamic model of a column

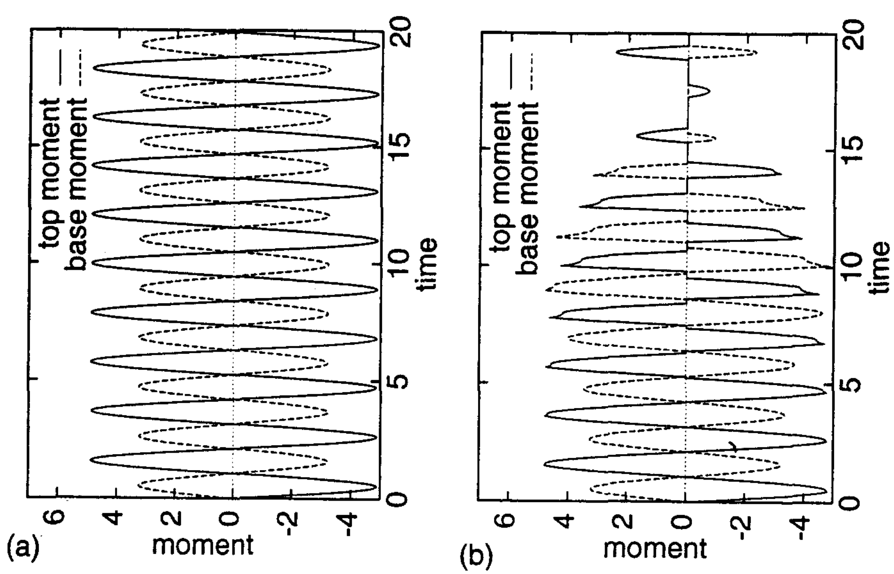

Figure 4: Response of the column model: (a) elastic, (b) progressive damage

in which the motion deviates exponentially from this periodic solution is possible. Note that the periodic excitation does not lead to a symmetric dynamic solution with equal moment magnitudes at the top and bottom. To obtain a symmetric dynamic solution, symmetric excitations at both the base and the top would have to be prescribed. But this would be unrealistic. For this reason, the dynamically loaded column will seldom find itself near the static bifurcation state. However, the motion of the column approaches a state corresponding to the limit point of the column statically loaded by the inertial forces, at which one of the hinges begins softening. In that case, dynamic instability with exponential growth of deviations from a periodic solution must be expected.

In Fig. 4a, the amplitude and period of the periodic excitation at base is selected so that the maximum moment exactly equals the peak moment M_p. Fig. 4b shows the solution under the assumption that the amplitude of excitation is slightly larger, which triggers softening response in the top hinge while the other hinge still behaves elastically. Now note that the peaks of the top moment cycles deviate progressively faster from the periodic solution and the deviation peaks grow approximately exponentially. This observation agrees with our previous general analysis of the initial dynamic response after the static limit point.

While the peaks of the top moment exponentially decline, the peaks of the base moment grow until the maximum bending moment is reached at the base (Fig. 4b). Then the peaks of this bending moment start declining, too, and the deviation from the maximum moment grows approximately exponentially. This is not surprising because the start of softening at the base results into another negative eigenvalue of the tangential stiffness matrix. So this example clearly verifies the exponential growth of very small imperfections after softening has begun, and shows the growth can be quite rapid.

It is interesting to observe that there is some similarity with the static bifurcation solution. First one hinge undergoes softening, then the other. Also note that in this case the solution would have a similar character even if the column were considered nonsymmetric.

6 Comments on Simplified Earthquake Analysis

The present results have important consequences for practical simplified analysis of earthquake response (Chopra, 1995). Such analysis is normally based on approximating the entire structure by an equivalent single-degree-of-freedom oscillator whose stiffness and mass are chosen so as to approximate the first vibration mode of the structure taking into account the stiffness reduction due to damage

(Seismic, 1986).

In the light of the present results, however, such an approach can be applicable only for moderate damage occurring when the softening hinges begin to form. It cannot be applicable to the analysis of complete collapse because the exponential growth of the deviation from the symmetric mode of response represents a mode that is very different from the initial response mode approximated by the single-degree-of-freedom oscillator.

Perhaps a simplified method based on an oscillator with two degrees of freedom, one representing an equivalent system for the initial first mode of vibration and the second representing an equivalent system for the deviation from the symmetric response after the static bifurcation state, could be developed.

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References

Bažant, Z.P. (1976) Instability, ductility and size effect in strain-softening concrete. **J. of Engrg. Mech.** ASCE, 102, 331–344.

Bažant, Z.P., and Jirásek, M. (1995) Softening-induced dynamic localization instability: Seismic damage in frames. Submitted to **J. of Engrg. Mech.** ASCE.

Bažant, Z. P., Pan. J.-Y., and Pijaudier-Cabot, G. (1987) Softening in reinforced concrete beams and frames. **J. of Struct. Engng.** ASCE 113, (12), 2333–2347, see also 2348–2364.

Chopra, A.K. (1995) Dynamics of structures: theory and applications to earthquake engineering. Prentice Hall, Englewood Cliffs, N.J.

Hunt, G.W., and Baker, G. (1993) Principles of localization in the fracture of quasi-brittle structure. Submitted to **J. of Mechanics and Physics of Solids.**

Jirásek, M. (1995) Elements for numerical analysis of softening damage in frames. Submitted to **Comm. Applied Num. Meth.**

Maier, G. (1971) Instability due to strain softening. In **Stability of Continuous Systems**, Proc. of IUTAM Symp. held in Karlsruhe, 1969, Springer-Verlag, Berlin, pp. 411-417.

Seismic design guidelines for essential buildings (1986). **Report** DA-TM-5-809-10-1, Departments of the Army, Navy and Air Force, Washington, D.C. (available from National Technical Information Service, Springfield, Virginia 22161).

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