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Fracture characteristics and micromechanical theory of rock as a quasibrittle material: Aperçu of recent advances

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ABSTRACT: Realistic characterization of rock fracture properties is essential for successful predictions of rock fracture under any situation, including the fragmentation in blasting. The paper presents a review of some interesting recent results on the characterization of rock fracture by point-wise, line-wise and diffuse models for the fracture process zone. The effect of the rate of loading and of crack growth on the fracture characteristics is also reviewed and the micromechanical aspects are discussed. Attention is then focused on the problem of the effect of microcracks on the global stiffness tensor of a microcracked material such as rock, and some new results which take into account the growth of cracks retaining their criticality are described. The lecture is documented by numerical results.

1 INTRODUCTION

Although no-tension models for rock failure are useful for some geotechnical engineering problems, analysis of rock failure requires fracture mechanics (Bažant, 1996). Proper characterization of the fracture properties of rock is a necessary tool for dependable predictions of rock failure, including rock fragmentation by blasting. The fracture behavior of rock is complicated by the fact that rock is a quasibrittle material, that is, a material in which the fracture tip is normally not sharp but is surrounded by a sizeable fracture process zone. This is best manifested by the fact that the size effect on the nominal strength of geometrically similar rock specimens does not follow the size effect of linear elastic fracture mechanics (LEFM) but exhibits a transitional size effect between plasticity and LEFM.

Various new mathematical formulations to describe the quasibrittle behavior of rock, as well as other materials such as concrete, ice, toughened ceramics and various composites, have recently been developed, and factors that significantly affect the quasibrittle response, such as the rate of loading and of fracture growth, have been intensely studied. To gain understanding of the fracture mechanism in a diffuse fracture process zone of a quasibrittle material, some interesting micromechanical models have been formulated.

The purpose of the present lecture is to provide an apperçu of recent advances in this subject, with a focus on the results recently obtained at Northwestern University. In addition, one new result, namely the extension of the theory of elastic constants of a randomly and uniformly microcracked elastic material to tangential stiffness calculation of a material with growing microcracks, will be presented in detail. It must be warned that the intent of the review that follows is not a broad coverage of all the results in the literature but an exposition selectively highlighting only some recent contributions.

2 REVIEW OF RECENT RESULTS

The fracture properties of a quasibrittle material can be described in three ways: (1) in a point-wise fashion, in which the energy dissipation in the fracture process zone is lumped into a point, the crack tip, (2) line-wise fashion, in which this energy dissipation is represented by stress-displacement relations for a cohesive crack, and (3) in a multi-dimensional diffuse fashion, representing a continuous smearing of the discrete microcracking and other inelastic phenomena such a frictional slip and fragment pull out in the fracture process zone.

2.1 *Point-wise characterization of fracture*

2.1.1 Size effect method and determination of R-curve

The size effect of quasibrittle fracture (Bažant, 1984b; Bažant, 1984a) can be effectively exploited to identify quasibrittle nonlinear fracture characteristic solely from measured maximum loads of geometrically similar specimens of sufficiently different sizes (Bažant and Pfeiffer, 1987; Bažant and Kazemi, 1990a; Bažant and Kazemi, 1990b; Bažant et al., 1991). The size effect is due to difference in scaling of energy release and energy dissipation of large fractures (and not to a possible fractal character of fracture (Bažant and Li, 1995)). The size effect law represents an asymptotic matching between the large size and small size asymptotic expansions of the size effect in quasibrittle fracture (Bažant and Li, 1995). By fitting the test results for specimens of different sizes, one can obtain size-independent as well as shape-independent values of the effective fracture energy, the effective length of the process zone, and the effective critical crack-tip opening displacement. Using the size effect law, one can also easily calculate the R-curve (resistance curve), which turns out to be geometry and shape dependent.

The main advantage of the size effect method is its simplicity and unambiguity. Assuming that specimens of positive geometry are used, measurements of the crack-tip locations, which are notoriously difficult in the case of a microfracturing material such as rock, are not necessary. Only the maximum loads of specimens of different sizes need to be measured, which means that a stiff machine and fast servo-control are not needed.

To eliminate the need for producing specimens of different sizes, the size effect method has recently been modified in a way which allows using specimens of only one size but with notches of different lengths, so as to provide a sufficient range of brittleness number (Bažant and Li, 1996b). This method uses also the value of the flexural strength of unnotched specimens. It is based on a universal size effect law — a generalization of size effect law that provides the transition from failures at large cracks to failures at no crack (Bažant, 1995), as recently derived by asymptotic analysis.

Extensive studies have demonstrated that the size effect method provides a simple yet realistic representation of the results on fracture properties of various types of rock (Takahashi, 1988; Schmidt and Lutz, 1979; Carpinteri, 1980; Schmidt, 1977; Hoagland et al., 1973; Bažant et al., 1991; Labuz et al., 1985).

2.1.2 Rate influence

The R-curve model, characterizing the variation of the critical energy release rate with the crack propagation length, has been generalized to describe both the rate effect and size effect observed in rock or other quasibrittle materials (Bažant and Jirásek, 1993). In this generalization, it is assumed that the crack propagation velocity depends on the ratio of the stress intensity factor to its critical value based on the R-curve, and that this dependence has the form of a power function with an exponent much larger than 1. The shape of the R-curve is determined, taking into account the specimen geometry, as the envelope of the fracture equilibrium curves corresponding to the maximum load values for geometrically similar specimens of different sizes. The formulation also allows taking into account the creep in the bulk of the specimen, however, this is not important for rock in most cases. Good representations of test data have been demonstrated for rock as well as concrete (in the case of concrete, the creep in the bulk is also very important, and causes a change of brittleness).

The tests which were used to calibrate the rate-dependent R-curve model used limestone specimens subjected to loading rates ranging over four orders of magnitude of the loading rate (Bažant et al., 1993). The loading rates were all in the static regime of response of the specimen, which simplified evaluation (the times to the peak load of specimens ranged from 2 seconds to 83,000 seconds). It is likely that the results can be extrapolated to dynamic loading rates in which the inertia and wave propagation effects are important. These tests utilized the size effect method with three specimen sizes.

A phenomenon of considerable interest for dynamic fracture analysis as recent tests have shown (Bažant and Gettu, 1992) is that a sudden increase of the loading rate causes a reversal of softening response to hardening response followed by a second peak, while a sudden decrease of the loading rate causes a decrease of the softening slope. These characteristics are described by the rate generalization of the R-curve model quite well (Bažant and Jirásek, 1993).

2.2 *Line-wise characterization of fracture*

A more realistic model for fracture is the cohesive crack model, in which the fracture properties are characterized by the dependence of the cohesive (crack-bridging) stress on the crack opening displacement. For concrete this model is also known as the fictitious crack model, pioneered by Hillerborg et al. (Hillerborg et al., 1976), and is based on the original ideas of Barenblatt (Barenblatt, 1962) and Dugdale (Dugdale, 1960).

This model has recently been formulated on the basis of energy principles and variational equations useful for numerical solutions have been presented (Bažant and Li, 1995). Using the energy formulations, the conditions of stability loss of a specimen or structure with a growing cohesive crack have been obtained from the condition of vanishing of the second variation of the complementary energy or the potential energy. They were found to have the form of a homogeneous Fredholm integral equation for the derivatives of the cohesive stress or crack opening displacement with respect to the crack length. Based on this equation, the criterion of stability limit of geometrically similar specimens or structures of different sizes was transformed to an eigenvalue problem, with the size of the specimen or structure as the eigenvalue. This formulation makes it possible to solve the size effect curve directly, without actually calculating the load-deflection curves of the specimens. One solves directly for the structure size for which a given relative crack length corresponds to stability loss, i.e., the maximum load. This formulation greatly simplifies studies of the size effect.

The cohesive (fictitious) crack model serves as the basis of the work-of-fracture method (Nakayama, 1965; Tatersall and Tappin, 1966) for measuring the fracture energy of rock or concrete. This method has recently been subjected to critical analysis which clarified its limitations (Bažant, 1987). It was shown that measuring the unloading compliance at a sufficient number of states on the post peak descending load-deflection curve, one can calculate the so-called pure fracture energy, representing the energy dissipated by the fracture process alone, excluding the energy dissipated by plastic frictional slips and fragment pull outs. However, this value of fracture energy is pertinent only if the material model, consisting of a fracture law and a constitutive law for structural analysis, takes into account separately the fracture-damage deformations and the plastic-frictional deformations. Otherwise, one must use the conventional fracture energy, which includes all the plastic-frictional energy dissipation in the fracture process zone. Either type of fracture energy should properly be determined by extrapolating the results of the work of fracture method to infinite specimen size, or else unambiguous, size-independent results cannot be obtained.

It was also shown that the work of fracture method can be improved by averaging the work done by fracture over only a central portion of the ligament. Other valuable advances were recently reported by Elices and Planas (1989), Elices et al. (1992), Hu and Wittmann (1991), and Planas et al. (1992).

For numerical purposes, the crack band model developed by Bažant (1982,1984a) and Bažant and Oh (1983, 1984) is normally more convenient in the finite element context. This model is essentially equivalent to the cohesive (fictitious) crack model and yields nearly the same results. The idea of the model is to replace a line crack with a crack band of a fixed width considered as a material property and assume a uniform distribution of cracking strain across the crack band such that the accumulated cracking strain be equal to the opening of the cohesive crack model.

2.2.1 Rate influence on cohesive crack model

Application of the cohesive crack model to dynamic fracture requires incorporating the time dependence, particularly the effect of loading rate and the rate of crack growth. It was shown (Bažant and Li, 1996a) that the rate effect can be realistically described by introducing a rate dependent softening law between the cohesive stress and the crack opening displacement (Bažant and Li, 1996a). The proper form of this softening law has been derived from the activation energy theory of the rate process of bond ruptures on the atomic level (Bažant, 1993).

It was shown that the phenomenon that causes the sudden reversal of softening to hardening when the loading rate is suddenly increased is the rate dependence of the softening stress-displacement law. The effect of viscoelasticity in the specimen or structure has also been incorporated into the cohesive crack model, but this is of little importance for rock.

2.3 *Multidimensional diffuse characterization of fracture and micromechanics*

Several new results will now be described, which are, however, limited in their present form to static failure of rock. Further generalization will be required to extend them to dynamic crack growth.

2.3.1 Effect of microcracks and their interactions on macroscopic stiffness

Ideally, for the purpose of structural analysis, fracture must be characterized on the continuum level. In the case of a quasibrittle material with a diffuse fracture process zone, this means that the model for the fracture process zone should be a multidimensional continuum which incorporates microcracking (as well as plastic-frictional phenomena) in a smeared manner. This involves two problems: (1) determination of the stiffness properties, including softening, under the assumption that the microcracking is distributed uniformly throughout a material element, which can be done by a constitutive stress-strain tensorial relation, and (2) interactions between microcracks which control the localization of damage due to microcracking. The second problem is at present the most challenging one and is the object of intense debates. Some new results on the first problem will be presented in detail in section 3 of the present paper.

Recently, a nonlocal continuum model for strain-softening was derived by micromechanical analysis of a macroscopically non-homogeneous (nonuniform) system of interacting and growing microcracks. Kachanov's version of the superposition method (Kachanov, 1985; Kachanov, 1987a) was used as the point of departure. Homogenization (or smearing) of the microcrack system was achieved by seeking a continuum field equation whose possible discrete approximation coincides with a matrix equation governing a system of interacting microcracks.

The result of this homogenization was a Fredholm integral equation for the unknown non-local inelastic stress increments, which involved two spatial integrals. One integral, which ensues from the fact that crack interactions are governed by the average stress over the crack length rather than the crack center stress, represents short-range averaging of inelastic macrostresses. The kernel of the second integral is the long-range crack influence function, which is a second-rank tensor and varies with directional angle (i.e., is anisotropic), exhibiting sectors of crack shielding and crack amplification. For long distances r, the weight function decays as r^{-2} in two dimensions and as r^{-3} in three dimensions.

Application of the Gauss-Seidel iteration method which can conveniently be combined with iteration in each loading step of a nonlinear finite element code, simplifies the handling of the nonlocality by allowing the nonlocal inelastic stress increments to be calculated from the local ones explicitly. This involves evaluation of an integral involving the crack influence function for which closed-form expressions are derived (for three dimensions they are based on the recent results of Fabrikant, 1990). Because the constitutive law for the microcracked material is strictly local, no difficulties arise with the unloading criterion or the continuity condition of plasticity. This micromechanical theory puts the previously proposed phenomenological nonlocal models for strain softening damage on a solid footing.

2.3.2 Micromechanical model for compression crushing with microbuckling

Compression failure, also called crushing, requires lateral expansion of the damaged material in the directions normal to the maximum compressive stress. Fracture mechanics and micromechanics of this behavior is a rather difficult subject which has recently received considerable attention. One recently proposed model which appears to give a good description of compression failure including the size effect is based on the idea of transverse propagation of a band of axial splitting cracks and microbuckling of the microslabs between the axial splitting cracks. This model has first been used for explaining the size effect observed experimentally in the breakout of boreholes in rock (Nesetova and Lajtai, 1973; Haimson and Herrick, 1986; Haimson and Herrick, 1989; Carter et al., 1992; Carter, 1992; Martin et al., 1994; Dzik and Lajtai, 1994). The analysis involved calculations of the energy release from the surrounding rock mass by approximating the failure zone as elliptical and using the Eshelby theorem of elasticity. This analysis, as well as the subsequent analysis of lateral propagation of a band of axial splitting cracks indicated that the asymptotic effect of size on the nominal strength can be described by a power law in which the nominal strength is approximately proportional to the -2/5 power of the characteristic size (Bažant and Xiang, 1996).

2.3.3 Spacing of cracks at their initiation

The spacing of parallel cracks in rock or other materials is a matter of stability of crack system and bifurcation in its evolution. Under static loading, parallel cracks such as cooling cracks tend to evolve in such a manner that every other crack closes and the intermediate ones open more widely while advancing further (Bažant and Ohtsubo, 1978; Bažant et al., 1979; Bažant and Cedolin, 1991). However, the initial spacing of cracks when they initiate is a matter of fracture characteristics of the material.

The initiation of parallel cracks in a half space under static loading has recently been studied by Li, Hong and Bažant (1995). It was shown that the following three conditions govern the initial spacing and the initial equilibrium length of parallel cracks: (1) the energy release rate of the initial cracks must be equal to its critical value, (2) the total energy dissipated by the formation of the initial cracks must be equal to the total energy release as a result of the finite initial cracks, and (3) the stress before the appearance of the initial cracks must be equal to the tensile strength of the material. From these three conditions, one can calculate the initial length of the cracks, their initial spacing, and the load at which the cracks form.

3 NEW RESULTS ON THE STIFFNESS TENSOR OF A MATERIAL WITH GROWING MICROCRACKS

Microcracks in a brittle material affect its stiffness, strength and toughness. Their evolution is a mechanism of failure. Prediction of the response of structures made of damaging materials such as rock (as well as concrete, ice, ceramics or composites), requires modeling of the effect of microcracks on the macroscopic constitutive law. The basic problem is the effect that a crack system statistically uniform in space has on the overall elastic constants of the material.

In rock mechanics, careful attention must be paid to the families of cracks, preexisting or man-made. They can have a strong influence on the response of the rock during blasting operations in tunneling, excavation and mining. The distribution of these cracks within the rock mass is usually characterized by aligned patterns with some preferred crack orientations that render the body macroscopically anisotropic. The crack density can vary from the case of dilute (non-interacting) cracks to highly concentrated (heavily interacting) cracks depending on the geological history and loading. On crack systems in rock, one may distinguish cracks of at least two typical scales: (1) microcracks of the order of the grain size, and (2) intersecting families of rock joints. For the former, the material can be treated as a continuum on the scale over 1 ft, and for the latter on the scale over 1 km. The continuum treatment means that the effect of a crack system statistically uniform in space can be captured by the macroscopic stiffness tensor of a continuum.

The problem of calculating the macroscopic stiffness tensor of elastic materials intersected by various types of random or periodic crack systems has been systematically explored during the last two decades and effective methods such as the self-consistent scheme (Budiansky and O'Connell, 1976; Hoenig, 1978), the differential scheme (Roscoe, 1952; Hashin, 1988), or the Mori-Tanaka method (Mori and Tanaka, 1973; Benveniste, 1987) have been developed. A serious limitation of the current knowledge is that all the studies have dealt with cracks that neither propagate nor shorten (Fig. 1a). This means that, in the context of response of a material with growing damage illustrated by the curve in Fig. 1a, the existing formulations predict only the secant elastic moduli (such as E_s in Fig. 1a). Such information does not suffice for calculating the response of a body with progressing damage due to cracking.

To calculate the response of a material with cracks that can grow or shorten, it is also necessary to determine the tangential moduli, exemplified by E_t in Fig. 1b. Knowledge of such moduli makes it possible, for a given strain increment, to determine the inelastic stress drop $\Delta\sigma_{cr}$ (Fig. 1b). This problem will be addressed in this paper. For a more detailed derivation see (Prat and Bažant, 1995).

Knowledge of the secant and tangential moduli is of course still insufficient to predict the response of a structure with growing cracks. It is now well known that softening damage caused by cracking tends to localize into cracking bands or other regions. The localization of cracking is caused and governed mainly by interactions among propagating cracks. The interactions cause that the average behavior of a representative volume of the material with cracks does not follow the local stress-strain curve for growing cracks but follows a slope that is either smaller or larger, as shown in Fig. 1c. This problem has recently been analyzed and an integral equation in space governing the nonlocal behavior of such material has been formulated (Bažant, 1994; Bažant and Jirásek, 1994; Jirásek and

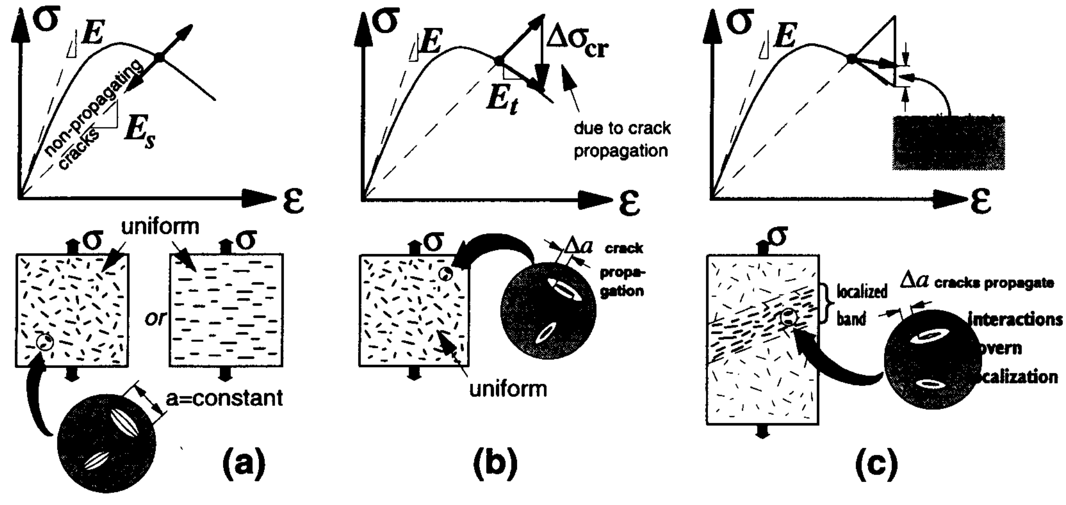

Figure 1. (a) Effective Secant Moduli; (b) Tangential moduli and stress increment due to crack growth; (c) Tangential stiffness of material with localizing cracks.

Bažant, 1994) on the basis of continuum smoothing of a matrix equation for crack interactions. However, a complete solution is beyond the scope of this paper.

3.1 *Equivalent elastic modulus of a material with uniformly distributed cracks*

We consider a representative volume V of an elastic material containing on the microscale many cracks (microcraks). On the macroscale, we imagine the cracks to be smeared and the material to be represented by an approximately equivalent homogeneous continuum whose local deformation within the representative volume can be considered approximately homogeneous over the distance of several average crack sizes. Let $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ be the average strain tensor and average stress tensor within this representative volume. To obtain a simple formulation, we consider only circular cracks of average radius $\bar{a}$.

To derive the tangential compliance tensor of the material on the macroscale, the cracks must be allowed to grow during the prescribed strain increment $\Delta\boldsymbol{\epsilon}$. This means that the energy release rate per unit length of the front edge of one crack, $\mathcal{G}_1$, must be equal to its critical value, i.e., to the fracture energy G_f of the material. For the sake of simplicity, we will enforce the condition of criticality of cracks only in an overall (weak) sense, by assuming that the average overall energy release rate of all the cracks within the representative volume equals their combined energy dissipation rate.

Let n be the number of cracks per unit volume of the material with cracks, characterizing the representative volume. The surface area of one circular crack of radius a is $A = \pi a^2$ and its change when the crack radius increases by δa is $\delta A = 2\pi a \delta a$. We assume we can replace the actual crack radii a by their average radius $\bar{a}$. The critical state of crack growth is obtained when the energy release rate per unit volume of the material, $\mathcal{G}_1$, equals the rate of energy dissipation by all the cracks in the unit volume, i.e.,

$$\frac{\partial \Pi^*}{\partial \bar{a}} = 2\pi \bar{a} n G_f \qquad (1)$$

where $\Pi^* = \Pi^*(\boldsymbol{\sigma}, \bar{a}) = \frac{1}{2}\boldsymbol{\sigma} : \mathbf{C}(\bar{a}) : \boldsymbol{\sigma}$ is the complementary energy per unit volume and $\mathbf{C}(\bar{a})$ is the macroscopic secant compliance tensor of the material with the cracks.

Calling $G^* = 2\pi n G_f$, the following equation for $\Delta\bar{a}$ is obtained:

$$\boldsymbol{\sigma} : \mathbf{C} : \Delta\boldsymbol{\sigma} + \left[\frac{1}{2}\boldsymbol{\sigma} : \frac{\partial \mathbf{C}}{\partial \bar{a}} : \boldsymbol{\sigma} - G^* \bar{a}\right] \Delta\bar{a} = 0 \qquad (2)$$

in which $\Delta\bar{a} = \bar{a} - \bar{a}_0 =$ is the average crack radius increment. Eqn. (2) can be easily solved numerically, and the current average crack size is obtained. The strains are then computed from $\boldsymbol{\epsilon} = \mathbf{C}(\bar{a}) : \boldsymbol{\sigma}$, which is explicit. Thus no iterations within the constitutive subroutines for structural analysis by incremental loading are required.

3.2 *Equivalent elastic modulus of a material with arbitrarily oriented cracks*

In the preceding section we have considered all the orientation of cracks to be equally probable, which must obviously lead to isotropic effective elastic properties. In reality, when the cracks are produced by a non-isotropic strain tensor such as is the case in rock mechanics problems, a preferential crack orientation must exist. So we will now assume that the crack orientations are not uniformly distributed but that there exist several families of cracks, each with a different orientation, different average crack size and different crack density.

Consider now that the elastic body is intersected by N families of random cracks, labeled by subscripts $\mu = 1, 2, \ldots, N$. Each crack family may be characterized by its spatial orientation $\boldsymbol{\nu}_\mu$, its average crack radius $\bar{a}_\mu$, and the number n_μ of cracks in family μ per unit volume of the material. Thus, the compliance tensor may be considered as the function $\mathbf{C} = \mathbf{C}\left(\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_N; n_1, n_2, \ldots, n_N\right)$. Approximate estimation of this function has been reviewed by Kachanov and co-workers (Kachanov, 1992; Kachanov, 1993; Sayers and Kachanov, 1991; Kachanov et al., 1994).

The incremental constitutive relation can be obtained by differentiation of Hooke's law, which yields:

$$\Delta\boldsymbol{\epsilon} = \mathbf{C} : \Delta\boldsymbol{\sigma} + \sum_{\mu=1}^{N} \frac{\partial \mathbf{C}}{\partial \bar{a}_\mu} : \boldsymbol{\sigma}\Delta\bar{a}_\mu + \sum_{\mu=1}^{N} \frac{\partial \mathbf{C}}{\partial n_\mu} : \boldsymbol{\sigma}\Delta n_\mu \quad (3)$$

where Δ denotes small increments over a loading step.

Our analysis will be restricted to the case when the number of cracks in each family is not allowed to change ($\Delta n_\mu = 0$, i.e., no new cracks are created and no existing cracks are allowed to close). With the number of cracks constant, the third term on the right-hand side of Eqn. 3 vanishes, yielding:

$$\Delta\boldsymbol{\epsilon} = \mathbf{C} : \Delta\boldsymbol{\sigma} + \sum_{\mu=1}^{N} \frac{\partial \mathbf{C}}{\partial \bar{a}_\mu} : \boldsymbol{\sigma}\Delta\bar{a}_\mu \quad (4)$$

The crack radius increments $\Delta\bar{a}_\mu$ must be determined in conformity to the laws of fracture mechanics. Let us assume that the cracks (actually microcracks) follow linear elastic fracture mechanics (LEFM). This assumption means that the energy release rates must be equal to the fracture energy of the material, G_f. To make the problem tractable, we impose the energy balance condition only in the overall, weak sense, which leads to the following N_g conditions for $\Delta\bar{a}_\mu > 0\ (1 \le \mu \le N_g)$:

$$\frac{\partial \Pi^*}{\partial \bar{a}_\mu} = 2\pi\bar{a}_\mu n_\mu G_f \quad (5)$$

in which N_g is the number of families of growing cracks. (Repetition of subscript μ in this and subsequent equations does not imply summation.)

When the cracks are shortening, their faces are coming in contact, which requires no energy. Therefore, for shortening cracks ($\Delta\bar{a}_\mu < 0$, $N_g < \mu \le N_s$),

$$\frac{\partial \Pi^*}{\partial \bar{a}_\mu} = 0 \quad (6)$$

where N_s is the number of all the families of growing and shortening cracks. For the remaining crack families ($N_s < \mu \le N$) no equations are necessary since for them $\Delta\bar{a}_\mu \equiv 0$.

Substitution of Eqns. (5) or (6) into

$$\frac{\partial \Pi^*}{\partial \bar{a}_\mu} = \boldsymbol{\sigma} : \mathbf{C} : \frac{\partial \boldsymbol{\sigma}}{\partial \bar{a}_\mu} + \frac{1}{2}\boldsymbol{\sigma} : \frac{\partial \mathbf{C}}{\partial \bar{a}_\mu} : \boldsymbol{\sigma} \quad (7)$$

(where $1 \le \mu \le N$) leads to the following incremental relations which must be satisfied by the crack radius increments (positive or negative, or 0):

$$\begin{aligned} &\boldsymbol{\sigma}:\mathbf{C}:\Delta\boldsymbol{\sigma} + \left(\frac{1}{2}\boldsymbol{\sigma}:\frac{\partial \mathbf{C}}{\partial \bar{a}_\mu}:\boldsymbol{\sigma} - 2\pi\bar{a}_\mu n_\mu G_f\right)\Delta\bar{a}_\mu = 0 \\ &\boldsymbol{\sigma}:\mathbf{C}:\Delta\boldsymbol{\sigma} + \frac{1}{2}\boldsymbol{\sigma}:\frac{\partial \mathbf{C}}{\partial \bar{a}_\mu}:\boldsymbol{\sigma}\Delta\bar{a}_\mu = 0 \\ &\Delta\bar{a}_\mu = 0 \end{aligned} \quad (8)$$

The first equation applies to the families of growing cracks ($1 \le \mu \le N_g$), the second to the families of shortening cracks ($N_g < \mu \le N_s$), and the third to the families of stationary cracks ($N_s < \mu \le N$).

The constitutive law of Eqn. (4) and the energy equilibrium conditions of Eqn. (8) together represent a system of $N+6$ equations for the increments of crack radii in N crack families of different orientations and 6 increments of stress tensor components, $\Delta\boldsymbol{\sigma}$. If the strain tensor increment $\Delta\boldsymbol{\epsilon}$ is prescribed, these $N+6$ unknowns can be solved from this system of equations. The tangential stiffness tensor can be obtained by solving the stress increments for all the cases in which a unit value is assigned to each component of $\Delta\boldsymbol{\sigma}$, with all the other components being 0.

To solve the problem, we must have the means to evaluate the effective secant stiffness $\mathbf{C}$ as a function of the vector crack radii. For its simplicity, we will use a technique developed by Sayers and Kachanov (1991) using the symmetric second-order crack density tensor $\boldsymbol{\alpha} = \sum_{\mu=1}^{N} n_\mu \bar{a}_\mu^3\, \boldsymbol{\nu}_\mu \otimes \boldsymbol{\nu}_\mu$ (Vakulenko and Kachanov, 1971; Kachanov, 1980; Kachanov, 1987b).

The effective secant compliance $\mathbf{C}$ can be derived from an elastic potential F which may be considered as a function of the crack density tensor $\boldsymbol{\alpha}$ in addition of the stress tensor $\boldsymbol{\sigma}$:

$$\boldsymbol{\epsilon} = \frac{\partial F(\boldsymbol{\sigma}, \boldsymbol{\alpha})}{\partial \boldsymbol{\sigma}} = \mathbf{C} : \boldsymbol{\sigma} \quad (9)$$

Also, $\mathbf{C} = \mathbf{C}^\circ + \mathbf{C}^{cr}$ in which $\mathbf{C}^\circ$ is the elastic compliance and $\mathbf{C}^{cr}$ is the additional compliance due to the crack system. The elastic potential $F(\boldsymbol{\sigma}, \boldsymbol{\alpha})$ can be expanded into a tensorial power series. According to the Cayley-Hamilton theorem, this expansion can always be reduced to a cubic tensor polynomial. Sayers and Kachanov (1991) proposed to approximate potential $\mathbf{F}$ by a tensor polynomial that is quadratic in $\boldsymbol{\sigma}$ and linear in $\boldsymbol{\alpha}$:

$$\begin{aligned} &F(\boldsymbol{\sigma}, \boldsymbol{\alpha}) = \\ &\frac{1}{2}\boldsymbol{\sigma} : \mathbf{C}^\circ : \boldsymbol{\sigma} + \eta_1(\boldsymbol{\sigma} : \boldsymbol{\alpha})\operatorname{tr}\boldsymbol{\sigma} + \eta_2(\boldsymbol{\sigma} \cdot \boldsymbol{\sigma}) : \boldsymbol{\alpha} \end{aligned} \quad (10)$$

in which η_1 and η_2 are assumed to depend only on the first invariant of $\boldsymbol{\alpha}$ (Sayers and Kachanov, 1991).

The additional secant compliance $\mathbf{C}^{cr}$ due to the cracks is written in terms of η_1, η_2 and $\boldsymbol{\alpha}$. The functions $\eta_1(\rho)$ and

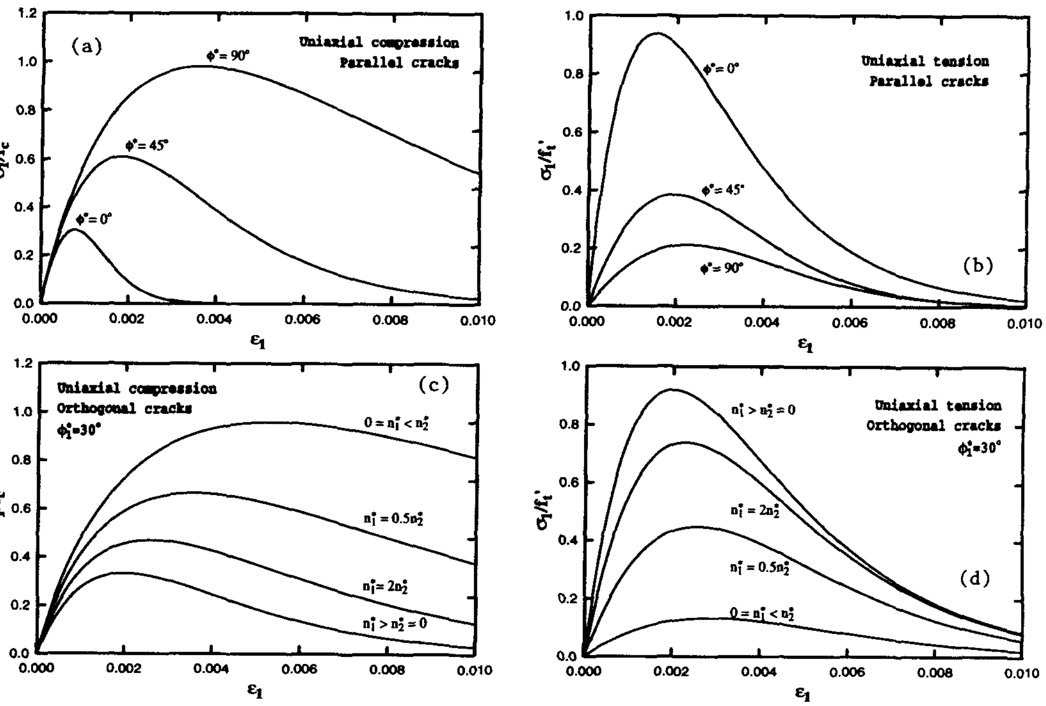

Figure 2. Results for a macroscopically orthotropic material: (a) one family of parallel cracks under uniaxial compression; (b) one family of parallel cracks under uniaxial tension; (c) two families of orthogonal cracks under uniaxial compression; (d) two families of orthogonal cracks under tension.

$\eta_2(\rho)$ can be obtained by taking the particular form of the preceding formulation for the case of random cracks, and equating the results to those obtained using, e.g., the differential scheme (Hashin, 1988):

$$\begin{aligned} \eta_1 &= \frac{3}{2\rho}\,\frac{3(E_{\text{eff}}-E)+4(\nu E_{\text{eff}}-E\nu_{\text{eff}})}{E_{\text{eff}}E} \\ \eta_2 &= \frac{6}{\rho}\left(\frac{1+\nu_{\text{eff}}}{E_{\text{eff}}}-\frac{1+\nu}{E}\right) \end{aligned} \tag{11}$$

where E_{eff} and ν_{eff} are the effective Young's modulus and Poisson's ratio for random cracks obtained using the differential scheme. Substituting η_1 and η_2 into $\mathbf{C}^{cr}$, the effective compliance tensor is obtained. Then $\Delta\bar{a}_\mu$ is calculated from Eqns. 4 and 8 for a given $\Delta\boldsymbol{\epsilon}$

3.3 *Examples*

Fig. 2a shows the results obtained when uniaxial compression is applied to an elastic material with a single family of parallel cracks. The angle of crack normals with the direction of compression is ϕ^*. The solution has been obtained for three different values of the angle: 0°, 45° and 90°. As expected, the calculated peak stress (strength) is the lowest when the cracks are parallel to the loading direction ($\phi^* = 0°$), and the material response is the most brittle as indicated by the steepest postpeak decline of stress. Fig. 2b shows the results obtained when uniaxial tension is applied. In that case, the minimum strength occurs when the cracks are perpendicular to the tension direction, as known from experimental evidence.

The analysis has been repeated for an elastic material with two families of orthogonal cracks, which in general are not aligned with the tension direction ($\phi_1^* = 30°$, $\phi_2^* = 120°$). The analysis has been performed for several combinations of the number of cracks n_1^* and n_2^* corresponding to the two crack families. These combinations are indicated in Figs. 2c and 2d. Fig. 2c depicts the results obtained for uniaxial compression and Fig. 2d the results for uniaxial tension. The analytical results obtained for both cases agree with the expected trends.

The model has also been used to simulate the response to biaxial stress applied to an initially isotropic material. Fig. 3a shows the stress-strain curves obtained for biaxial tests with different stress ratios σ_1/σ_2. These curves again agree with the observed behavior of brittle materials such as rock (or concrete). Furthermore, combining the peaks of the curves for different stress ratios, the biaxial failure envelope (Fig. 3b) is obtained. The envelope agrees well with experimental results. In particular, the ratios of the uniaxial tensile strength and of the biaxial compression

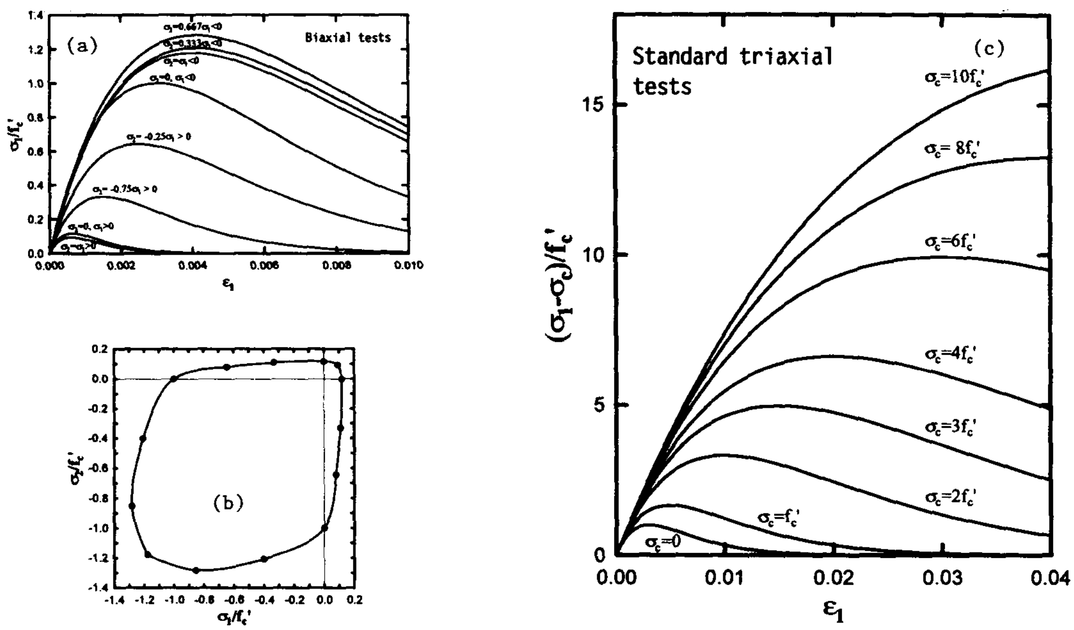

Figure 3. Multiaxial tests: (a) biaxial tests, stress-strain curves; (b) biaxial failure envelope; (c) triaxial tests, stress-strain curves.

strength to the uniaxial compression strength have values that are normal for some rocks or concrete.

Finally, the model has been verified for the case of standard triaxial tests, in which a certain hydrostatic pressure of maximum value σ_c is applied first and then uniaxial compression is superimposed. The analytical results are presented in Fig. 3c, for several values of σ_c ranging from 0 to $10f_c'$ where f_c' is the uniaxial compression strength. So we see that, at least qualitatively, the new model is able to capture the decreasing brittleness of the material for increasing hydrostatic pressure. Eventually this approach is likely to lead to an improvement of the microplane model (Bažant et al., 1996a; Bažant et al., 1996b), a very effective and versatile model for fracturing materials.

4 CONCLUDING REMARKS

As is clear from the preceding exposition, fracture of rock is a complex phenomenon with many facets, which is only now beginning to be reasonably well understood. A number of effective models for rock fracture, varying from simple or sophisticated, have been outlined. Attention has then been focused on the micromechanical aspects, whose understanding is essential for the development of realistic models. The micromechanical modeling of fracture of rock or other quasibrittle materials is still in its infancy, and should represent the focus of further research. One difficult problem which deserves thorough further study if the problem of localization of microcracking into large macroscopic fractures, and its continuum description.

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