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Statistical and fractal aspects of size effect in quasibrittle structures: Conspectus of recent results

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ABSTRACT: The purpose of the lecture is to critically review the statistical theory of size effect and to highlight some new results on quasibrittle materials, particularly the role of statistical heterogeneity of the material in the size effect due to energy release. It is shown that, in quasibrittle materials, a nonlocal generalization of Weibull's probabilistic concept becomes necessary. According to this generalization, the statistical size effect in structures failing after large stable crack growth asymptotically disappears as the structure size increases and the nonlocal statistical theory reduces to the Bažant's deterministic type of size effect due to energy release. Furthermore, attention is focussed on some new results of asymptotic energy release analysis showing the recent hypothesis that an invasive fractality of crack surfaces causes the size effect on nominal strength of structures cannot be valid. A Weibull-type statistical analysis is then outlined in some detail to critically examine another, more recent, version of the fractal hypothesis—namely that a lacunar fractality of microcrack distribution in the fracture process zone causes the size effect. It is concluded that the hypothesis of lacunar fractality cannot explain the size effect either. Finally, various pertinent experimental data (including the results of some large scale tests of sea ice fracture in the Arctic) are reviewed and some numerical simulations with a large random lattice model are demonstrated in the oral presentation. The lecture concludes with suggestions of profitable research directions.

1 Introduction

The problem of scaling is the most fundamental aspect of every physical theory. It has received the most prominent attention in the mechanics of fluids. In the mechanics of materials and structures, the size effect has recently been recognized as a major problem but is as old as the theory itself. The earliest studies of length effect on the strength of ropes date back to Leonardo da Vinci (The Notebooks 1945) and Galileo Galilei (1638) in the 16th and early 17th centuries. Mariotte (1686), late in the 17th century, was the first to attribute the observed size effect to the randomness of material strength—a concept whose modern statistical treatment, based on the extreme value distribution, had to wait for the development of extreme value statistics (Tippett 1925, Peirce 1926, Fréchet 1927, Fischer and Tippett 1928, von Mises 1936), culminating with the work of Weibull (1939; also 1949, 1951, 1956). Weibull's theory dominated all the subsequent thought until recently.

During the 1970's, however, it became apparent that, in quasibrittle materials such as concrete, rock, ice, and many 'high-tech' materials such as fiber or particulate composites and toughened ceramics, there exists another, deterministic, type of size effect, caused by energy release. This quasibrittle size effect is caused by the relatively large size of the fracture process zone of distributed cracking, and is of two different kinds:

- The size effect due to the initial growth of a large fracture process zone at crack initiation from a smooth surface (Hillerborg et al. 1976, Petersson 1981), and
- the size effect due to the (very desirable) capability of quasibrittle structures to grow large cracks in a stable manner prior to maximum load.

The latter kind is the size effect that matters for brittle failures of reinforced concrete structures because the purpose of reinforcement is to prevent failures at crack initiation.

Even though the size effect due to energy release is basically deterministic, material randomness certainly has some influence. A unified theory, amalgamating this size effect with the Weibull size effect (Bažant and Xi 1991b), will be reviewed and discussed in the first part of this lecture.

Recently, a third possible explanation of size effect was advanced by Carpinteri (1994a,b) and Carpinteri et al. (1995a,b,c, 1996a,b,c)—the fractal nature of crack surfaces and microcrack distributions. Observations have indeed shown that, within a certain range of scales, the fracture surfaces in many materials, including quasibrittle materials, exhibit partly fractal characteristics, and considerable progress in their understanding has already been made (Mandelbrot et al., 1984; Brown, 1987; Mecholsky and Mackin 1988; Cahn, 1989; Peng and Tian, 1990; Saouma et al., 1990; Bouchaud et al., 1990; Chelidze and Gueguen, 1990; Long et al., 1991; Måløy et al., 1992; Mosolov and Borodich, 1992; Borodich, 1992; Lange et al., 1993; Xie, 1987, 1989, 1993; Xie et al. 1994, 1996; Saouma and Barton, 1994; etc.). A causal relationship between the fractal nature of cracks on the microscale and the scaling law on the macroscale has been proposed (Carpinteri, 1994a,b; Carpinteri and Ferro, 1994; Carpinteri and Chiaia, 1995; Carpinteri et al. 1993, 1995;). But this connection has been based merely on intuitive analogy and geometric arguments. It has not been established in terms of mechanics.

The original hypothesis of Carpinteri et al. (1994) was that the size effect on the nominal strength of geometrically similar quasibrittle structures is caused by the fractal morphology of crack surfaces. This hypothesis, which does not have statistical aspects, was shown invalid at the 1994 IUTAM Symposium in Torino (Bažant 1996a, 1994; also Bažant 1995a,b, 1996, and in detail Bažant 1997b) on the basis of two arguments:

1. The correct prediction ensuing from the fractal crack curve hypothesis by generalization of asymptotic energy release analysis (Bažant 1997a) yields a size effect prediction that clearly disagrees with experimental results (Fig. 1b,c).
2. The crack curve fractality cannot matter on the global scale because the fractal curve dissipates much less energy (by several orders of magnitude) than the microcracking and frictional-plastic slips in the typical fracture process zone of a quasibrittle material (Fig. 1a).

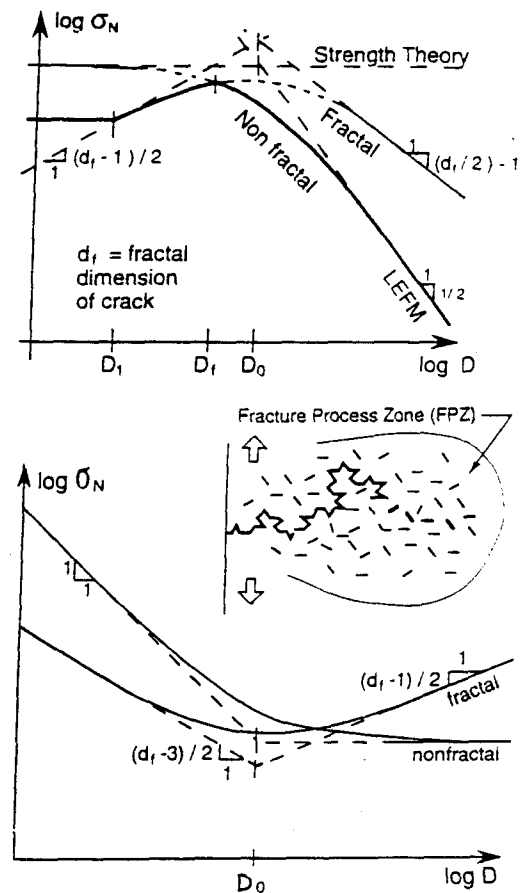


Figure 1: Size effect curves predicted by nonfractal and fractal energy-based analyses, for failures after large crack growth ((a) top) or at crack initiation ((b) bottom); and fractal crack curve and its fracture process zone with distributed cracking ((c) middle).

Subsequently, a modified fractal hypothesis was proposed by Carpinteri et al. (1995b) and Carpinteri and Chiaia (1995)—namely that the size effect is caused by the lacunar (rarifying) fractality of the distribution of microcracks in the fracture process zone, rather than by the invasive (densifying) fractality of the crack curve. This hypothesis, however, necessarily leads to Weibull-type statistical size effect, and therefore suffers of the same limitations (Bažant 1997b). The mechanical and statistical argument refuting the lacunar fractal hypothesis for scaling is outlined and discussed in the second part of this lecture.

The oral presentation closes by reviewing various pertinent experimental data (including the results of some large scale tests of sea ice fracture in the

Arctic) and some large-scale numerical simulations with a random lattice model, and by suggesting profitable research directions.

2 Nonlocal Modification of Weibull Statistical Theory for Quasibrittle Structures

The development of the statistical theory of size effect on the nominal strength of structures based on the concept of random material strength was in principle completed by Weibull (1939). The Weibull theory has been enormously successful in applications to metal structures embrittled by fatigue. However, it took until the 1980's to realize that this theory does not explain the size effect in quasibrittle structures failing after a large stable crack growth.

The Weibull theory rests on two basic hypotheses:

1. The structure fails as soon as one small element of the material attains the strength limit.
2. The strength limit is random and the probability P_1 that the small element of material does not fail at a stress less than σ is given by the following Weibull cumulative distribution:

$$\varphi(\sigma) = \left\langle \frac{\sigma - \sigma_u}{\sigma_0} \right\rangle^m \quad (\sigma \geq \sigma_u \approx 0) \quad (1)$$

This distribution applies to the classical problem of a long chain or cable, analyzed by Weibull. It also applies to any statically determinate structure consisting of many elements (for example a truss consisting of bars), which fails if one element fails. But it does not apply to statically indeterminate structures and multidimensional bodies that do not fail at crack initiation.

Weibull's theory is correct only if the multidimensional structure fails as soon as one small element of the material fails. Such a sudden failure occurs in fatigue-embrittled metal structures, in which the critical flaw at the moment the sudden failure is triggered is still of microscopic dimensions compared to the cross-section size. But this is not the case for concrete structures and other quasibrittle structures which are designed to fail only after a large stable crack growth. For example, in the diagonal shear failure of reinforced concrete beams the critical crack grows over 80% to 90% of the cross-section size before the beam becomes unstable and fails. During such large stable crack growth, enormous stress redistributions occur and cause a large

release of stored energy, which produces a large deterministic size effect.

The size effect in Weibull theory arises from the fact that, the larger the structure, the greater the probability of encountering a small material element of a given small strength. By considering the joint probability of survival of all the small material elements in the structure, one obtains for the structure strength a probability integral of a similar form as that for a long chain or for a series coupling of many elements (Tippett 1925, Peirce 1926, Fréchet 1927, Fischer and Tippett 1928, von Mises 1936):

$$\ln(1 - P_f) = \int_V \varphi[\sigma(\mathbf{x})] dV(\mathbf{x})/V_r \quad (2)$$

in which P_f = failure probability of the structure, V = volume of the structure, V_r = small representative volume of the material whose strength distribution is given by $\varphi(\sigma)$, and \mathbf{x} = spatial coordinate vector. By virtue of the fact that the Weibull distribution is a power law (and that σ_u may be neglected), the aforementioned probability integral always yields for the size effect a power law. It is of the form

$$\sigma_N = k_0 V^{-1/m} = k_0 D^{-n/m} \quad (3)$$

where k_0 = constant characterizing the structure shape, and n = number of dimensions of the structure (1, 2 or 3). For two-dimensional similarity ($n = 2$) and typical properties of concrete, the exponent is approximately $n/m = 1/6$.

The fact that the scaling law of Weibull theory is a power law implies that there is no characteristic size of the structure, and thus no characteristic material length (Bažant 1984), which is also obvious from the fact that no material length appears anywhere in the formulation. This observation makes the Weibull-type scaling suspect for the case of quasibrittle structures whose material is highly heterogeneous, with its heterogeneity characterized by a non-negligible material length.

To take into account stress redistributions, various phenomenological theories of load sharing and redistribution in a system of parallel elements have been proposed. Although they are useful if the redistributions and load-sharing are relatively mild, they are insufficient to describe the large stress redistributions caused by large stable crack growth. They lack the fracture mechanics aspects of the problem.

One might wish to capture the stress redistribution due to large fracture by substituting the near-tip stress field of LEFM into the probability integral

in (2). However, for normal values of the Weibull modulus m , the integral would then diverge. So this is not a remedy.

The Weibull theory can, however, be extended to capture large stress redistributions approximately—by introducing a nonlocal generalization (Bažant and Xi, 1991), in which the probability integral (2) is replaced by the following integral:

$$\ln(1 - P_f) = k_1 \int_V \varphi [E \bar{\epsilon}(\mathbf{x})] dV(\mathbf{x})/V_r \quad (4)$$

The stress at a given point in the structure is here replaced by the average (over a certain neighborhood) of the strain field, $\bar{\epsilon}$ (times the elastic modulus E , to get a quantity of the stress dimension). In other words, the failure probability at a certain point x of the structure is assumed to depend not on the stress (stress according to the continuum theory) at that point but on the average strain in a certain neighborhood of the point, as in nonlocal theories for strain localization in strain-softening materials.

With this nonlocal generalization, the analytical evaluation of the integral (4) seems prohibitively difficult, however it is easy to obtain the asymptotic behavior for $D \rightarrow \infty$ and $D \rightarrow 0$. Also, for $m \rightarrow \infty$, the solution should approach the size effect law based on energy release. It was shown that a simple formula that interpolates between these three asymptotic cases, i.e., achieves asymptotic matching, is as follows (Bažant and Xi, 1991):

$$\sigma_N = \frac{\sigma_P}{\sqrt{\beta^{2n/m} + \beta}}, \quad \beta = \frac{D}{D_0} \quad (5)$$

This formula is sketched in Fig. 2, which also shows the aforementioned asymptotic scaling laws. They turned out to be the same as the Weibull type scaling law for small sizes (line of slope $-m/n$), and the LEFM scaling law for large similar cracks and large sizes (line of slope $-1/2$). According to this result, the scaling law of the classical Weibull theory should be applicable for sufficiently small structures. However, comparisons with test data for concrete show that the deterministic size effect law which begins by a horizontal asymptote, and the size effect law in (5) which begins by an asymptote of slope $-m/n$, both fit the test data about equally well, relative to the scatter of measurements.

It is interesting that the effect of material randomness completely disappears for large sizes, as revealed by the fact that the large size asymptote has the LEFM slope of $-1/2$. How can it be physically explained?

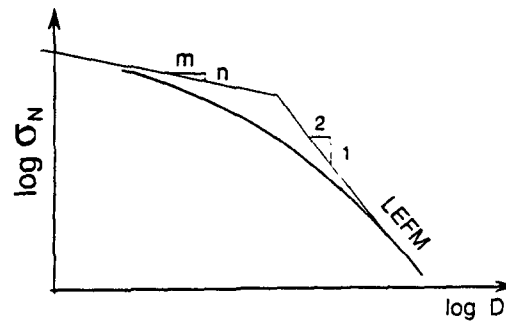


Figure 2: Scaling law according the nonlocal generalization of Weibull theory for failures after large crack growth

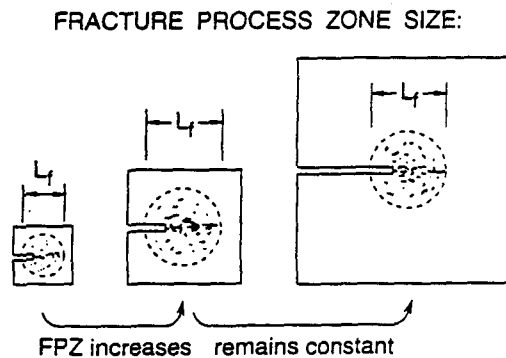


Figure 3: Changes of fracture process zone size with increasing structure size.

The reason is that, when the structures are sufficiently large, a further increase of the structure size is not accompanied by any increase in the size of the fracture process zone (Fig. 3). The Weibull-type probability integral in (4) is taken over the entire structure, however, the only significant contribution to the integral comes from the fracture process zone. Since the fracture process zone does not increase with an increase of the structure size, it is obvious that the failure probability should not be affected by a further increase of the structure size if it is already large.

3 Lacunar fractality of microcracks: Can it cause size effect?

Having discussed Weibull theory, we are ready to tackle the lacunar fractality of microcracks, which is illustrated in Fig. 4). From distance we see one crack, but looking closer we see it consists of several cracks with gaps, and looking still closer we see

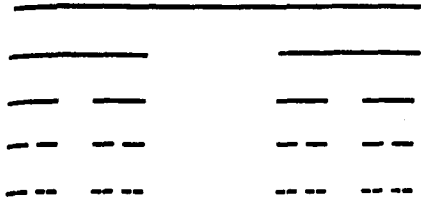


Figure 4: Top: Lines of microcracks as lacunar fractals, at progressive refinements; bottom: 'MFSL' law proposed by Carpinteri et al. (1995).

that each of these cracks consists of several smaller cracks with gaps, and so forth. Refinement to infinity generates a Cantor set or a fractal set whose fractal dimension d_f is less than the Euclidean dimension of the space (which is 1 for cracks in a line; Fig. 4).

The argument that lacunar (or rarefying) fractality is the cause of size effect in quasibrittle structures (Carpinteri and Chiaia, 1995) went as follows. The fractal dimensions of the arrays of microcracks are different at small and large scales of observation. For a small scale, the fractal dimension D_f is distinctly less than 1, and for a large scale it is nearly 1. For the failure of a small structure the small scale matters, and for the failure of a large structure the large scale matters. Therefore, there should be a transition from a power scaling law corresponding to small scale fractality to the power scaling law corresponding to the large scale fractality, the latter having exponent 0 for the strength, i.e., no size effect. Thus, it was argued, the size effect should be given by a transitional curve between the two asymptotes of slope $-1/2$ and 0 shown in Fig. 5. The slope of the initial asymptote was assumed to be $-1/2$. This size effect was described by the so-called the 'multifractal' scaling law (MFSL) (Carpinteri et al. 1993, 1995a,b,c)

$$\sigma_N = \sqrt{A_1 + \frac{A_2}{D}} \quad (6)$$

in which A_1 and A_2 are constants. It was shown that some test data for concrete can be reasonably well described by this formula (although the data can be equally well described by Bažant's size effect law based on energy release provided that there is a residual strength or that the failure occurs at crack initiation; Bažant 1997a).

There are, however, test data that clearly disagree with the MFSL Law, Eq. (6). Many test data exhibit in the logarithmic size effect plot an initial slope much less than $-1/2$, particularly for specimen sizes that are as small as possible for the given

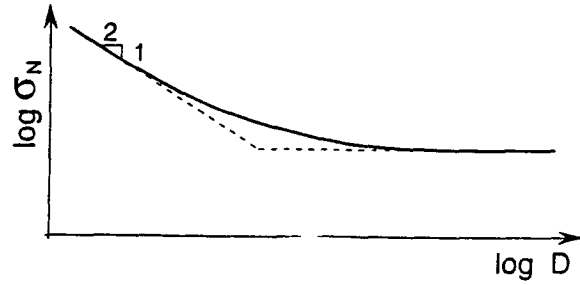


Figure 5: Size effect formula called the 'multifractal scaling law' (MFSL) proposed by Carpinteri et al. (1995).

size of aggregate. Many data approach an asymptote of slope $-1/2$ at very large sizes. Also, there are many data that exhibit a negative rather than positive curvature in the plot of $\log \sigma_N$ and $\log D$. These features disagree with the MFSL law.

More seriously, at closer scrutiny there are also mathematical and physical reasons why the lacunar fractality cannot be the source of the observed size effect. If the failure is assumed to be controlled by lacunar fractality, that is by microcracks, it obviously implies that the failure occurs at crack initiation, in which case the mathematical formulation must be akin to Weibull theory. Labeling the aforementioned small and large scales of observations by superscripts A and B , the Weibull distributions of the strength of a small material element in the fracture process zone with lacunar microcracks may be written as

$$\varphi[\sigma(\mathbf{x}); d_f^A] = \left\langle \frac{\sigma_N S(\xi) c_f^{1-d_f^A} - \hat{\sigma}_u^A}{\hat{\sigma}_0^A} \right\rangle^m \quad (7)$$

$$\varphi[\sigma(\mathbf{x}); d_f^B] = \left\langle \frac{\sigma_N S(\xi) c_f^{1-d_f^B} - \hat{\sigma}_u^B}{\hat{\sigma}_0^B} \right\rangle^m \quad (8)$$

Here the stress in the small material element of random strength has been written as $\sigma = \sigma_N S(\xi)$, in which S is the same function for all sizes of geometrically similar structures, and $\xi = \mathbf{x}/d$, for the non-fractal (non-lacunar) case. For the fractal (lacunar) case, this is generalized as $\sigma = \sigma_N S(\xi) c_f^{1-d_f}$ because the stress of the material element, in the case of lacunar microcracks, must be considered to have a non-standard, fractal dimension. Obviously, the Weibull constants $\hat{\sigma}_0$ and $\hat{\sigma}_u$ must now be considered to have fractal dimensions as well, but Weibull modulus m must not.

An equation of the type of Eq. (7) or (8) was written by Carpinteri et al., however, further analysis consisted of geometric and intuitive arguments. We will now sketch a recently published mechanical

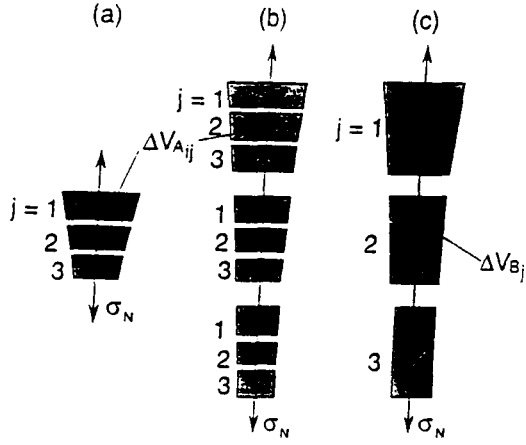


Figure 6: Subdivision of: (a) a small structure into small elements, (b) a large structure into small elements, and (c) a large structure into large elements.

analysis (Bazant, 1997b).

In Weibull theory (failure at initiation of macroscopic fracture), every structure is equivalent to a long bar of variable cross section (Bazant, Xi and Reid 1991); Fig. 6. Carpinteri et al.'s argument means that a small structure is subdivided into small material elements (Fig. 6a), while a large structure is subdivided into proportionately larger material elements (Fig. 6c). However, this approach to the failure mechanism of two structures made of the same material violates the principle of objectivity.

The large elements of the larger structure shown in Fig. 6c must be divisible into the small elements considered for the structure in Fig. 6a, which are the representative volumes of the material for which the material properties are defined. This follows from the requirement of objectivity of the definition of the material properties. Such intermediate subdivision into the small elements is shown in Fig. 6b. Therefore, it must be possible to calculate the failure probability of the largest structure on the basis of the refined subdivision into the small elements, as shown in Fig. 6b, or else it would imply that the small and large structures are not made of the same material.

It may now be noted that the Weibull failure probability P_f of the large structure subdivided into large elements $j = 1, 2, \dots, N$, and the failure probability P_{fj}^B of the large element of the large structure subdivided into small elements B_A , may be written as follows

$$-\ln(1 - P_{fj}^B) = \sum \varphi(\sigma_N S_{ij}^A; d_f^A) \Delta V_{Aij} / V_r \quad (9)$$

$$-\ln(1 - P_f) = \sum_j \varphi(\sigma_N S_j^B; d_f^B) \Delta V_j^B / V_r \quad (10)$$

Now the crucial point: Since for the same material one may subdivide each element B of the large structure into the small elements A , the following relation must hold:

$$\begin{aligned} -\ln(1 - P_f) &= -\sum_j \ln(1 - P_{fj}^B) \quad (11) \\ &= \sum_j \sum_i \varphi(\sigma_N S_{ij}^A; d_f^A) \Delta V_{ij}^A / V_r \end{aligned}$$

Equating this to (10), we must conclude that, in order to meet the requirement of the objectivity of the material properties, the Weibull characteristics on scale A and B must be different and must be such that

$$\varphi(\sigma_N S_j^B; d_f^B) = (\Delta V_j^B)^{-1} \sum_i \varphi(\sigma_N S_{ij}^A; d_f^A) \Delta V_{ij}^A \quad (12)$$

Eqs. (11) and (12) imply that consideration of different scales cannot yield different scaling laws. The same power law (in the case of zero Weibull threshold) must result from the hypothesis of lacunar fractality of microcrack distribution, regardless of the scale considered. The material properties do not depend on whether we look at them on a small scale or on a large scale.

Thus, the scaling law of a structure failing at the initiation of fracture from a fractal field of lacunar microcracks must be identical to the scaling of the classical Weibull theory. This means that the 'MFSL' scaling law does describe what is has been intended to describe. The only new aspect offered by the lacunar fractal hypothesis is that the values of Weibull parameters depend on the lacunar fractality. This dependence would have to be taken into account if the values of these parameters were predicted by micromechanics. But as long as the Weibull parameters are determined by experiments (which is the only approach feasible in practice today), the lacunar fractality of microcracks can have no effect on the scaling law.

4 Conclusions

1. To capture the effect of randomness of material strength on the size effect on nominal strength of structure, a nonlocal generalization of Weibull theory may be used.

2. The hypothesis of lacunar fractality of microcrack distribution in the fracture process zone cannot explain the size effect.

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