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Volume 10

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The IUTAM Bookseries publishes the proceedings of IUTAM symposia under the auspices of the IUTAM Board.

# IUTAM Symposium on Scaling in Solid Mechanics

Proceedings of the IUTAM Symposium held  
in Cardiff, UK, 25–29 June, 2007

Edited by

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## Recent Progress in Energetic Probabilistic Scaling Laws for Quasi-Brittle Fracture

Zdeněk P. Bažant and Jia-Liang Le

**Abstract** Rational determination of safety factors necessitates establishing the probability density function (pdf) of the structural strength. For perfectly ductile and perfectly brittle materials, the proper pdf's of the nominal strength of structure are known to be Gaussian and Weibullian, respectively, and are invariable with structure size and geometry. However, for quasibrittle materials, many of which came recently to the forefront of attention, the pdf has recently been shown to depend on structure size and geometry, varying gradually from Gaussian pdf with a remote Weibull tail at small sizes to a fully Weibull pdf at large sizes. The recent results are reviewed, and then mathematically extended in two ways: (1) to a mathematical description of structural lifetime as a function of applied (time-invariable) nominal stress, and (2) to a mathematical description of the statistical parameters of the pdf of structural strength as a function of structure size and shape. Finally, recent experimental data are analyzed and applicability of the present theory is verified.

**Keywords** Probabilistic mechanics · extreme value statistics · structural strength · cohesive fracture · scaling · size effect

### 1 Introduction

The uncertainty in the understrength (or capacity reduction) parts of safety factors, which are still essentially empirical, is much larger than the typical errors of modern computer analysis of structures. This problem is of paramount importance for quasibrittle structures. Its resolution would yield greater benefits than most refinements in computational mechanics.

Purely empirical, statistically based, safety factors are adequate for structures whose failure is either purely ductile or purely brittle because the type of cumulative

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probability distribution function (cdf) of structural strength is independent of structure size and geometry, and is either Gaussian (normal) or Weibullian.

Not so for structures consisting of quasibrittle materials, which include, at normal scale, concrete, fiber-polymer composites, tough ceramics, rocks, sea ice, wood, etc., at normal scale, and many more at the scale of MEMS and thin films. For such structures, which are the focus of this study, the cdf continuously varies from nearly Gaussian to Weibullian as the structure size increases, and also depends on the structure geometry.

Quasibrittle structures are characterized by a fracture process zone that is not negligible compared to the cross-section dimensions of the structure. As firmly established by now, the mean strength of quasibrittle structures failing at fracture initiation does not scale as a power law but varies with the structure size as a gradual transition between two asymptotic size effect laws of power law type – one of them deterministic (or energetic), based on stress redistribution due to a large fracture process zone, and the other statistical, based on the weakest-link model and Weibull theory or random material strength.

In this study, it is argued that the safety factors for such structures, which have generally been considered as size independent and purely empirical, must also be considered to depend on the structure size as well as shape. Furthermore, the dependence of structural lifetime on structure size (at fixed nominal stress) must be considered to deviate, for such structures, from the power law predicted by Weibull theory, and the type of cdf of structural lifetime must be considered as size dependent. The safety factors must ensure an extremely low failure probability,  $\leq 10^{-6}$ .

For such a low probability, direct experimental verification by strength histograms is impossible. Therefore, a physically based theory whose experimental verification is indirect is required.

Extensive review of background works and related studies is given in [3].

## 2 Conspectus of Main Results

Recently it has been shown [2, 3] that the cdf of strength can be derived from the transition rate theory and the stress-dependence of activation energy of interatomic bond breaks (Fig. 1a). The analysis indicates that the far-left tail of every cdf of strength of a representative volume element (RVE) of any material must be a power law. For ductile (plastic) materials, the power-law tail is so remote that it plays no role, but not for quasibrittle materials. The cdf of strength of quasibrittle structures of positive geometry (which are the structures failing at fracture initiation) can be modelled as a chain (or series coupling) of RVEs (Fig. 1e).

It is demonstrated that the RVE must be modelled by neither a chain nor a bundle. Rather, it must be statistically represented by a hierarchical model consisting of bundles (or parallel couplings) of only 2 long sub-chains, each of them consisting of sub-bundles of 2 or 3 long sub-sub-chains of sub-sub-bundles, etc., until the nano-scale of atomic lattice is reached. The power-law character of the cdf tail is

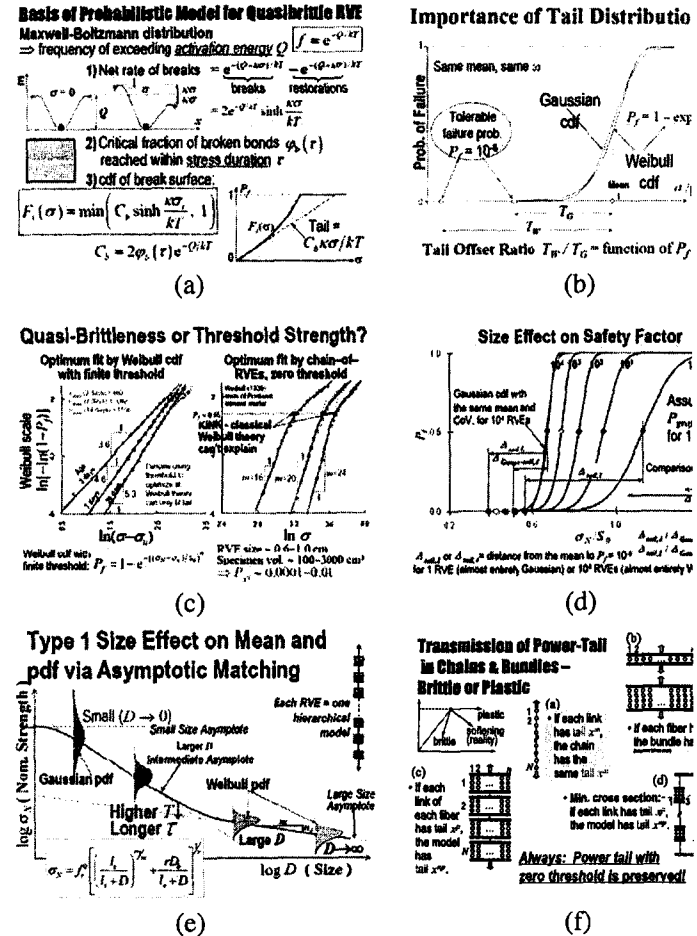


Fig. 1 Explanations of concept and properties of activation energy based chain-of-RVEs n Skewing of interatomic potential by applied stress  $\sigma$ , and resulting cdf of strength of nanos section for specified load duration  $\tau$ . (b) Gaussian (normal) and Weibull cdf for the same variance. (c) Histograms of Weibull's (1939) classical direct tension tests of cement mort 100 specimens are grouped in each point, fitted by Weibull distribution with nonzero (left) and by the present theory (right). (d) Variation of cdf of quasibrittle structures with size. (e) Curve of mean size effect of quasibrittle structure failing at macro-crack initiation variation of strength distribution. (f) Transmission of power-tail in chains and bundles

indestructible. It is transmitted through all the scales from nano to macro while its exponent is gradually raised from 1 on the atomistic scale to a value equal, on the RVE scale, to the Weibull modulus (typically between 10 and 50) (Fig. 1f).

The physical meaning of Weibull modulus is shown to be the minimum of the sum of strength powers among all possible cuts separating the hierarchical model into two halves, which should be equal to the number of dominant cracks needed to break the representative volume element (RVE) of material. Thus the model indicates the Weibull modulus  $m$  to be governed by the packing of inhomogeneities within a RVE. It is also shown that the RVE cannot be defined in the classical sense but must be understood as the smallest material volume whose failure causes failure of the whole structure (of positive geometry).

The model indicates that the cdf of RVE strength must have a broad Gaussian (or error function) core, onto which a power-law tail of an exponent equal to the Weibull modulus is grafted at the failure probability of about 0.001, if the structure is quasibrittle [2–4]. The model predicts how the grafting point, separating the Gaussian and Weibullian parts, moves to higher failure probabilities as the structure size increases, and also how the grafted cdf depends on the temperature, lifetime (or load duration, loading rate) and activation energy (which in turn is affected by aggressive chemical environment). On a large enough scale (equivalent to at least 5000 RVEs), quasibrittle structures must follow the Weibull distribution with, necessarily, a zero threshold. Thus the cdf of structure strength changes from predominantly Gaussian for small sizes to predominantly Weibull for large sizes (Fig. 1e).

It is widely agreed that engineering structures must generally be designed to ensure failure probability  $P_f \leq 10^{-6}$ . Since the point of  $P_f = 10^{-6}$  is, for the Weibull distribution, about twice as far from the mean than it is for the Gaussian distribution of the same mean and variance (Fig. 1b), the understrength part of the safety factor may change greatly when passing from the Gaussian cdf for small sizes to the Weibull cdf for large size (Fig. 1d). The consequences for the necessary safety factor may be serious. They are unique to quasibrittle structures. They have not been considered in the design of large concrete structures or large composite parts of aircraft made of composites.

On the other hand, the coefficient of variation (CoV) of quasibrittle structures, unlike perfectly brittle structures, decreases with structure size until the Weibull cdf is reached. This behavior may partly or fully offset the need for a larger safety factor Fig. 1(d).

The experimental histograms with kinks, which were previously thought to require the use of a finite threshold, are shown to be fitted much closer by the proposed chain-of-RVEs model with a zero threshold (Fig. 1c). For not too small structures, the model is shown to represent, in the mean sense, essentially a discrete equivalent of the previously developed nonlocal Weibull theory [7], and to match the mean size effect law previously obtained from this theory by asymptotic matching (Fig. 1e) [1].

The chain-of-RVEs model (Fig. 1e) can be verified and calibrated from the mean size effect curves, as well as from the kink locations (Fig. 1c) on experimental strength histograms for sufficiently different specimen sizes. The former is more effective. Strength histograms for specimens of one size only are not

sufficient for fully calibrating the theory. The mean stochastic response agrees with the cohesive crack model, crack band model and nonlocal damage models [5, 6]. The first few statistical moments of cdf for various sizes agree with the nonlocal Weibull theory. This indicates that, due to the finite size RVE, the present theory also captures the energetic size effect (of Type 1, [1]), which is caused by stress redistribution.

### 3 Review of Size Effect in Weakest Link Model and Its Asymptotics

We will consider geometrically similar structures of different sizes  $D$ , representing the characteristic dimension of the structure. We will restrict consideration to structures of positive geometry. This is a broad class of structures, for which the derivative of the energy release rate with respect to the crack length at constant load  $P$  is positive. These are structures that fail (under load control) as soon as the full fracture process zone (FPZ) forms and a distinct continuous macro-crack begins to grow. Let  $\sigma_N = P/bD$  = nominal stress in a structure, where  $P$  = applied load (or parameter of the load system) and  $b$  = structure width. For geometrically similar elastic or elasto-plastic structures,  $\sigma_N$  at maximum load (or the nominal strength of structure) is independent of the structure size, and, therefore, a decrease of  $\sigma_N$  with the structure size is called the size effect.

From the viewpoint of failure statistics, a structure of positive geometry may be modeled as a chain, which is known as the weakest link model. In that case that the structure fails as soon as the FPZ, roughly equal to one RVE, is fully formed). For such structures, the representative volume element (RVE) must be defined as the smallest material volume whose failure causes the whole structure to fail. The size of the RVE can be considered equal to the width of the FPZ and typically equals 2–3 material inhomogeneity sizes. If one RVE fails, the whole structure fails, i.e., the strength of the chain is decided by its weakest element, or link, which is called the weakest link model. In our interpretation, each link corresponds to one RVE, and so we have a chain-of-RVE model.

It has been shown that the strength of one RVE of a quasibrittle material must have a composite cdf having a broad Gaussian core onto which a Weibull tail is grafted at the probability of about  $P_g \approx 10^{-3}$  (called the grafting probability) [2–4]. Since, in the weakest-link model,  $\sigma_N = \sigma$  = stress in each link (i.e., one RVE), the Gaussian core may be written as:

$$P_1 = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-x^2/2} dx, \quad x = \frac{\sigma_N - \mu}{s_0} \quad (1)$$

where  $\mu$  = mean,  $s_0$  = standard deviation, and  $\Phi(x)$  is the error function representing the standard (unit) Gaussian (or normal) cdf. Because  $P_g$  is, for a single RVE, very small, the Weibull tail is nearly identical to a power function, i.e.

is rescaled horizontally and vertically to be normalized. According to Eq. (2), if the strength of each link in a chain is Weibullian up to stress  $\sigma_w$ , corresponding to link failure probability  $P_1(\sigma_w)$ , then the cdf of the strength of the whole chain is Weibullian for all  $P_f \leq P_{N_g} = 1 - [1 - P_1(\sigma_w)]^N$ .

In the remaining part for  $P_f > P_{N_g}$ , the cdf at increasing  $\sigma$  closely approaches the Gaussian distribution if  $N$  is small, but the Gumbel distribution if  $N \rightarrow \infty$ . For large  $N$  the Weibullian part occupies almost the entire cdf (i.e.,  $P_{N_w} \rightarrow 1$ ), and so the Gumbel part is irrelevant.

Hence, we may assume that, approximately, the strength cdf for any  $N$  is a graft of Weibull cdf onto a Gaussian cdf, with the grafting point given approximately by

$$P_{N_g} \approx 1 - (1 - P_{1g})^N, \quad P_{1g} = P_1(\sigma_w) \quad (12)$$

If  $P_{N_g}$ ,  $\bar{\sigma}_N$ ,  $\omega$  are known for any  $N$ , the grafted Weibull-Gaussian cdf can be constructed similarly to Eqs. 50–56 in [3]. The entire cdf being known, one can then calculate the load for which the structural failure probability is, e.g.,  $10^{-6}$ . Integrating this cdf with the pdf of the load, one can also obtain the structural failure probability for a given distribution of the load.

## 6 Size Effect on Structure Lifetime

All of the foregoing analysis applies for constant temperature and a fixed load duration  $\tau$ . We will now explore the question of size dependence of lifetime  $\tau$  of quasibrittle structures under a given nominal stress  $\sigma_N$ .

The transition state theory with the concept of activation energy has been used to show that the left tail of the cdf of the strength of interatomic bonds must have the form  $F(\sigma) = (C_b \kappa / kT) \sigma =$  power function of stress  $\sigma$  with exponent 1 [2, 3]; where  $T$  = absolute temperature,  $k$  = Boltzmann constant,  $\kappa$  = coefficient of linear dependence of activation energy on  $\sigma$ , and  $C_b$  = constant. Based on this fact it is further shown that, for various (though constant)  $T$  and  $\tau$  values, the nominal strength  $\sigma_N/s_0$  in the argument of failure probability  $P_1$  of one RVE must be replaced by  $\sigma_N/s_0 R(\tau, T)$  where

$$R(\tau, T) = \frac{\Lambda(\tau_0) T}{\Lambda(\tau) T_0} e^{\left(\frac{1}{T} - \frac{1}{T_0}\right) \frac{Q}{k}} \quad (13)$$

where  $T_0$ ,  $\tau_0$  = reference values of  $T$  and  $\tau$ , for which  $R = 1$ , and function  $\Lambda$  indicates how the stress for which the atomic thermal vibrations produce a contiguous surface of a break in the material nanostructure scales with the load duration [3] (where, for the sake of simplicity, one may set  $\Lambda(\tau) = r\tau =$  linear function of  $\tau$ ;  $r =$  constant). Hence, for a finite chain of RVEs, Eq. (2) must be generalized as:

$$P_N(\sigma_N, \tau, T) = 1 - \{1 - P_1[\sigma_N/s_0 R(\tau, T)]\}^N \quad (14)$$

From this we can obtain not only the pdf of structural strength as a function of applied  $\sigma_N$ , i.e.  $p_f = [dP_N/d\sigma_N]_{T,\tau}$ , but also the pdf of strength as a function of load duration  $\tau$ , i.e.  $p_N(\tau) = [dP_N/d\tau]_{T,\sigma_N}$  which can be calculated from Eqs. (13) and (14). This distribution agrees with the requirements that, for  $\sigma_N \rightarrow 0$ , one has  $P_N = 0$  and  $\tau \rightarrow \infty$ ; while for  $\sigma_N \rightarrow \infty$ , one has  $P_N = 1$  and  $\tau \rightarrow 0$ .

The mean structural lifetime (or durability) as a function of  $\sigma_N$  and  $T$  may be calculated as

$$\bar{\tau}(N, \sigma_N, T) = \int_0^\infty (1 - P_N) d\tau = \int_0^\infty \{1 - P_1[\sigma_N/s_0 R(\tau, T)]\}^N d\tau \quad (15)$$

$$\text{and for } N \rightarrow \infty: \bar{\tau}(N, \sigma_N, T) = \int_0^\infty e^{-N P_1[\sigma_N/s_0 R(\tau, T)]} d\tau \quad (16)$$

The last equation follows by setting  $P_1 = x/N$  and noting that  $\lim_{N \rightarrow \infty} (1 - x/N)^N = e^{-x}$ . If we assume that  $R$  is linear in  $\tau$ , i.e.  $\Lambda(\tau) = r\tau$ , and note that  $\Gamma(1/m)/m = \Gamma(1 + 1/m)$ , then for  $N \rightarrow \infty$ :

$$\bar{\tau} = \int_0^\infty e^{-N(\sigma_N \tau / s_0 \tau_0 R_0)^m} d\tau = \tau_0 \frac{s_0}{\sigma_N} \frac{T}{T_0 N^{1/m}} e^{\left(\frac{1}{T} - \frac{1}{T_0}\right) \frac{Q}{k}} \Gamma\left(1 + \frac{1}{m}\right) \quad (17)$$

Note that the mean lifetime for  $N \rightarrow \infty$  decreases as a power law of structure size and is inversely proportional to the applied stress. For finite  $N$ , however, there is a deviation from the power law.

For the coefficient of variation  $\omega_\tau$  of structural lifetime under load  $\sigma_N$  and at temperature  $T$ , one has

$$\omega_\tau^2 = \frac{1}{[\bar{\tau}(\sigma_N, T)]^2} \int_0^\infty [\tau(\sigma_N, T)]^2 dP_N[R(\tau, T)\sigma_N] - 1 \quad (18)$$

$$\text{for } N \rightarrow \infty: \omega_\tau^2 = \frac{\Gamma(1 + 2/m)}{\Gamma^2(1 + 1/m)} - 1 \quad (19)$$

This is the same coefficient of variation as for the strength at fixed load duration, and is governed solely by Weibull modulus.

To obtain the small size asymptotics of  $\bar{\tau}$  and  $\omega_\tau$ , asymptotic matching similar as before may be used. For small  $N$ , the cdf of  $\tau$  is again Gaussian, except for the tail of probability  $<$  circa 0.001, and with increasing  $N$ , a Weibull tail grows into the Gaussian core until, for  $N >$  circa 5000, it occupies essentially the entire cdf of  $\tau$ .

To determine the approximate complete pdf of lifetime, one must begin with the grafted cdf of  $\sigma_N$  at fixed  $\tau$ , which gives the (cumulative) failure probability for any chosen  $\tau$  and  $\sigma_N$ . Its differentiation with respect to  $\tau$  yields the desired grafted pdf of lifetime for a structure of any size  $N_{eq}$ . Again, the grafting point is found to move from left to right as the structure size increases.

## 7 Closing Comments

The mathematical model established in this article for the effect of structure size and geometry on the probability distribution of structural lifetime at constant load is not yet verified experimentally. The verification and calibration is left for a forthcoming journal article.

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# The Fractal-Statistical Nature of Size-Scale Effects on Material Strength and Toughness

Alberto Carpinteri and Simone Puzzi

**Abstract** The size-scale effects on the mechanical properties of materials are a very important topic in engineering design. Three different modeling approaches have been proposed and analyzed at least, i.e. the statistical, the energetical and the fractal one. Aim of this paper is to revisit the fractal approach and to reject the most recurrent criticisms against it. Moreover, we will show that it is wrong to set the fractal approach to size-scale effects against the statistical one, since they are deeply connected. More in detail, by analyzing a fractal distribution of micro-cracks in the framework of Extreme Value theory, we will obtain a scaling law for tensile strength characterized, in the bi-logarithmic plot, by the slope  $-1/2$ . Conversely, by considering a fractal grain size distribution in a grained material, we will obtain a scaling law for fracture energy characterized, in the bi-logarithmic plot, by the positive slope  $1/2$ . These slopes are the natural consequence of perfect self-similarity of the flaw (or grain) size distribution. Eventually, the theoretical results regarding the link between fractals and statistics will be confirmed by numerical simulations.

**Keywords** Size-scale effects · self-similarity · fractals

## 1 Introduction

Since the pioneering paper by Mandelbrot et al. [9] on the fractal character of the fracture surfaces in metals, the fractal features in the deformation and failure of materials have been investigated by several Researchers. Self-affinity of roughness of fracture surfaces has been found in a wide range of materials, from metals to wood, from ceramics to concrete, from rock to polymers (see, e.g. the review paper [10]). Since fractality had been discovered, several Authors tried to connect it with the size-scale effects on fracture energy [11–17].

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F.M. Borodich (ed.), *IUTAM Symposium on Scaling in Solid Mechanics*,  
IUTAM Bookseries 10 DOI 10.1007/978-1-4020-9033-2\_14,  
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