Crack Band Model for Fracture of Geomaterials

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SYNOPSIS
Due to their heterogeneity, fracture in rocks as well as the artificial rock — concrete — propagates with a deepened band of microcracks at the front. The progressive formation of the microcracks is described by a triaxial stress-strain relation which exhibits a gradual strain-softening. The stiffness matrix of a material intersected by a system of parallel continuously distributed cracks is obtained in the limit. The area under the stress-strain curve, multiplied by the width of the crack band (fracture process zone) represents the fracture energy. The resulting fracture theory is characterized by three independent parameters, the fracture energy, the tensile strength, and the width of the crack band, latter being empirically found to approximately equal five-times the grain size in rock. The formulation lends itself easily to a finite element analysis, which is employed to calculate the theory by fitting various test data on rock fracture available in the literature. Excellent agreement is achieved both with the Austin load data and the B-curves data. A linear fracture theory, which is obtained as the limit for very large sizes of the structure and the finite elements, and the effect of the choice of element size are discussed. Moreover, similarity between the fracture of rocks and of concretes is emphasized and the results obtained in parallel investigations of concrete are summarized. Finally, fracture analysis from the viewpoint of stress-localization instability is given.

INTRODUCTION
Due to their heterogeneous microstructure and the relatively low strength of interface bonds, many geomaterials, including most rocks as well as the artificial rocks — concretes, exhibit fracture behavior which markedly differs from that of metals, glass, polymers and other materials. In materials for which fracture mechanics was developed first, the crack tip is surrounded by a nonlinear zone which is rather small, and linear fracture mechanics is then applicable. Later nonlinear ductile fracture mechanics was developed for certain metals and other materials in which the nonlinear zone surrounding the crack tip is large compared to the dimensions of the specimen (Fig. 1a). An often unpronounced yet important characteristic of these theories is that the fracture process zone, defined as the zone in which progressive microcracking or void formation causes a decrease of stress at increasing strain, remains small.

The fracture of geomaterials differs chiefly in the fact that not only the nonlinear zone but also the fracture process zone is large compared to the dimensions of the specimen or structure (Fig. 1b). At the same time, compared to the theories for ductile fracture of metals, we detect one simplifying feature, namely that the nonlinear zone is not much larger than the fracture process zone, permitting us to consider, as an approximation, that all material surrounding the fracture process zone is elastic. Due to this fact, there is no need to employ the J-integral in the analysis; indeed, nonlinear behavior is found only inside the fracture process zone, in which the J-integral cannot be contour-independent as a result of strain-softening and the fact that fracture energy is being consumed all over this zone.

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In the modelling of fracture in finite element analysis, there are two possibilities: either the cracks are considered to open at the interface of finite elements, or they are modelled as bands of continuously distributed (cracked, dispersed) cracks intersecting the entire finite element. The second approach, introduced in the mid-1960's by Kern, and Age and Sneedall, has prevailed for large finite element meshes. While the interelement crack approach requires doubling the number of nodes as the cracks pass through them, the second approach has the advantage of introducing a global crack system which, when combined with a global stiffness matrix, can be used to determine the stress fields. The crack direction is simply characterized by the direction of the axis of material anisotropy. Such a crack band can propagate through an element mesh in an arbitrary direction if we accept the approximation of a smooth curved crack band by a zig-zag band. In the present lecture, we outline an extension of this, by now classical, approach to the modeling of nonlinear fracture, placing emphasis on the applications to rocks.

**FULLY CRACKED MATERIAL**

Throughout this lecture, we explore cartesian coordinates $x, y, z$, and the cracks being assumed to be normal to the axis $z$. The normal stress and strain components may be grouped into the column matrices

$$\sigma = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \end{bmatrix}, \quad \epsilon = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \epsilon_z \end{bmatrix}$$

where $\sigma$ denotes the stress tensor, and the strain are assumed to be linearized, or small. The elastic stress-strain relation for the normal components may then be written as

$$\epsilon = \frac{1}{E} \sigma$$

where $E$ is the modulus of elasticity.

In which

$$E = \begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{12} & E_{22} & E_{23} \\ E_{13} & E_{23} & E_{33} \end{bmatrix}$$

$\frac{1}{E}$ is the stiffness matrix of the uncracked material.

If the elastic material becomes intersected by continuously distributed cracks normal to $z$, the stress-strain relation is known (e.g., Sain and Schnobrich, 1973) to take the form

$$\epsilon = \frac{1}{E} \sigma$$

$$\sigma = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$\sigma_{cr}$

$$p_{cr} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This matrix, representing the stiffness matrix of a fully cracked material, is derived from the condition that the stress normal to the cracks must be $0$, assuming the material between the cracks to behave as an uncracked elastic material (this is a simplification, because often even the material between the cracks may be damaged by presence of discontinuous microcracks).

**PROGRESSIVE MICROCRACKING**

To describe progressive development of microcracks in the fracture process zone, we need to formulate a stiffness matrix which continuously changes from the form given in Eq. 3 to that in Eq. 5. This objective is not very easy to achieve by direct reasoning, since every element of the $E_{ii}$ stiffness matrix changes. It was found (Bafron, 1989), that in such a case the compliance matrix $C_i$, which is

$$C_i = \frac{1}{\epsilon}$$

where

$$\epsilon = \frac{1}{\sigma}$$

It appears that one needs to consider only one element of the compliance matrix to change:

$$C_{ij} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Since the following statement (theorem) has been proven (Bafron, 1989):

$$\frac{G_{cr}}{G_0} = \frac{1}{\sigma}$$

i.e., the foregoing stiffness matrix of a fully cracked material (Eq. 5) represents the limit of the inverse of the compliance matrix with parameter $i$ in Eq. 8 as this parameter tends to $0$. With regard to numerical programming, it should be noted that instead of setting $E = 0$ and $\epsilon = 0$, we assign small number (e.g., $10^{-15}$) so as to allow the computer to carry out the inversion of the matrix numerically. The result is a stiffness matrix like that in Eq. 5 except that extremely small numbers ($10^{-15}$) are obtained (instead of $0$). For the programmer, this is actually an easier way to write the program than to directly set up Eq. 5.

Comparing now the compliance matrices in Eqs. 1 and 8, we see that a continuous transition from a crack-free state to a fully cracked state may be very simply obtained by a continuous variation of parameter $i$, which we call the cracking parameter. The limiting cases are

- uncracked: $i = 1$
- fully cracked: $i = 0$

The variation of the cracking parameter, $i$, may be calibrated on the basis of a uniaxial tension test. From tests carried out on extremely stiff machines or with a stabilization by parallel stiff bars (e.g., Evans and Marathe, 1965), and employing sufficiently small
In the absence of significant rotation of the principal stress directions during the passage of the fracture process zone through a fixed section justifies another simplifying assumption which is implicit in our preceding formulation. It is a fact that the total stress-strain relations which are employed (Eq. 8) are path-independent. In reality, all inelastic behavior is of course path-dependent. Nevertheless, the assumption of path-independent behavior is acceptable for practical purposes of the study of ductile fracture, in which the application of Ricci's hypothesis is contingent upon the validity of this assumption.

A characteristic feature of the matrix properties for progressive microcracking (Eq. 7) is the absence of Poreson effect with regard to the cracking. This feature may be justified on physical grounds if we assume all the microcracks to be normal to axis. This is certainly a simplifying assumption, and in reality we must assume a distribution of orientations of the microcracks, the orientation normal to axis being the predominant one. If inclined microcracks were considered, then it would be necessary to also change the off-diagonal terms in Eq. 7 as the formation of microcracks advanced.

The difference of the actual strain $e_{ij}$ at the straintensoriating branch from the strain predicted for an uncracked material (Fan, 1981) is the strain arising from the interference between the crack band, which is the difference between the actual strain and the strain predicted for an uncracked material. This difference is the strain arising from the interference between the crack band and the uncracked material.

$$e_{ij} = e_{ij}^{(a)} - e_{ij}^{(u)}$$

For $u = 0$ prevent:

$$e_{ij}^{(a)} = e_{ij}^{(u)}$$

where $e_{ij}^{(a)}$ is the average strain and $e_{ij}^{(u)}$ is the strain at the crack band.
The case when the slope of the strain-softening branch is very small is also inadmissible for practical purposes. Therefore, the size of the finite element should be distinctly less than the values ω₀ given by Eq. 23. Since typically C3 = -C35 it appears suitable to use finite elements of width

\[ h = \omega_0/4 \]  

(24)

**VERIFICATION BY TEST DATA**

The foregoing mathematical model, which was first developed and applied with a great deal of success to concrete (Balsamo, 0h, 1981), has been fitted by 0h (1982) to a large number of test data on large structures available in the literature. The optimum fits achieved are shown by the solid lines in Figs. 4-10. For comparison, the best possible fits according to the classical linear fracture mechanics are also shown in these figures, as the dashed lines. The material parameters corresponding to the solid lines are available in Table 1. All fits were calculated by the finite element method, using rectangular finite elements such that consist of four constant strain triangles. A plane stress state was assumed for all calculations. The stress-strain relation we just developed was assumed to hold for all finite elements but using small enough loading steps the softening state was reached only within one or two of finite elements. The tangent stiffness was assumed to be the same for all four triangles comprising a rectangular element and was determined from the average of the strains in the four triangles.

In a preceding analysis (Balsamo, 0h, 1981) of twenty-two test series on the fracture of concrete reported in the literature—quite a large statistical sample indeed—it was discovered that the optimum width of the crack (fracture process zone) is roughly w₅/3, where w₅ represents the maximum size of the aggregate in concrete. It is for this width w₅ that the area under the stress-strain curve yields the correct value of the fracture energy needed to obtain good fits of the test data. Finding in mind, the fits of all fracture test data presented here were sought under the restriction that the ratio w₅/3 where w₅ = grain size, be the same for all test data for the various rocks considered, and only the values of C₂ and C₄ were considered to vary from rock to rock (Table 1). This analysis led to the following rather useful result:

\[ \omega_0 = \frac{w_5}{3} \]  

(25)

Due to this relation, as well as the previously derived energy relation (Eq. 21), our nonlinear fracture theory is a two-parameter theory, the two material parameters to be determined by experiment being C₂ and C₄. We should, however, consider Eq. 25 as tentative since it rests mainly on the experimental results of Janssen et al. (1973); the statistical sample available for rock in the literature is much smaller than that available for concrete.

Test data in Figs. 4-7 (Schmidt, 1976; Carpentieri, 1980; Schmidt, Lutes, 1978, for isolated limestone, Carrara marble, Colorado oil shale, and Weatherby granite, give the maximum loads Pmax achieved in the fracture tests. These values are reported in terms of the ratio Pmax/P₀, where P₀ is the maximum load which follows by applying the bending theory. In Fig. 4, P₀ = 2851/0.5, maximum load based on bending theory calculation for an uncracked beam, where 5 = beam depth, b = beam thickness, L = beam span, and f = tensile strength. The same definition of P₀ was used for Figs. 5 and 6. For the tests in Fig. 7, P₀ = wL, where w = specimen width, and b = its thickness. The maximum loads Pmax were obtained with the finite element code simply by increasing the displacements at the loading points in small increments, calculating at each loading step the crack opening, and thus identifying the maximum reaction. The linear theory results, shown by the dashed lines, were calculated according to the blunt crack band approach (Janssen, 1979, 1980), considering a sudden stress drop and a finite element version of the energy release criterion.

Fig. 8 shows the fits of the data by Janssen et al. (1973) and Schmidt and Lutes (1979) on the b-curve (resistance curve), in which the apparent fracture energy determined from the tests is plotted as a function of the crack extension (in slow crack growth). According to the classical linear fracture mechanics, this apparent fracture energy would be constant, and the dashed line is horizontal and is made to pass through the terminal measured values at which the apparent fracture energy value stabilized.

To evaluate the errors, we may construct the plot of τ = Pmax/P₀ vs. x = X/P₀, in which Pmax = measured max P₀ = theoretical P₀, and P₀ = failure load based on strength as defined before. All the data points from Figs. 4-7, the number of which totals N = 10, are plotted in this manner in one figure (Fig. 9).

Furthermore, in Fig. 10, the same plot is shown for the results of linear theory. Since the ratio x generally decreases with the size of the specimen, the points at the right of the plot pertain to small specimen and those at left bottom to large specimens. If our theory were perfect, then the plot of τ vs. x would have to be a straight line having slope 1.0 and passing through the origin. The regression line of the data P₀ = s², must have a slope very close to 1.0 if the optimum fits have been correctly determined. The errors, i.e., the vertical deviations of data points Y from the regression line, may be characterized by the coefficient of variation, u:

\[ u = \sqrt{\frac{\sum (Y - \bar{Y})^2}{N}} = 1 \]  

(26)

where s = standard deviation and \( \bar{Y} \) = mean value. From Fig. 9 we calculate:

For our fracture theory (Fig. 9): u = 0.106

For linear fracture theory (Fig. 10): u = 0.452

For the strength criterion: u = 0.796

The last value for the strength criterion is simply a statistics of the population of P₀/P₀ — values. An additional statistical characteristic, one may calculate the coefficient of variation of the population of the values of X = Pmax/P₀; these are

![Fig. 5](image5.png)  
Fig. 5 Fit of the Maximum Load Data of Carpentieri (1980) for Carrara Marble.

![Fig. 6](image6.png)  
Fig. 6 Fit of the Maximum Load Data of Carpentieri (1980) for Colorado Oil Shale.
For our fracture theory: \( \omega = 0.069 \) \((28)\)

For linear fracture theory: \( \omega = 0.393 \) \((29)\)

For a statistical analysis of the errors in the N-curves, it is appropriate to normalize the fracture energy with regard to the internal force transmitted by the fracture process zone, which is roughly proportional to \( G_f / \omega \), where \( G_f \) is the fracture energy. Accordingly, we may plot the values of \( G_f / \omega \) taken from Fig. 8. On the Y-axis we use the measured value \( G_f \) of \( G_f \), and on the X-axis we use the theoretical value \( G_f \) of \( G_f \). Again, if the theory were perfect, this plot would have to be a straight line \( Y = aX^b \), with \( a < 0 \) and \( b > 1 \), and thus a linear regression may be carried out. The standard error for the vertical deviation from the regression line may then be calculated to be

For our fracture theory: \( a = 0.064 \) \((29)\)

For linear theory: \( a = 0.322 \) \((29)\)

Figs. 9–10 also show as dashed lines the 95% confidence limits corresponding to \( a \) and \( b \). These curves are hyperbolas but due to the large size of our statistical sample they are almost straight lines at a vertical distance \( \pm 1.96 \sigma \) units from the regression line.

From this statistical analysis we conclude that our nonlinear theory is capable of satisfactorily describing the available experimental evidence on the fracture of rock (Oh, 1982).

**LINEAR THEORY FOR LARGE BODIES AND STRUCTURES**

In Eq. 23, we gave an upper limit for the finite size that can be used for our nonlinear fracture theory. This does not mean however that one cannot use larger finite elements. The finite elements can be made larger than the value in Eq. 23, but then one must consider a sudden stress drop. In this case, which is the case of bodies or structures far larger than the

Table 1: Parameters for Test Data

<table>
<thead>
<tr>
<th>Test Series</th>
<th>( f_s ) (psi)</th>
<th>( E ) (kpsi)</th>
<th>( G_f ) (lb/ft)</th>
<th>( d_e ) (in.)</th>
<th>( \omega ) (in.)</th>
<th>( G_{in} ) (lb/ft)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Schmidt (1976) No. 1</td>
<td>356</td>
<td>3.130</td>
<td>0.069</td>
<td>0.078</td>
<td>0.393</td>
<td>0.073</td>
</tr>
<tr>
<td>2.</td>
<td>356</td>
<td>3.130</td>
<td>0.068</td>
<td>0.078</td>
<td>0.393</td>
<td>0.073</td>
</tr>
<tr>
<td>3.</td>
<td>356</td>
<td>3.130</td>
<td>0.066</td>
<td>0.078</td>
<td>0.393</td>
<td>0.073</td>
</tr>
<tr>
<td>4.</td>
<td>356</td>
<td>3.130</td>
<td>0.066</td>
<td>0.078</td>
<td>0.393</td>
<td>0.073</td>
</tr>
<tr>
<td>5.</td>
<td>356</td>
<td>3.130</td>
<td>0.066</td>
<td>0.078</td>
<td>0.393</td>
<td>0.073</td>
</tr>
<tr>
<td>6.</td>
<td>356</td>
<td>3.130</td>
<td>0.066</td>
<td>0.078</td>
<td>0.393</td>
<td>0.073</td>
</tr>
<tr>
<td>7.</td>
<td>356</td>
<td>3.130</td>
<td>0.066</td>
<td>0.078</td>
<td>0.393</td>
<td>0.073</td>
</tr>
<tr>
<td>8.</td>
<td>356</td>
<td>3.130</td>
<td>0.066</td>
<td>0.078</td>
<td>0.393</td>
<td>0.073</td>
</tr>
<tr>
<td>9.</td>
<td>356</td>
<td>3.130</td>
<td>0.066</td>
<td>0.078</td>
<td>0.393</td>
<td>0.073</td>
</tr>
<tr>
<td>10.</td>
<td>356</td>
<td>3.130</td>
<td>0.066</td>
<td>0.078</td>
<td>0.393</td>
<td>0.073</td>
</tr>
</tbody>
</table>

Note: \( psi = 6895 \) N/cm², \( lb/ft = 175.1 \) N/m, \( in. = 25.4 \) mm, kpsi = 1000 psi.

* Asterisk indicates numbers estimated by calculation; without asterisk as reported.

**Young's modulus provided by authors or other references.**
agreement with the exact solutions for sharp cracks and approximate these solutions just as well as the interelement crack approach. This has been demonstrated by Balfanz and Codolino (1979) and one of these demonstrations is shown in Fig. 13 in which the load parameter is plotted versus the crack length. The specimen on the coordinate was a rectangular panel with a center crack, loaded by a uniform stress at top and bottom. The calculation has been carried out for three different mesh sizes shown in Fig. 12; with finite element sizes in the ratio 1:2:3.

The exact solutions are slightly different for each case because the size of the specimen for the three meshes was not exactly the same.

The method which is currently used in all large finite element codes is to determine propagation of distributed (smoothed) cracking from one element to another on the basis of the tensile strength criterion. It has been shown for a long time that such a calculation cannot converge to correct results, since refinement of the element size to 0 leads to infinite stress concentrations just ahead of the front cracked element, so that the load needed for further extension of the crack tends always to 0. It has not been possible, however, to recognize that the use of the strength criterion can lead to very large errors. As demonstrated by Balfanz and Codolino (1980), the difference in the results can be as large as 100% when the finite element sizes differ as 1:2:1. This is demonstrated by the numerical finite element results in Fig. 13, where the failure load needed for further extension of the crack band is plotted for the same panel as in Fig. 12 against the length of the crack band. The curves obtained for meshes A, B, C of finite element sizes 1:2:1, are seen to be very far apart, whereas the curves for the finite element results obtained with the energy criterion for the abrupt stress drop agree with each other, the difference being negligible and tending to 0 as the mesh is refined.

CRACKS IN STEEL DIRECTION IN THE MESH
In a general situation the crack direction would not be parallel to the mesh line, as in the foregoing analysis of test specimens. A smoothly curved crack or crack band is then conveniently represented as a zigzag crack band in the finite element mesh; see Fig. 14. Referring to the notation defined in this figure, the effective width of the crack band to be used in the fracture theory is, for a square mesh of step b,

$$w_i = b_i$$

where $$b_i$$ is the angle of the crack direction with the mesh line. This condition simply follows from the requirement of an equal area of the zigzag crack band and a smooth crack band in the direction of the cracks.

TENSILE GENERALIZATION AND CURVED SOFTENING
In presence of tensile principal stresses in more than one direction, the preceding stress-strain relation (Eq. 6.6) may be generalized as follows:

$$\sigma \equiv \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{bmatrix}$$

where $$\varepsilon_x$$, $$\varepsilon_y$$, $$\varepsilon_z$$ are the principal strains at the point of interest.

To model real fracture processes more accurately, we should consider that the softening stress-strain diagram is generally curved and smoothly curved. Furthermore, when this is done, it is clear that the directions of the principal stress axes rotate during the progressive straining that leads to fracture. For this purpose, we need to generalize Eq. 32 into a tensorial form, guaranteeing fulfillment of tensors’ invariance conditions. This may be accomplished by the following second-order tensile stress-strain relation

$$\varepsilon_{ij} = \frac{1}{2} \left( \frac{\sigma_{ij}}{c_{ij}} \right)$$

where

$$c_{ij} = \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{bmatrix}$$

In which

$$c_{ij} = c_{ij}^{ijkl} \eta_k \eta_l$$

In which subscripts $$i, j, k, l$$ refer to cartesian coordinates $$x_i$$ ($$i = 1, 2, 3$$), $$\varepsilon_{ij}$$ is the tensor of second compliance and $$C_{ijkl}$$ is the tensor of elastic compliances, expressed for an isotropic material in terms of $$\eta_i$$ and $$b$$ is a scalar parameter. It may be checked that the term $$c_{ijkl}$$ satisfies only the diagonal coefficients in the compliance matrix and that Eq. 23 reduces to Eq. 22 when the principal stresses and strain directions coincide and principal coordinates are used. In particular, Eqs. 35-36 yield the following expressions for the cracking parameters

$$\sigma_{ijkl} = \frac{1}{2} \left( \frac{\varepsilon_{ijkl}}{c_{ijkl}} \right)$$

Parameter $$b$$ may be in general by a scalar function of stresses and strains, and by comparison with the typical shapes of the tensile softening curve a suitable form appears to be

$$b = \frac{1}{2} \left( \frac{\varepsilon_{ijkl}}{c_{ijkl}} \right)$$

In which case Eq. 35, when written for uniaxial tension, reduces to the form

$$\sigma = \frac{1}{2} \left( \frac{\varepsilon}{c_{ijkl}} \right)$$

where $$b$$ constant (depending on the strength of concrete).

We should also note that due to the use of an exponential function, the stress-strain diagram in compression is different from that in tension, and for sufficiently large exponent coefficient $$c$$ (a constant) the occurrence of strain-softernning in compression may be suppressed.
making the compression behavior linear. We should however realize that Eq. 34 still cannot describe the complete behavior in compression; in particular, Eq. 34 does not reflect the plastic component of deflections under high inelastic compression and the microfracturing under high compression stresses parallel to the microcracks.

It is possible that there exist other tensorial generalizations than that in Eqs. 35–37. If we restrict attention to quadratic terms in strain, Eq. 34, it is not. One may check this by trying all possible permutations of subscripts in the term $\epsilon_{ij} \epsilon_{kl}$. We should especially observe that the terms of the type $\epsilon_{ij} \epsilon_{kl}$ are admissible since they do not reduce to Eq. 34. On the other hand, the use of loading surfaces $\sigma_{ij} \sigma_{kl} = 0$, always leads to terms of the type $\epsilon_{ij} \epsilon_{kl}$ and $\epsilon_{ij} \epsilon_{kl}$, which always produces terms of the form $\epsilon_{ij} \epsilon_{kl}$ rather than $\epsilon_{ij} \epsilon_{kl}$ when quadratic terms are considered. We thus arrive at an interesting conclusion: The use of loading surfaces cannot provide a correct compliance matrix for the description of progressive microcracking which leads to complete fracture.

This means, especially, that the plastic-fracturing theory (Bažant, Klin, 1979), formulated to describe microcracking under compressive and shear loadings, cannot be extended to formulate tensile strain-softening up to complete fracture. This theory was based on the use of loading surfaces in the strain and stress spaces.

The picture of the stress-strain diagram given by Eq. 37 is shown in Fig. 15. To describe the fact that compressive stresses $\sigma_{ij}$ and $\sigma_{ij}$ in the directions parallel to the crack plane reduce the tensile stress (peak stress), it may be appropriate to further introduce parameter $\beta$ as a function of the principal stresses in the directions parallel to the crack plane.

**STRAIN-LOCALIZATION INSTABILITY**

The formation of fracture through a gradual deformation of a finite fracture process zone should properly be described by an instability of a nonlinear continuum, in which a uniformly distributed strain localizes into a softening band of finite width $\psi_0$, at the boundary of which there is a jump in the value of strain. With regard to shear failures, the concept of strain-localization in tension, with particular attention to the effect of geometric nonlinearities. A stability analysis of strain-localization in tension, with particular attention to finite size bodies and to the effect of differences between the tangent loading modulus and the tangent unloading modulus in the strain-softening regime, was presented by Bažant (1976) and Bažant and Panasiuk (1978). Fundamental to the stability analysis is the sign of the energy change due to strain localization.

Consider the work $\mathcal{W}$ which must be supplied externally to the structure to advance the crack by distance $\Delta a$. According to Bažant and Cebolín (1979, 1980) this work may be expressed as

$$
\Delta W = \mathcal{W}_e = \mathcal{W}_0 + \mathcal{W}_1
$$

where

$$
\mathcal{W}_0 = \int_0^\infty \frac{1}{2} \sigma_{ij} \epsilon_{ij} - \frac{1}{2} \epsilon_{ij} \epsilon_{kl} \sigma_{kl} \, dV
$$

and

$$
\mathcal{W}_1 = - \int_0^\infty \frac{1}{2} \sigma_{ij} \epsilon_{ij} \, dV
$$

Here $\mathcal{W}_0$ is the volume of the element into which the crack band extends during the jump $\Delta a$ in length, and $\mathcal{W}_1$ is the boundary of this element; $\mathcal{W}_0$ is the strain energy released from volume $\mathcal{V}$ due to the formation of microcracks, and $\mathcal{W}_1$ is the energy transferred into volume $\mathcal{V}$ (fracture process zone) from the surrounding structure. $\sigma_{ij}$ and $\epsilon_{ij}$ are the initial stresses and strains at the boundary $\delta_0$ before fracture, and $\epsilon_{ij}$ are the values of stress and strain at the boundary $\delta_0$ during fracture. $\mathcal{V}$ are the displacements at the boundary $\delta_0$ before fracture and $\mathcal{V}$ are the displacements during fracture; $\epsilon_{ij}$ are the boundary tractions which balance the initial stresses $\sigma_{ij}$ in the surrounding structure as the stresses in volume $\mathcal{V}$ are reduced from $\sigma_{ij}$ to $\sigma_{ij}$. In case of plane strain and isotropic material, we have

$$
\sigma_{ij} = E \epsilon_{ij}, \quad \sigma_{ij} = E(1-\nu^2)
$$

If the required work $\Delta W$ is positive, then no change can happen if this work is not externally supplied, i.e., the crack band is stable (does not advance). If $\Delta W$ is negative, no work needs to be supplied but work is released by the fracture, i.e., fracture can occur spontaneously and the energy release $\Delta W$ goes into kinetic energy.

If $\Delta W > 0$, we have a critical state in which fracture can occur statically, since no excess energy is available to produce kinetic energy. Thus, the stability conditions are:

$$
\Delta W > 0 \quad \text{stable}
$$

$$
\Delta W = 0 \quad \text{critical (unstable)}
$$

$$
\Delta W < 0 \quad \text{unstable}
$$

Following previous work (Bažant, 1976), it is quite instructive to analyze in this manner the failure of a uniformly stressed specimen subjected to uniaxial tension (Fig. 15c). We may imagine such a specimen to serve as an approximate model for volume $\mathcal{V}$ of the fracture process zone. We assume the specimen to be loaded through a spring of constant $C$ which models either the spring constant of a testing machine or the elastic support of the fracture process zone by the surrounding structure. Let the cross section of the specimen be $A = 1$. The appearance of the crack band in the specimen may be considered as a sudden finite jump by distance $\Delta a = 1$ in which the front of the crack band moves from the left to the right face of the specimen (Fig. 15c). For a uniform stress state in the specimen (of length $l$), Eqs. 38 and 39 take the form

$$
\Delta W = \frac{1}{2} C \Delta a = \mathcal{W}_0 - \mathcal{W}_1
$$

where

$$
\mathcal{W}_0 = \frac{1}{2} C \Delta a
$$

and

$$
\mathcal{W}_1 = - \frac{1}{2} C \Delta a
$$

In which

$$
\Delta W = \frac{1}{2} C \Delta a - \frac{1}{2} C \Delta a
$$

In this case, the work $\Delta W$ is negative, and the specimen is unstable under the given load, which agrees with the experimental observations.
The value of stresses \( \sigma_0 \) at which the instability begins may be determined from Eq. 46 by substituting Eqs. 46-47. For the instability which leads to complete failure, i.e., \( \sigma_0 = \sigma_{0c} \), (stress is reduced to 0), we thus obtain the condition

\[ \sigma_0 = \frac{E}{2(1-\nu)} \left( \frac{\sigma_{0c}}{E} \right)^2 \]

(51)

while for an infinitesimally small stress change \( \delta \sigma \) we obtain

\[ \frac{\delta \sigma}{E} = \frac{1}{2(1-\nu)} \left( \frac{\sigma_{0c}}{E} \right)^2 \delta \varepsilon \]

(52)

If we know the dependence of tangent modulus \( E_0 \) and the unloading modulus \( E_u \) on strain \( c \) for the dependence of average tangent modulus \( E_{\bar{c}} \) and the averaging unloading modulus \( E_{\bar{u}} \) on strain \( c \), we can determine from these equation the points of instability, as has been done by Bazant (1976). Eq. 52 thus gives the point on the stress-strain curve at which strain-localization into the crack begins, and Eq. 51 gives the point where the stress-strain curve starts to deviate from linear elasticity.

- From Eq. 52 we may observe that for support by a very soft spring, i.e., \( k \rightarrow 0 \), or for a very long specimen, i.e., \( L \rightarrow \infty \), the strain-localization starts to \( \sigma_{0c} \) and often is very small.
- From Eq. 51 we may observe that for support by a very soft spring, i.e., \( k \rightarrow 0 \), the strain-localization starts to \( \sigma_{0c} \) and often is very small.

The stress-strain curve becomes unstable when the strain \( c \) exceeds a critical value. The strain \( c \) is a critical value of the strain at which the crack begins to propagate. When the crack begins to propagate, it is not necessarily equal to the peak stress.

In case the instability which produces fracture happens right at the peak stress point, Eq. 49 provides

\[ \min \frac{0.25}{2(1-\nu)} \left( \frac{\sigma_{0c}}{E} \right)^2 \]

(50)

which is constant and represents the fracture energy value corresponding to the value we used in our fracture model to fit this data. The value of \( \sigma_{0c} \) is characterized by the cross-hatched area in Fig. 15c limited by the unloading and the softening branches beginning from the peak stress point.

In contrast to tension, the shear or compression loading of gratings may lead to only a partial strain soften- ing after which a plateau corresponds to a certain constant stress. Fig. 4(2c) (gradient curve) illustrates such a situation. It is easy to carry out a completely analogous stability analysis from which it follows that this type of behavior may also be characterized in terms of fracture mechanics or analysis of strain localization, as is done by Rice and Palmer (1973). Such a calculation shows that the fracture energy is given by the area under the stress-strain curve which is not limited by a plateau, but by the strain at which the material has the residual strength. Fig. 14a shows the area corresponding to the maximum fracture energy and the area giving the apparent fracture energy when instability starts later in the strain-softening process. All that is necessary to do in the precise calculation is to replace the jump from \( \sigma_0 \) or from \( \sigma_0 \) by the jump from these values to \( \sigma_{0c} \).

CONCLUSION

From the analysis presented in this lecture, we may conclude that, due to their heterogeneity, the fracture of many geomaterials, including rocks as well as the artificial rock concrete, should be treated in terms of a three-parameter fracture mechanics, involving not only the fracture energy, but also the strength and the width of the fracture process zone. The latter parameter can be determined, e.g., from the failure of a sample of the size of the inhomogeneity (grain in rock or maximum asperity in concrete), which leaves us with a two-parameter nonlinear fracture mechanics as a basis for bridge approximation. Such a theory is in satisfactory agreement with the experimental data. It is highly significant that no simplification of our treatment of fracture is approached through strain-softening models, which is considered as an obstacle. The strain-softening may also be regarded as the phenomenon of strain-localiza- tion in a strain-softening continuum, in which fracture is nothing else but a strain-localization instability.

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