Micropolar Medium as a Model for Buckling of Grid Frameworks

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ABSTRACT

Attention is focused on large rectangular frameworks of constant mesh size and constant properties of members in each direction. The framework is considered to be under initial axial loads. A continuous approximation for the expression of potential energy is formulated and, postulating an equivalent micropolar continuum under initial stress, differential equations of equilibrium in terms of displacements and rotations are derived. Expressions for stresses, couple stresses and constitutive relations are also presented.

INTRODUCTION

Although the methods for the analysis of buckling of frames are theoretically well-known, buckling of truly large grid frameworks, such as high-rise buildings, is practically intractable with the classical methods since the size of problem overtaxes the capacity of computers presently available. In practice, the assessment of stability is restricted, as a rule, to local behavior of columns within the frame, and very slender buildings are either avoided, in order to ensure that investigation of the overall loss of stability is not important, or very rigid bracings (or stiffening walls) are provided, which secures stability even without the framework. Nevertheless, even in such structures the response to horizontal forces is affected by the initial loads in the columns. In the future development toward higher, lighter and slenderer structures, it can be expected that even the overall stability modes with axial extensions of columns will become an important consideration in design.

Experience with the exact solutions of the overall behavior of large frames indicates that the displacements and rotations of joints usually vary relatively smoothly from floor to floor and bay to bay. This suggests that a certain continuum approximation can be used as one method of overcoming the difficulties.

The development of various theories of structured continua has been accomplished only recently [1, 2, 3]. Their common characteristic feature is the existence of couple stresses and asymmetric shear stresses. As will be shown later, an
appropriate approximation to a grid framework is Eringen's micropolar medium [3], characterized by the dependence of the elastic potential on the gradient of microrotation and the difference between micro- and macrorotation, in addition to the dependence on the symmetric part of the displacement gradient as in classical elasticity.

The possibility of applying these theories as approximations to frameworks and lattices has been mentioned in many papers. Some very general discussions were made, e.g., by Wózniak [4]. First specific treatment was presented by Banks and Sokolowski [5]. In their paper, however, the special case of a Cosserat continuum, in which the micro- and macrorotations are equal [7], was assumed. This model is, however, inadequate because the microrotation, which corresponds to the rotation of joints in a framework, and the macrorotation, which characterizes the rotation of a line connecting two adjacent joints, are in general unequal. A further significant contribution was made by Askar and Cakmak [6] who considered a rectangular gridwork with diagonals and correctly arrived at a micropolar medium. However, their model is also not fully consistent because certain important terms in the expression of elastic potential, namely those which contain second derivatives of microrotation but can be transformed on integration by parts to terms with first derivatives only, have been neglected. Buckling and deformations of frameworks under initial stress probably have not yet been treated in this light. The intent of the present paper is to formulate a consistent continuous analogy for such problems.

**POTENTIAL ENERGIES OF GRIDWORK AND CONTINUUM**

Consider a member of a planar framework (Figure 1) which is initially straight and in equilibrium under a large axial force, \( P^0 \). Assume that small end moments \( M_a, M_b \), shear force \( T \) and axial force \( P \) is superposed at the ends of member,

![Diagram of a grid framework](image)

**Fig. 1**

Incremental forces and deformations of a member of the framework

which thus undergo small rotations \( \phi_a, \phi_b \), lateral displacements \( v_a, v_b \), and longitudinal displacements \( u_a, u_b \). As is well-known [7], the following relationship then applies:

\[
\begin{bmatrix}
M_a \\
M_b \\
P
\end{bmatrix} =
\begin{bmatrix}
ks, ksc, 0 \\
ksc, ks, 0 \\
0, 0, E
\end{bmatrix}
\begin{bmatrix}
\phi - \phi_a \\
\phi - \phi_b \\
u_a - u_b
\end{bmatrix}
\]

(1)
where $\hat{\gamma} = (v_a - v_b)/L$; $L =$ initial length of member; $\hat{\gamma} = \varphi_a$ and $\hat{\gamma} = \varphi_b =$ rotations relative to ab; $k = EI/L$; $E' = EA/L$; $I$ and $A =$ inertia moment and area of the cross-section; $E =$ Young's modulus. Coefficients $s$ and $c$ are functions of $P_0$, called stability functions. The expressions and tables for these functions are available in the literature [7]. For a zero axial force, $s = 4$, $c = 1/2$.

The expression for the incremental strain energy $U_1$ of a single member is

$$U_1 = \frac{1}{2} \left[ M_a (\varphi_a - \hat{\gamma}) + M_b (\varphi_b - \hat{\gamma}) + P(u_b - u_a) \right] - P_0 (l \hat{\gamma}^2/2)$$

(2)

plus a linear term $P_0 (u_b - u_a)$ which need not be considered because it governs only the initial equilibrium. The value $(l \hat{\gamma}^2/2)$ represents, with an error $O(\hat{\gamma}^4)$, the axial extension of the member due to small incremental lateral displacements $v_a$, $v_b$. If the expressions for $M_a$ and $M_b$, and $P$ according to Equation (1) are substituted, Equation (2) may be brought, after rearrangements, to the form:

$$U_1 = \frac{1}{2} E'(u_b - u_a)^2 + \frac{1}{2} L (\varphi_a - \varphi_b)^2 + k s' (\varphi_a - \varphi_b) (\hat{\gamma} - \varphi_a) - P_0 \frac{1}{2} L \hat{\gamma}^2$$

(3)

where

$$s' = s (1 + c)$$

(3a)

Consider now a plane rectangular grid framework with members parallel to Cartesian axes $x$ and $y$ (Figure 2). Assume that the properties in each direction are uniform, including the value of the axial forces. Quantities related to the directions $x$ and $y$ will be distinguished by subscripts $x$ and $y$. Subscripts $x$ or $y$ preceded by a comma will denote partial derivatives, e.g., $v_x = \partial v/\partial x$,

$$v_{x} = \partial^2 v/\partial x^2.$$

The individual joints will be referred to by subscripts $i$ and $j$ expressing the number of the vertical or the horizontal row of members (Figure 2). The displacements of joint $(i, j)$ in the $x$- and $y$-directions will be denoted as $u_{i,j}, v_{i,j}$, and its rotation as $\varphi_{i,j}$. Here a comma between the subscripts does not refer to a derivative.
The transition from a discrete to a continuous system may be achieved by defining (sufficiently smooth) continuous functions \( u, v, \varphi \) and \( f_x, f_y, m \) of the variables \( x, y \), such that their values in points \( (x_i, y_j) \) are sufficiently close to the values \( u_i, j, v_{i, j}, \varphi_{i, j} \), and \( \tilde{f}_{x, i, j}, \tilde{f}_{y, i, j}, \tilde{m}_{i, j} \), respectively. The latter three values represent prescribed incremental loads and moments applied in the joint and \( f_x, f_y, m \) are the equivalent incremental distributed loads and moments per unit area of the gridwork.

The smoothing operation, by which the continuous approximation of gridwork may be obtained, consists in introducing the continuous functions \( u, v, \varphi \) into the expression for potential energy and neglecting higher order derivatives in the Taylor series expansions of \( u, v, \varphi \). This, of course, justified only if the change of \( u, v, \varphi \) from joint to joint is sufficiently small.

The incremental strain energy \( U_x \) contained in a pair of horizontal members between the joints \( (1 - j, j) \) and \( (i + j, j) \) is a sum of two expressions of form (2). Expanding the values of \( u, v, \varphi \) in joints \( (i + 1, j) \) and \( (i, j) \) in Taylor series about the point \( (i, j) \) yields the following continuum approximation:

\[
U = \frac{1}{2} \sum_{x,x}^{X} L_x x^2 + \frac{1}{2} \sum_{x,y}^{X} L_{xy} xy + \frac{1}{2} \sum_{y,y}^{Y} L_y y^2 + \frac{1}{2} \sum_{x,x}^{X} E_x x^2 + \frac{1}{2} \sum_{y,y}^{Y} E_y y^2 + 2k_x \sum_{x,x}^{X} (v_x - \varphi)^2 + 2k_y \sum_{y,y}^{Y} (v_y - \varphi)^2
\]

(4)

In this expression, the terms with higher than first derivatives of \( u, v, \varphi \) have in general been dropped. An exception must be made however, with the term \( \varphi \), because integration by parts in the expression for energy of the whole structure converts this term into a term with first order derivatives. (This point has been overlooked in Reference [8].) It is sufficiently close to the term \( \varphi_{x,x} \) an agreement with the continuum approximation derived from the equilibrium equations of a joint could not be reached. The legitimacy of dropping the terms with other combinations of higher derivatives, with regard to integration by parts, can be easily verified.

The incremental strain energy \( U_y \) stored in a pair of vertical members meeting in the joint \( (i, j) \) can be expressed in a similar manner. The strain energy corresponding to the area \( L_x L_y \) of the frame is \( (U_x + U_y)/2 \).

The incremental potential energy of the whole structure, \( dU \), approximately equals

\[
\int \int (U + U_x - f_x u - f_y v - mp) \, dx \, dy = \frac{dU}{2L_x L_y}
\]

(5)

minus the work of the loads applied at the boundary of frame. Integrating the terms involving the products \( \varphi_{x,y} \) and \( \varphi_{y,y} \) by parts (or applying the Green's theorem), the integral (5) takes on the form:

\[
\int \int U \, dx \, dy = \int \int \left( \sum_{x}^{X} d_x + \sum_{y}^{Y} d_y + pf_x u - f_y v + mp \right) \, dx \, dy = \frac{dU}{2L_x L_y}
\]

(6)

where

\[
U = \left[ \frac{1}{2} \sum_{x,x}^{X} L_x x^2 + \frac{1}{2} \sum_{y,y}^{Y} L_y y^2 - \frac{1}{2} \sum_{y,y}^{Y} E_y y^2 \right] + \frac{1}{2} \sum_{x,x}^{X} E_x x^2 + \frac{1}{2} \sum_{y,y}^{Y} E_y y^2 + 2k_x \sum_{x,x}^{X} (v_x - \varphi)^2 + 2k_y \sum_{y,y}^{Y} (v_y - \varphi)^2
\]

(7)

plus a certain contour integral of terms involving products \( \varphi_{x,y} \) and \( \varphi_{y,y} \). Expression \( U \) can be regarded as the specific incremental elastic potential of the continuum approximating the framework.

Inspecting Equation (7) to determine the mutually independent variables of which \( U \) is a function, the special case of our continuum for \( f_x = f_y = 0 \) is found to represent the micropolar medium as defined by Eringan [1]. This also shows that the classical Cosserat's medium [1] is insufficient, while theories more general
than micropolar medium are unnecessarily complex \([2, 3]\).

**DIFFERENTIAL EQUILIBRIUM EQUATIONS**

The first variation of the incremental potential \(\delta \psi\) may be written in the form:

\[
\delta \psi = \iint \left[ \frac{\partial E_{11}}{\partial u} \delta u + \frac{\partial E_{12}}{\partial v} \delta v - \frac{\partial E_{22}}{\partial x} \right] dx \, dy + \left[ \int \int \left( \frac{\partial k_s}{\partial x} \delta \varphi \right) \right] dx \, dy
\]

plus a certain contour integral expressing the work of prescribed boundary loads. If in Equation (8) the terms containing derivatives of the variations are integrated by parts (or if Green's theorem is applied), the condition that \(\delta \psi = 0\) for any \(\delta u, \delta v\) and \(\delta \varphi\) results in the following differential equations:

\[
\begin{align*}
L^{2}_{x} \left( \frac{\partial^2 u}{\partial x^2} \right) + k_s \left( \frac{\partial^2 u}{\partial y^2} \right) + \frac{\partial^2 u}{\partial x \partial y} + f &= 0 \\
L^{2}_{y} \left( \frac{\partial^2 v}{\partial y^2} \right) + k_s \left( \frac{\partial^2 v}{\partial x^2} \right) - \frac{\partial^2 v}{\partial x \partial y} + f &= 0
\end{align*}
\]

(9a, 9b)

\[
\begin{align*}
2k_s \left( \frac{\partial \varphi}{\partial x} \right) + \left( \frac{\partial^2 \varphi}{\partial x^2} \right) + k_s \left( \frac{\partial^2 \varphi}{\partial y^2} \right) + \frac{\partial^2 \varphi}{\partial x \partial y} + ml &= 0
\end{align*}
\]

(9c)

where \(s^n\) and \(s^n\) are defined as follows (omitting subscript \(x\) or \(y\)):

\[
s^n = 2s^1 - \pi \frac{2v}{R_B^2}, \quad R_B = EI \pi^2 / L^2
\]

(9d)

Equations (9a)-(9c) represent the differential equations of equilibrium in terms of displacements and rotations for the continuous medium approximating the framework.

**CONSTITUTIVE RELATIONS FOR MICROPOLAR CONTINUUM**

The components of stress, \(\sigma_{xx}, \sigma_{yy}, \sigma_{xy}, \tau_{xy}, \) and couple stress, \(m_x, m_y,\) for a micropolar medium in plane stress may be defined with the help of the specific potential energy as is indicated in the following relations:

\[
\begin{align*}
\sigma_{xx} &= \frac{1}{2} \frac{\partial^2}{\partial x^2} (\psi_x - \varphi) = \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} \right) (\psi_x - \varphi) \\
\sigma_{yy} &= \frac{1}{2} \frac{\partial^2}{\partial y^2} (\psi_y - \varphi) = \frac{1}{2} \left( \frac{\partial^2}{\partial y^2} \right) (\psi_y - \varphi) \\
\sigma_{xy} &= \frac{1}{2} \frac{\partial^2}{\partial x \partial y} (\psi_x - \varphi) = \frac{1}{2} \left( \frac{\partial^2}{\partial x \partial y} \right) (\psi_x - \varphi) \\
\tau_{xy} &= \frac{1}{2} \frac{\partial^2}{\partial x \partial y} (\psi_y - \varphi) = \frac{1}{2} \left( \frac{\partial^2}{\partial x \partial y} \right) (\psi_y - \varphi) \\
m_x &= \frac{1}{2} \frac{\partial^2}{\partial y^2} \varphi = \frac{1}{2} \left( \frac{\partial^2}{\partial y^2} \right) \varphi \\
m_y &= \frac{1}{2} \frac{\partial^2}{\partial x^2} \varphi = \frac{1}{2} \left( \frac{\partial^2}{\partial x^2} \right) \varphi
\end{align*}
\]

in which also the expressions obtained after substitution of Equation (7) are introduced. The values \(\sigma_{xx}, \sigma_{yy}\) represent initial stresses in the micropolar medium, \(\sigma_{xy} = \frac{\partial \psi_x}{\partial y} + \frac{\partial \psi_y}{\partial x}\). The reason for their appearance in the stress definitions is that the work of the initial stress on the incremental
displacement $u$ is $\sigma_{yx} u$, $\epsilon_{yx}$.) Equations (10) have the significance of stress-strain relationships of the micropolar medium.

It is of interest to investigate the relationship of the above stresses to the internal forces in members of the frame. To this end, let us consider their values at the midspan (Figure 3). According to the equilibrium conditions of the member shown in Figure 1,

$$ T = \frac{N_a + N_b}{L} - P$$

Then, using Equation (1) and considering the equilibrium of the half-length of the member in Figure 1, the internal forces in the midspan can be obtained as follows:

$$ M = \frac{N_b - N_a}{2} - \frac{1}{2} k_s (l - c) (\phi_b - \phi_a)$$

$$ N = E'(u_b - u_a) = - F, \quad T = N\left[ s''(v_b - v_a)/L - s'(\phi_a + \phi_b)\right]/L$$

$$ M = \left( M_b - M_a \right)/2$$

where $N$ is the axial force, $T$ is the shear force and $M$ is the bending moment at the midspan which is taken, by definition, about the point located on the straight line connecting the ends of member in the deformed position. Notice that these values characterize the forces and moments as well:

$$ M_a = -M - (T + P) l/2, \quad M_b = M - (T + P) l/2$$

Expanding $u$, $v$, $\phi$, $\psi$, and $\phi_a$, $\psi_b$ in Taylor series and dropping all terms containing higher than first order derivatives, the following expressions are obtained:

$$ \begin{align*}
N_x &= L E' u_x, \\
T_x &= k_s s'' v_x/L_x = 2 k_s s'' u_x/L_x, \\
M_x &= \frac{1}{2} L k_s (1 - c) \phi_x, \\
N_y &= L E' v_y, \\
T_y &= k_s s'' u_x/L_y, \\
M_y &= \frac{1}{2} k_s s'' (1 - c) \phi_y.
\end{align*}$$

$$ \left\{ \begin{array}{l}
N_x = L E' u_x, \\
T_x = k_s s'' v_x/L_x = 2 k_s s'' u_x/L_x, \\
M_x = \frac{1}{2} L k_s (1 - c) \phi_x, \\
N_y = L E' v_y, \\
T_y = k_s s'' u_x/L_y, \\
M_y = \frac{1}{2} k_s s'' (1 - c) \phi_y.
\end{array} \right\}$$

Fig. 3

Internal forces at the midspan and their analogy with the stresses and couple stresses acting on an element of a micropolar continuum.
These expressions may be regarded as the continuous counterparts of the internal forces (11) at the midspan. Values of all functions in these expressions ought to be evaluated for the midspan.

Comparing expressions (12) and (10), it follows that

\[
\begin{align*}
N_x &= L \sigma_{yx}, & N_y &= L \sigma_{xy}, \\
T_x &= L \sigma_{xx}, & T_y &= L \sigma_{xy}, \\
N_x &= -L_m (1 - c_x)/(2c_x), & M_y &= -L_m (1 - c_y)/(2c_y)
\end{align*}
\]

(13)

For a medium without initial stress, the latter of these relationships reduces to

\[
\begin{align*}
N_x &= -\frac{1}{2} L_m x, & M_y &= -\frac{1}{2} L_m x y 
\end{align*}
\]

(14)

It is interesting to note that, in Equation (14), \( M_x \) is not equal to the resultant of the couple stresses \( m_{xx} \) over length element \( L_y \) in the micropolar medium but rather equals minus half of it. (The formulation in Reference[5] implies incorrectly that \( M_x = L_m x \).) With varying initial stress, the ratio \( m_{xx}/M_x \) changes. The reason for the lack of any simple, intuitive correspondence between \( m_{xx} \) and \( M_x \) lies obviously in the fact that \( M_x \) varies along the member. By contrast, \( T \) and \( P \) are constant within each member and \( N_x \) or \( T_x \) do represent the results of stresses \( \sigma_{xx} \) or \( \sigma_{yy} \) over the length element \( L_y \).

Expressions for stresses, Equations (10), and their relations to internal forces in framework, Equation (13), allow to formulate the boundary conditions of micropolar bodies approximating grid frameworks. The boundary conditions can, of course, be also deduced from the first variation of the full expression for potential energy.

CONCLUDING REMARKS

The equations presented above fully define the analogy between a grid framework and a micropolar medium under initial stress.

The equations of equilibrium could have been, alternatively, also derived by determining the continuum approximation to the equations of equilibrium of a joint in the framework (Figure 2). It has been verified that such a procedure does indeed yield the same results. For the correct expression of couple stresses, however, the potential energy approach is inevitable.

Application to practical problems is left to a subsequent paper.

REFERENCES


see also "Bending and Stability Problems with Lattice Structures," No. 6, pp. 781-796.

