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NONLOCAL ANALYSIS OF STABLE STATES AND STABLE PATHS OF PROPAGATION OF DAMAGE SHEAR BANDS

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ABSTRACT

Compression failure due to propagation of shear bands is studied by finite elements, using Mohr-Coulomb's plasticity with a degrading yield limit and normality rule. Strain-softening is modeled through a negative value of the plastic hardening modulus. To serve as a localization limiter, the yield limit is assumed to depend on a nonlocal plastic strain, while the elastic response is local. It is found that the model is capable of producing shear bands inclined with regard to the mesh, provided the element size is smaller than about 30% of the characteristic length of the material. If the element size is reduced further, the results appear to converge. Calculations indicate bifurcations of the response path after the peak load point. The correct path is identified on the basis of Bažant's second-order work criterion for path stability, and it is found that the stable response path corresponds to a breakdown of symmetry, with asymmetric shear bands of which one is active and another one unloads. The results indicate the need to include checks for path stability in finite element programs for strain-softening.

INTRODUCTION

Checks for path bifurcations and determination of the stable path after bifurcation are an important ingredient of finite element programs for the analysis of damage in structures. A preceding paper in this volume [1] established, on the basis of the second law of thermodynamics, the basic stability criteria of inelastic systems which generalize the well-known criteria of Hill [2] for a stable equilibrium state and for bifurcation of equilibrium path.

It was shown that for irreversible structural systems distinction must be made between stable states and stable paths. An equilibrium state is stable if any admissible deviation from this state decreases its internal entropy, S_{in}; i.e., $\Delta S_{in} < 0$. The tracing of an equilibrium path of the structure may be imagined to consist of infinitesimal disruptions of equilibrium followed by infinitesimal approaches toward new equilibrium states. The condition of stable paths is that these approaches to a new equilibrium state must maximize the increment ΔS_{in} of the internal entropy as compared to all other equilibrium paths emanating from a common infinitely close bifurcation point. It was shown also that the path which is stable minimizes the second-order work done during the increments along the path if the displacements are controlled, and maximizes this work if the loads are controlled.

For inelastic structures, examination of stability of equilibrium states is generally insufficient. The reason is that inelastic structures can exhibit a spurious, unstable path which consists entirely of stable equilibrium states. The finite element programs can follow such a spurious unstable path without any indication of trouble, e.g., without any divergence of iterations in the loading steps. Therefore, distinction between stable states and stable paths is particularly important.

In the present follow-up paper, the thermodynamic criterion of stable path is applied to finite element analysis of a structure which is destabilized by material strain-softening due to yield limit degradation [3,4]. To avoid spurious localization of damage energy dissipation to a zone of zero volume, the nonlocal continuum approach is used. The particular form of the nonlocal approach is a recently developed nonlocal continuum with local strain [4,5,6], which has been shown to be computationally efficient and easily programmable. In this concept, the only state variables which are nonlocal are those which are associated with strain-softening, and all the other variables, especially those which determine the elastic response, including unloading and reloading, are local. In the particular case of yield limit degradation as a function of the effective plastic strain, the nonlocal variable is the yield limit [3,4].

Attention is focused in the present paper on the problem of shear band formation and propagation. As will be seen, the nonlocal continuum with local strain is an effective approach to such problems, and among various possible equilibrium solutions the correct one can be obtained on the basis of the thermodynamic criterion of stable path.

NONLOCAL PLASTICITY WITH YIELD LIMIT DEGRADATION

Strain-softening which is associated with degradation of material stiffness and is manifested by a decrease of the unloading slope is properly modeled according to continuum damage mechanics or some type of fracturing theory. Strain-softening which is not accompanied by any loss of material stiffness, i.e., the unloading slope is given by the elastic modulus, is properly modeled as a decrease of the yield limit of the material. In practice, both types of strain-softening are usually combined, which is reflected in models such as the plastic-fracturing theory. In the present investigation we consider, for the sake of simplicity, only the strain-softening associated with the decrease of the yield limit. There exist various materials for which the yield limit degradation seems to be a realistic approach; e.g., low strength concretes or soils stabilized by cement grout.

Strain-softening of any type is an unacceptable concept for a local continuum. This follows from stability analysis [7-9] which shows that strain-softening, at least in certain situations, tends to localize into a zone of vanishing volume if the finite element mesh is refined to zero size. The consequence is that the structure is predicted to fail with a zero energy dissipation, an unacceptale conclusion. Therefore, strain-softening must be restricted to a zone of a certain minimum size which is

a material property. This can be achieved by various mathematical formulations called localization limiters. The most elementary type of such a formulation is the crack band model [10-12], in which the finite element size is restricted to a certain minimum value which is related to the characteristic dimension of the in-homogeneties in the material. This approach has been shown to yield good results in problems of nonlinear fracture mechanics and has been found to be essentially equivalent in these problems to the discrete crack models with a softening cohesive zone characterized by a stress displacement relation [13,14]. The latter models, however, are not completely general since they lack a condition for the minimum spacing of discrete cracks.

For the case when the finite elements are much larger than the characteristic size of the localization zone, models in which a strain-softening localization band is embedded in the finite element have been developed [15-18].

A more general and versatile approach is the nonlocal continuum. This concept, originally introduced on the basis of statistical analysis of heterogeneous materials, has been widely applied in elasticity; see e.g., Eringen and Edelen [19]. Application of the nonlocal concepts to strain-softening was proposed by Bazant, Belytschko and Chang [20]. The early form, however, was relatively unwieldy, requiring a complicated mesh in which the finite elements were imbricated. It also exhibited certain spurious zero energy instability modes which had to be suppressed by artificial means. These problems were recently circumvented by the concept of nonlocal damage or, more generally, the nonlocal continuum with local strain [4-6]. In this approach the main idea is that only those variables which cause strain-softening should be subjected to a nonlocal description. In particular, the total strain remains a local variable. In such a case, the variational derivation of the differential equations of motion or equilibrium and of the boundary conditions from the principle of virtual works, leads to field equations and boundary conditions which are of the standard form [6,7], while in the previous nonlocal theories additional terms arose in the field equations and boundary conditions.

The concept of nonlocal continuum with local strain has been successfully applied to the classical incremental plasticity [4,5]. In this case, the variable which introduces strain-softening is the plastic strain increment $\dot{\varepsilon}^p$, defined as the plastic shear strain increment which is equivalent to the plastic increment tensor in terms of work. This strain is processed through an averaging operator denoted by < > to obtain the nonlocal plastic strain increment:

$$<\dot{\varepsilon}^p(\underset{\sim}{x})> = \frac{1}{V_r(\underset{\sim}{x})} \int_V \alpha(\underset{\sim}{s} - \underset{\sim}{x})\, \dot{\varepsilon}^p(\underset{\sim}{s})\, dV \qquad (1)$$

in which

$$V_r(\underset{\sim}{x}) = \int_V \alpha(\underset{\sim}{s} - \underset{\sim}{x})\, dV \qquad (2)$$

where V is the volume of the body; $\underset{\sim}{x}$ and $\underset{\sim}{s}$ are the coordinate vectors; and $\alpha(\underset{\sim}{x})$ is the given weighting function, which is an empirical material property. Although from the physical viewpoint the shape of this function does not seem to have a great influence and a uniform weighting function for a certain finite zone can be used, it appears to be computationally more efficient if the weighting function is smooth. A suitable form of this function is the Gaussian error density function

$$\alpha(\underset{\sim}{x}) = e^{-(k|\underset{\sim}{x}|/\ell)^2} \qquad (3)$$

in which one has for 1, 2 or 3 dimensions:

$$\begin{aligned} &1\text{ D} : \quad |\underset{\sim}{x}|^2 = x^2, && k = \sqrt{\pi} \\ &2\text{ D} : \quad |\underset{\sim}{x}|^2 = x^2 + y^2, && k = 2 \\ &3\text{ D} : \quad |\underset{\sim}{x}|^2 = x^2 + y^2 + z^2, && k = (6\sqrt{\pi})^{1/3} \end{aligned} \qquad (4)$$

ℓ is the characteristic length, a material property related to the size of the material inhomogeneties. It defines in 1 D the length of the segment, in 2 D the diameter of the circle, and in 3 D the diameter of the sphere which has the same volume as the function $\alpha(\underset{\sim}{x})$ extending to infinity. The characteristic length plays the role of localization limiter and approximately equals the minimum thickness of the strain-softening zone that can develop.

For a finite body, the error density function extends beyond the boundary of the body. The region outside the body is deleted from the integration domain in both Eqs. 1 and 2. This fact causes V_r to depend on location $\underset{\sim}{x}$.

For numerical finite element computations, the integrals in Eqs. 1 and 2 are approximated by finite sums over all the integration points of all elements. However, only the integration points whose contribution is significant ($\alpha(\underset{\sim}{s} - \underset{\sim}{x}) > 0.01$) need to be included in the summation.

The numerical implementation of this nonlocal model for the plastic yield limit degradation has been described in detail by Bazant and Lin [5]. The computer program is obtained from a classical incremental plasticity program [21] by replacing in the calculation of the current yield limit the local effective plastic strain increment with the nonlocal effective plastic strain increment, and incorporating an averaging subroutine to calculate the latter quantity for every integration point of every finite element.

NUMERICAL RESULTS

As an example, the failure process of a concrete block loaded in compression has been studied by finite elements. The Mohr-Coulomb yield criterion, which allows for unequal strength limits in tension and compression and can describe internal friction, has been selected. The flow rule has been assumed to be associated, i.e., satisfying the normality condition. Softening has been introduced by changing the value of the plastic hardening modulus H' which is normally positive, to a negative value, called the softening parameter. The plasticity model was characterized by the following data: Young's elastic modulus : $E = 21 \times 10^9$ N/m^2; Poisson's ratio : $\nu = 0.15$; cohesion : $C = 5 \times 10^6$ N/m^2; angle of internal friction : $\phi = 45^\circ$; softening parameter : $H' = -4.45 \times 10^9$ N/m^2. The dimensions of the block were 30 x 30 x 54cm; the compression was parallel to the longest side. The characteristic length ℓ was assumed to be 12cm. This value corresponds to approximately 5-times the maximum aggregate size of a concrete which could be used in an actual experiment (d_a = 24 cm).

Calculations have been made with two regular meshes, mesh 1 with a finite element size h_1 = 3cm, and mesh 2 with h_2 = 6cm. Two kinds of boundary conditions, namely lubricated (laterally sliding) ends and bonded ends have been considered, and the loading as well as the reaction was assumed to be applied through rigid plates. The loading in compression was implemented by prescribing the increments of vertical displacements at the top end, which were all equal. Four-node isoparametric finite elements and a constant stiffness algorithm [21] have been used for computation. The plane strain hypothesis was assumed.

As an imperfection from which the plastic zone initiates, it has been assumed that there is a weak zone in the middle of the specimen, as shown by the shaded finite elements in Figs. 2a, 3a, 4a and 5a. In these elements the cohesion value was assumed to be 1% less than in the remaining finite elements.

Mesh 1, Sliding Boundary

Calculations indicate that failure occurs through the formation of two symmetric inclined shear bands crossing each other. Fig. 2b represents the force-displacement curve (dashed line). Fig. 2c shows the pattern of the symmetric shear bands just after the peak load. The stars represent the points in which there is active plastic loading in the strain-softening range, and the circles represent the points which are elastically unloading or reloading after previously experiencing plasticity.

It may be noted that the width of the two shear bands obtained is approximately equal to the characteristic length, ℓ.

In another calculation, all the parameters were the same except that two interior nodes have been slightly displaced laterally so as to introduce asymmetry. The results reveal that just after the peak load one of the two diagonal shear bands unloads while the other remains active; Fig. 2d. The force-displacement curve (solid line in Fig. 2b) is common for the two calculations up to the peak-load point, at which a bifurcation occurs. The bifurcation is caused by a breakdown of symmetry in the specimen response, and it is a consequence of instability due to strain-softening of the material.

As shown in Ref. 1, the path which occurs after the bifurcation point must minimize, for the present conditions of control displacement, the second-order work $\Delta W = \delta f\, \delta u/2$ where f is the applied force on top of the specimen and u the prescribed displacement. Calculations show that a smaller value of ΔW is obtained for the asymetric response mode; see Fig. 2b. This indicates [1] that the symmetric response mode represents a spurious, unstable path, even though it consists of a succession of stable equilibrium states, as has been checked on the basis of the second-order work criterion of stability [1]. The stable response path after the peak-load point consists of the propagation of a single asymmetric inclined shear band.

Mesh 1, Bonded Boundary

The same analysis has been carried out for a specimen in which the nodes at top and bottom ends cannot displace laterally. A regular square mesh of elements is again used. The analysis leads to the asymmetric response mode of two intersecting inclined shear bands, of which one exhibits continued strain-softening and the other undergoes elastic unloading after the peak-load point; see Fig. 3b,c. The asymmetric response has been obtained here even without assuming any asymmetric initial imperfection, merely as a consequence of the numerical round-off of errors, which occur with a slight asymmetry.

A second calculation has been performed with enforced symmetry, by considering only one-half of the finite element mesh with the boundary conditions of symmetry on the vertical symmetry plane. This calculation produces a symmetric response mode with two intersecting inclined shear bands which are both active even after the peak-load point. Certain numerical manipulations of symmetry had to be introduced in the computer program in this case, in order to correctly evaluate the nonlocal variables near the plane of symmetry.

Again, the condition of minimum second-order work along the path indicates that the stable response path is the asymmetric one, which leads after the bifurcation point to a steeper descent.

Mesh 2, Sliding Boundary

The results obtained with the coarser mesh appear to be different from those described previously. After the spread of plasticity from the assumed weak zone, the specimen fails with active plasticity only (i.e., strain-softening). No unloading is observed in this case, even if the loading steps are extremely refined. The force displacement curve is similar to that observed previously (Fig. 4b) but the failure pattern is completely different as no shear band is observed (Fig. 4c).

The same result is obtained if the two internal nodes mentioned previously are considered to be initially slightly displaced.

Mesh 2, Bonded Boundary

The same calculation has also been performed with a regular mesh, and a very stiff response of the specimen has been obtained in this case. The peak load is much higher (the dashed line in Fig. 5b), and no tendency to a shear band is observed (Fig. 5c).

If the two aforementioned internal nodes are assumed to be initially slightly displaced, still the same kind of response is observed (Fig. 5d), except that a few more points which undergo unloading are present and the load deflection curve lies, consequently, lower.

MESH SIZE AND SHEAR BAND

From the results obtained with the coarse mesh (Mesh 2), one may conclude that a shear band cannot be modeled with a nonlocal plasticity theory if the element size h is greater than a certain fraction of the characteristic length ℓ. In particular, the numerical results show that:

$$\left.\begin{array}{ll} \text{for } h > \ell/n : & \text{modeling of shear band is impossible} \\ \text{for } h < \ell/n : & \text{modeling of shear band is possible} \end{array}\right\} \quad (5)$$

in which n approximately equals 3.5.

Further numerical studies indicated that this conclusion remains valid for different choices for the characteristic length ℓ. Preliminary results further reveal that the solutions obtained with the fine mesh (Mesh 1) are very similar to those with a still finer mesh, for which h = 2cm. This seems to confirm that after a certain minimum refinement of the mesh, the finite element solution converges for further mesh refinement.

If a coarse finite element mesh has to be used in calculations, a different approach to the modeling of shear bands has to be adopted. A shear band must be embedded in the finite elements, occupying only a small portion of the element. This can be done, e.g., in the manner shown previously by Pietruszczak and Mroz [15], Willam, Hurlbut and Sture [16], and Ortiz, Leroy and Needleman [17].

The problem with the use of coarse finite elements is formally identical to that encountered in the modeling of tensile fractures. Such fractures, too, can also be modeled with large finite elements in which a discontinuous shape function is embedded, as has been shown by Droz [22].

CONCLUSIONS

Nonlocal plasticity theory with a degrading yield limit is a computationally effective model for studying the failure behavior of concrete structures in compression. The calculations show that there exists a path bifurcation associated with a breakdown of symmetry of the shear bands which form. The correct response path after the bifurcation can be determined on the basis of the second-order work criterion for stable paths [1]. Since the equilibrium states on both bifurcation branches of the equilibrium path are stable, and iterations in the loading steps converge well, a check for path stability is an important ingredient of a finite element program for strain-softening.

The nonlocal approach, which prevents the localization of strain-softening, makes it possible to model shear bands propagating through finite element meshes in arbitrary inclined direction, apparently without any mesh bias. The condition is that the element size must not be larger than approximately 30% of the characteristic length ℓ of the material.

Further studies, however, need to be made to investigate the effect of mesh size and inclination in greater depth.

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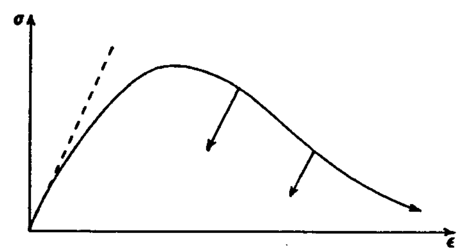

Figure 1. Unloading characteristics of a plasticity model with degrading yield limit.

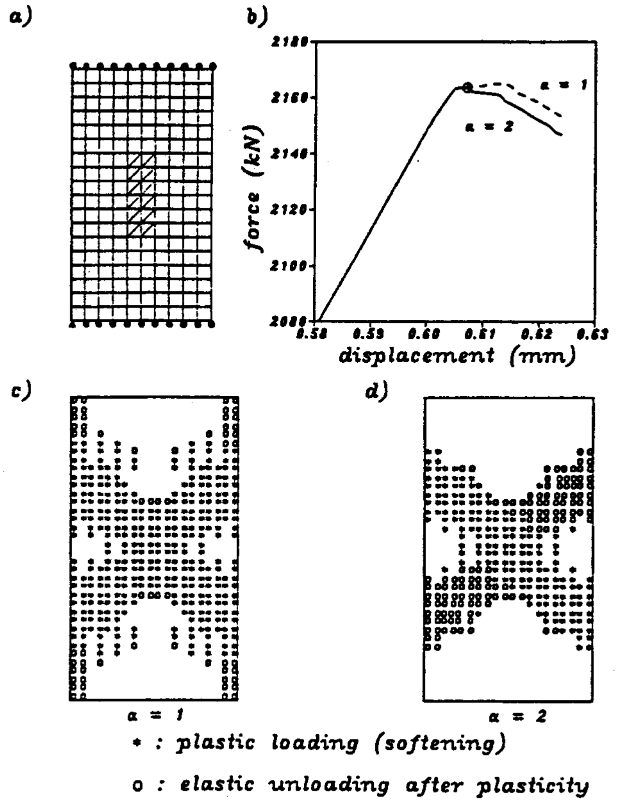

Figure 2. Mesh 1, with sliding boundaries, and the force-displacement curves with failure patterns after the peak load point.

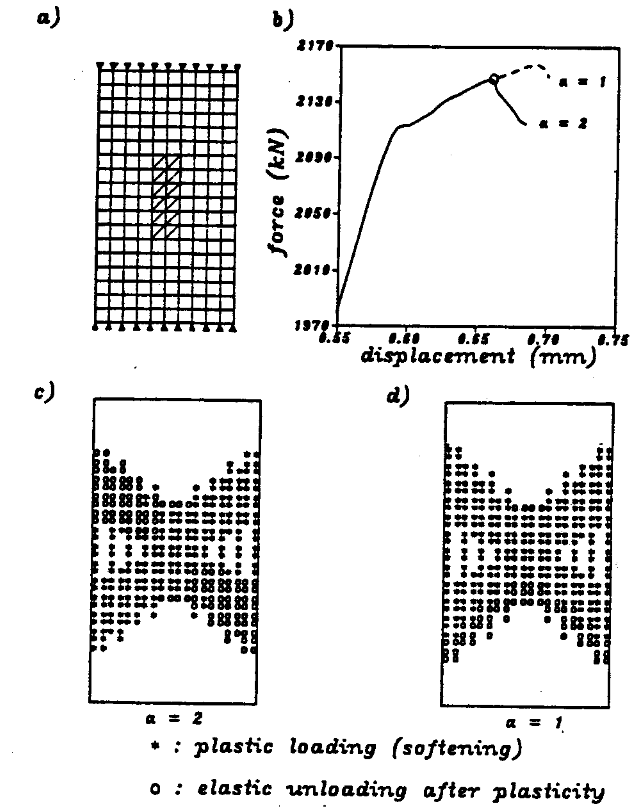

Figure 3. Mesh 1, with bonded boundaries, and force-displacement curves with failure patterns after the peak load point.

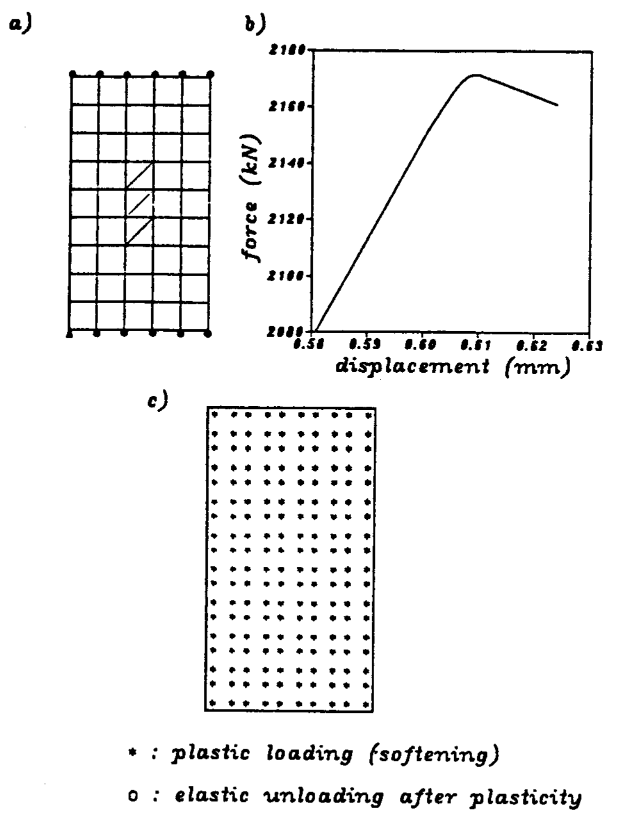

Figure 4. Mesh 2, with sliding boundaries and force-displacement curve with failure pattern after the peak load.

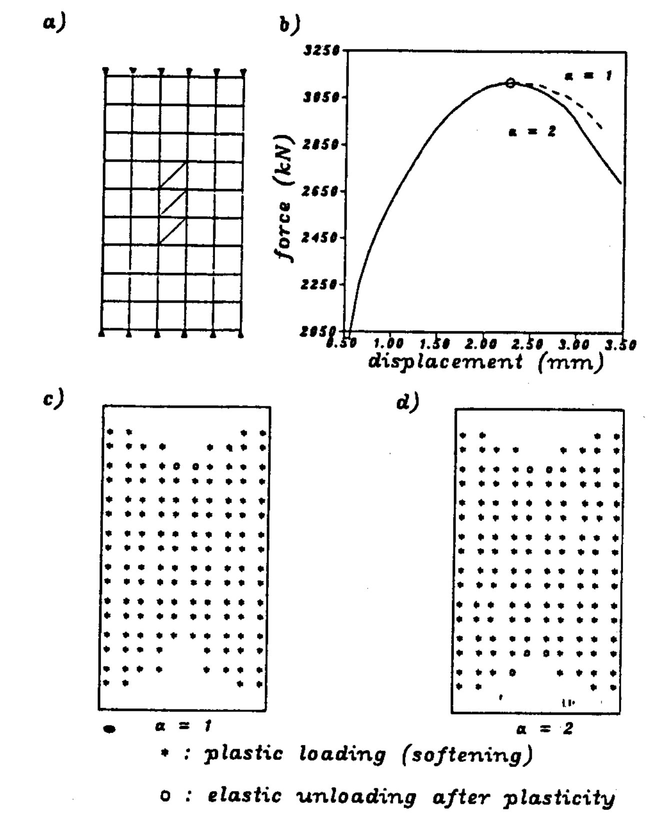

Figure 5. Mesh 2, with bonded boundaries, and force-displacement curves with failure patterns after peak load.