

# COMPUTATIONAL PLASTICITY

## *Models, Software and Applications*

BIFURCATIONS AND THERMODYNAMIC CRITERIA OF STABLE  
PATHS OF STRUCTURES EXHIBITING PLASTICITY  
AND DAMAGE PROPAGATION

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### ABSTRACT

The lecture presents an overview of the recent results obtained at Northwestern University in stability analysis of structures exhibiting plasticity or damage. Stability is analyzed on the basis of thermodynamics and it is shown that a distinction must be made between stability of state and stability of path. It is possible that the immediate post-bifurcation states on every path are stable, and then the stable path, i.e. the path which actually occurs, is that which maximizes the internally produced entropy increment along the equilibrium path. This criterion reduces to the condition that the second-order work of controlled displacements minus the second-order work of controlled loads must be minimized. Stability of iterations in a step-by-step algorithm depends only on the stability of state and has nothing to do with stability of path. The proposed concept of stable path is shown to be in agreement with what is known from Shanley's theory of plastic buckling. Numerical finite element applications to various structures exhibiting plasticity with damage are presented. They deal with the problem of multiple plastic strain-softening shear bands or multiple interacting propagating damage fronts. Generally it is found that stable response path is that for which the damage is localized, e.g. into a single shear band or a single propagating damage front.

### 1. INTRODUCTION

Material damage, which may be manifested either by degradation of the yield limit or by degradation of the material stiffness tensor, or both, has a profound destabilizing influence on structures. It leads to bifurcations of the loading path, whose number for a large structural system can be very high. The detection of these bifurcations in large-scale finite element analysis of structures and the selection of the correct, i.e. stable, post-bifurcation path represents a challenge in computational plasticity. The purpose of the present lecture is to give an

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overview of the recent researches in this field by a team of investigators at Northwestern University, in which some new advances have been achieved both in the basic stability concept and in its numerical implementation in finite element codes. The principal new result is the concept of a stable post-bifurcation path, which must be distinguished from the concept of a stable equilibrium state, and a general, thermodynamically based criterion of stable path applicable to arbitrary loading, in which both displacement control and load control can be mixed. The lecture presents only the essential results with numerical applications, while for details and evaluations the original recent papers need to be consulted [1-5].

## 2. THE CONCEPTS OF STABLE STATE AND STABLE EQUILIBRIUM PATH

### 2.1 Thermodynamic Approach

We consider an arbitrary discrete or discretized structure characterized by the column matrix of generalized displacements  $q = (q_1, \dots, q_n)^T$ , and the associated column matrix of forces  $f = (f_1, \dots, f_n)^T$ . The force increments are related to the displacement increments as  $df = K(\nu)dq$  where  $K$  is a  $(n \times n)$  square matrix of incremental stiffnesses of the structure which are based either on the isothermal or the isentropic (adiabatic) tangential moduli of the material, and  $\nu$  is the unit vector of the direction of  $dq$  in the  $n$ -dimensional displacement space. The stiffness matrix in general is a function of  $\nu$ , however, normally it is constant within sectors of various  $\nu$  directions. In plasticity, the reason for the dependence of  $K$  on the orientations  $\nu$  is the difference in stiffness between loading and unloading.

In the case of inelastic behavior, stability of a structure cannot be analyzed on the basis of the principle of minimum potential energy, which follows from Lagrange-Dirichlet theorem and Lyapunov stability definition for dynamic systems. Direct application of the dynamic stability concept is extremely difficult in the case of inelastic structures, principally because of discontinuous jumps in the values of  $K$  due to transitions between loading and unloading. Far simpler, and from the physics viewpoint more fundamental, is to approach stability from the viewpoint of thermodynamics.

The stability of any thermodynamic system, including a structure, is determined by the second law of thermodynamics, which tells us that if there exists a deviation from the initial equilibrium state in which the entropy of the system increases, the deviation must happen, and if the entropy decreases, the deviation cannot happen. The implication of the second law is that if we calculate the entropy change for all possible (kinematically admissible) deviations from the initial equilibrium state and find the entropy increment to be always negative, no deviation from the initial state can happen, and so the initial equilibrium state is stable. On the other hand, if we find that at least one deviation yields

a positive increment of entropy, such an increment must happen, spontaneously, and so the initial state is not stable. This serves as the fundamental thermodynamic criterion of stability, and for the case of elastic systems one can show that this criterion leads to the principle of minimum potential energy.

A further conceptual difficulty, however, arises in the case of inelastic systems. Inelastic behavior, by definition, is a dissipative process, and therefore, as long as further inelastic deformations are not impossible, an inelastic system cannot be in an equilibrium state, from the strict thermodynamic viewpoint. This fact has been widely recognized in the theory of constitutive relations, and the dissipative processes which produce inelastic response have been characterized by internal variables, whose values are typically far from thermodynamic equilibrium even though the structure may be in mechanical equilibrium. Unfortunately, however, the method of internal variables makes it impossible to analyze stability of an inelastic structure. By introducing internal variables, which is equivalent to admitting that the system is not in thermodynamic equilibrium, the question of stability cannot even be posed. Stability is a concept which makes sense only for the states of thermodynamic equilibrium.

To resolve this conceptual difficulty, the concept of tangentially equivalent elastic structure has been introduced [1,2]. This is an elastic structure whose elastic stiffnesses are equal to the tangential stiffnesses  $K(\nu)$  of the actual inelastic structure at each stage of loading. Obviously, the tangentially equivalent elastic structure is different at every stage of loading, and is also different for each different sector of loading directions  $\nu$  for which different values of  $K$  are applicable. If there is mechanical equilibrium, the tangentially equivalent elastic structure is in thermodynamic equilibrium (the actual inelastic structure is not), and the question of stability can then be posed. Without this concept, the question of stability is meaningless.

Introduction of a tangentially equivalent elastic structure also resolves another conceptual difficulty having to do with the constitutive equation. For states at or near thermodynamic equilibrium, the properties of a thermodynamic system are characterized by an equation of state (a well known example is the equation of ideal gas). Inelastic constitutive equations describe material properties at states which are far away from thermodynamic equilibrium, since the dissipative properties have not yet terminated. Therefore, inelastic constitutive equations are not equations of state, from the thermodynamic viewpoint, and so equilibrium thermodynamics cannot be applied. On the other hand, the stress-strain relations or the force-displacement relations for the tangentially equivalent elastic structure do represent equations of state, and it is this fact which makes application of equilibrium thermodynamics legitimate [1,2,6].

## 2.2 Criteria in Terms of Entropy Change and Second-Order Work

The stability analysis can be extended to arbitrary structural systems which are not thermally isolated from the surroundings. As shown by Guggenheim [7], one may separate from the total entropy increment  $\delta S$  the external entropy increment due to influx of heat from the surroundings, in which case the second law of thermodynamics requires the positiveness of the internally produced entropy increment  $(\delta S)_{in}$ . Thus, a structure is stable (i.e., remains in its initial state) if  $(\delta S)_{in} < 0$  for all possible  $\delta q$  or  $\delta f$ . From this condition it can be shown that stability depends on the second-order work

$$\delta^2 W = -T(\Delta S)_{in} - \frac{1}{2} \delta \underline{f}^T \delta \underline{g} - \frac{1}{2} \delta \underline{g}^T \underline{K}(\underline{\nu}) \delta \underline{g} \quad (1)$$

in which  $T$  = absolute temperature. The structure is stable if this expression is positive for all possible deviations from the initial state of mechanical equilibrium (positive definiteness), and is unstable if the foregoing expression is negative at least for one possible deviation from the initial state (in which case this expression is indefinite) [1,2]. It can further be shown that, if geometrical nonlinearities of strain are insignificant, the second-order work can alternatively be calculated as

$$\delta^2 W = \int_V \frac{1}{2} \delta \underline{\sigma}^T \delta \underline{\epsilon} \, dV - \int_V \frac{1}{2} \delta \underline{\epsilon}^T \underline{C} \delta \underline{\epsilon} \, dV \quad (2)$$

where  $V$  = volume of the structure,  $\underline{\epsilon}$ ,  $\underline{\sigma}$  =  $6 \times 1$  column matrices of small strains and stresses, and  $\underline{C}$  =  $6 \times 6$  tangential moduli matrix of the material (which is different for loading and unloading). For finite element computations, Eq. 1 is normally more efficient than Eq. 2. For the special case of fixed boundary displacements, the foregoing criterion of stability of equilibrium was in the theory of plasticity postulated by Hill [8], albeit without thermodynamic derivation.

Information on the stability of an equilibrium state is not all that is needed in inelastic structural analysis. To decide which equilibrium response path will occur after a bifurcation, a narrower concept of stable equilibrium path needs to be introduced [1,2]. The problem is that, in inelastic structures, bifurcations of the equilibrium response path can happen at a stable state, and the immediate post-bifurcation states on all the equilibrium paths emanating from the bifurcation point can be stable. So, stability of state does not decide which post-bifurcation path will be followed. This situation has been demonstrated to occur even in the elementary case of Shanley bifurcation in plastic column buckling, in which both the undeflected and deflected equilibrium states immediately above the tangent modulus load are stable states, even though, as discovered by Shanley in 1947, the undeflected equilibrium state cannot be attained in a continuous loading process, i.e. the column must deflect once the tangent modulus load is exceeded.

The decision as to which post-bifurcation path occurs must be based on the analysis of the approach to new post-bifurcation equilibrium states rather than the deviations from the initial equilibrium state at bifurcation. Because the structure is inelastic, analysis of the approach toward a new equilibrium state is not equivalent to the analysis of deviations from the initial equilibrium state (the deformations are irreversible). Progress along the equilibrium response path of a structure may be imagined as a series of infinitesimal deviations from equilibrium and restorations of equilibrium. Now, with regard to the restorations of equilibrium, another form of the second law of thermodynamics, enunciated by Gibbs, needs to be applied:

When there are several possible equilibrium states infinitely close to an initial nonequilibrium state, the structure will approach that state for which the entropy (in the general case the internally produced entropy) is maximized.

Using the concept of a tangentially equivalent elastic structure, it has been shown that, on approach to the new equilibrium states on post-bifurcation branches ( $\alpha = 1, 2, \dots$ ), the internally produced entropy increment is expressed as

$$T(\Delta S)_{in} = -\Delta Z = -\frac{1}{2} \delta \underline{q}_A^T \delta \underline{f}_A^{(\alpha)} + \frac{1}{2} \delta \underline{f}_B^T \delta \underline{q}_B^{(\alpha)} = \text{Max}_{(\alpha)} \quad (3)$$

Here we consider the general case in which some of the displacements, represented by  $\delta \underline{q}_A$  are controlled, and the remaining forces, represented by  $\delta \underline{f}_B$ , are controlled; i.e., we decompose the column matrices of displacement and forces as  $\underline{f}^{(\alpha)} = (\underline{f}_A^{(\alpha)}, \underline{f}_B^{(\alpha)})^T$ ,  $\underline{q}^{(\alpha)} = (\underline{q}_A, \underline{q}_B^{(\alpha)})^T$ ; ( $\alpha$ ) labels the quantities associated with the post-bifurcation branches numbered as  $\alpha = 1, 2, \dots$ . Only the response variables carry the labels ( $\alpha$ ), while the prescribed variables are the same for all the branches.

An interesting new result expressed by Eq. 3 is that the stability of path is not governed by the second-order work of force increments on displacement increments, but by the difference between two second-order works, namely the second-order work of prescribed displacement increments along the equilibrium path on the corresponding force increments, minus the second-order work of the prescribed force increments along the equilibrium path on the corresponding displacement increments. The difference of these two second-order works must be maximized, as a consequence of the second law of thermodynamics applied to the approach to the new equilibrium state [1,2].

In the special case of displacement control only, Eq. 3 yields for stable path the following criterion:

$$\delta^2 W = -T(\Delta S)_{in} = \Delta Z = \frac{1}{2} \delta \underline{f}^{(\alpha)T} \delta \underline{g} - \frac{1}{2} \delta \underline{g}^{(\alpha)T} \underline{K}^{(\alpha)} \delta \underline{g}^{(\alpha)} = \text{Min}_{(\alpha)} \quad (4)$$

while in the case of load control only, Eq. 3 yields the criterion:

$$\delta^2 W = -T(\Delta S)_{in} - \Delta Z - \frac{1}{2} \delta q^{(\alpha)T} \delta \underline{f} - \frac{1}{2} \delta \underline{f}^{(\alpha)T} \delta q^{(\alpha)} - \text{Max}_{(\alpha)} \quad (5)$$

Eq. 5 as well as 3 seems to be a new result, while Eq. 4 has been used by several investigators before, albeit without a thermodynamic proof; e.g. [9-17].

### 2.3 Bifurcation

The well known criterion of equilibrium path bifurcation of an inelastic structure can be obtained as the special limiting case of the foregoing thermodynamic analysis if the tangential stiffness matrix varies continuously during the progress of loading along the equilibrium path. The first bifurcation is known to occur in such a manner that the loading increments along both the main equilibrium path and the secondary path occur with the structure undergoing loading. So the same matrix,  $K_{(2)}$ , applies for two different loading increments  $\delta q^{(1)}$  and  $\delta q^{(2)}$ , which means that the equations  $K \delta q^{(1)} = \delta f$  and  $K \delta q^{(2)} = \delta f$ , for prescribed increments  $\delta f$ , must be simultaneously satisfied for two different displacement increments  $\delta q^{(1)}$  and  $\delta q^{(2)}$ . Subtracting these two equations, one concludes that the equation  $K (\delta q^{(2)} - \delta q^{(1)}) = 0$  must be satisfied for a nonzero vector in the parenthesis, which means that matrix  $K$  must be singular. This is a well known condition, following from Hill's method of linear comparison solids. It should be noted, however, that if the tangential moduli of the material change along the equilibrium loading path discontinuously, the first bifurcation need not occur when  $K$  is singular.

Bifurcations of equilibrium path can generally be avoided by introducing small imperfections, which of course always exist in reality. When imperfections are present, the foregoing theoretical analysis appears to be unnecessary. However, analysis of imperfect systems is more complicated than the analysis of perfect systems. Furthermore, in principle, one would have to examine all possible imperfections, and just by considering only some kinds of imperfection one cannot be sure that no bifurcation occurs near a path that is not excited by the imperfection considered. Moreover, study of the perfect system is important since it can provide information on the entire structure of the solution near the bifurcation state for all possible imperfections.

### 2.4 Stability of Iterations of Loading Step

With regard to the step-by-step solution algorithms in which the solution of every loading step is iterated many times, it need to be observed that stability of these iterations depends exclusively of the state of the structure, and has nothing in common with the stability of the equilibrium response path. In other words, it is possible, if the structure is stable, that the iterations converge to a stable state, but not to the stable path, i.e. converge to the wrong path which cannot be followed by the structure in

practice. The reason is that, while iterating the solution of a certain loading step, one considers only deviations from the initial equilibrium state and makes no comparison of changes on approach to every possible final state.

In numerical analysis of structures with plasticity or damage, it is frequently assumed that if the iterations converge, the solution must be correct. But this is not true.

## 3. EXAMPLES OF APPLICATIONS

### 3.1 Elastoplastic Buckling

The most instructive simple application of the foregoing criteria is provided by buckling of elastoplastic columns. Consider a simply supported perfect column of length  $l$ , consisting of two rigid bars connected at midheight by an elastoplastic link of vanishing length (Fig. 1). As the axial compressive load,  $P$ , is increased, column shortens, but beginning with the tangent modulus load of Shanley,  $P_t$ , it can also deflect. It has been shown [1] that

$$\begin{Bmatrix} l \delta f_1 \\ 2h \delta f_2 \end{Bmatrix} = 2\eta P_t l \begin{bmatrix} 1 + \xi - \frac{2P_0}{\eta P_t} & 1 - \xi \\ 1 - \xi & 1 + \xi \end{bmatrix} \begin{Bmatrix} X \\ Y \end{Bmatrix} \quad (6)$$

in which  $X = \delta q_1/l$ ,  $Y = \delta q_2/2h$ ;  $q_1$  = midspan deflection,  $q_2$  = axial relative displacement of the ends,  $h$  = cross section thickness,  $P_0$  = initial axial load before buckling; and for  $Y > X$  (loading only):  $\xi = 1$ ,  $\eta = 1$ ; for  $-X \leq Y \leq X$  (loading - unloading):  $\xi = \xi_u$ ,  $\eta = 1$ ; for  $Y < -X$  (unloading only):  $\xi = 1$ ,  $\eta = \xi_u$ ,  $\xi_u = E_u/E_t$ ;  $E_u$ ,  $E_t$  = tangent modulus and unloading modulus of the material in the elastoplastic link. From Eq. 6 one obtains

$$-T(\Delta S)_{in} = \delta^2 W(X, Y) = \frac{1}{2} (\delta f_1 \delta q_1 + \delta f_2 \delta q_2) = \quad (7)$$

$$\frac{P_t l \eta}{\xi + 1} \left[ ((\xi + 1)Y - (\xi - 1)X)^2 + 4\xi \left( 1 - \frac{(\xi + 1)P_0}{2\xi P_t \eta} \right) X^2 \right]$$

which represents the second-order work of load increments on the displacement increments. Eq. 7 looks like a quadratic form, but it is not a quadratic form because parameters  $\xi$  and  $\eta$  change discontinuously, and because  $X$  appears with an absolute value. As a consequence, the plot of the second-order works (entropy surface) versus the nondimensional displacements  $X$  and  $Y$  (Fig. 1) is a patch-up of quadratic surfaces rather than a single quadratic surface. The quadratic surfaces are joined continuously, with a continuous slope, but with discontinuous curvature. Stability is indicated by the fact that every path from the origin in Fig. 1 leads downward. If not, the state is unstable. It is now interesting to note that at Shanley's tangent modulus load,  $P_0 = P_t$ , the surface indicates stability, despite the fact that

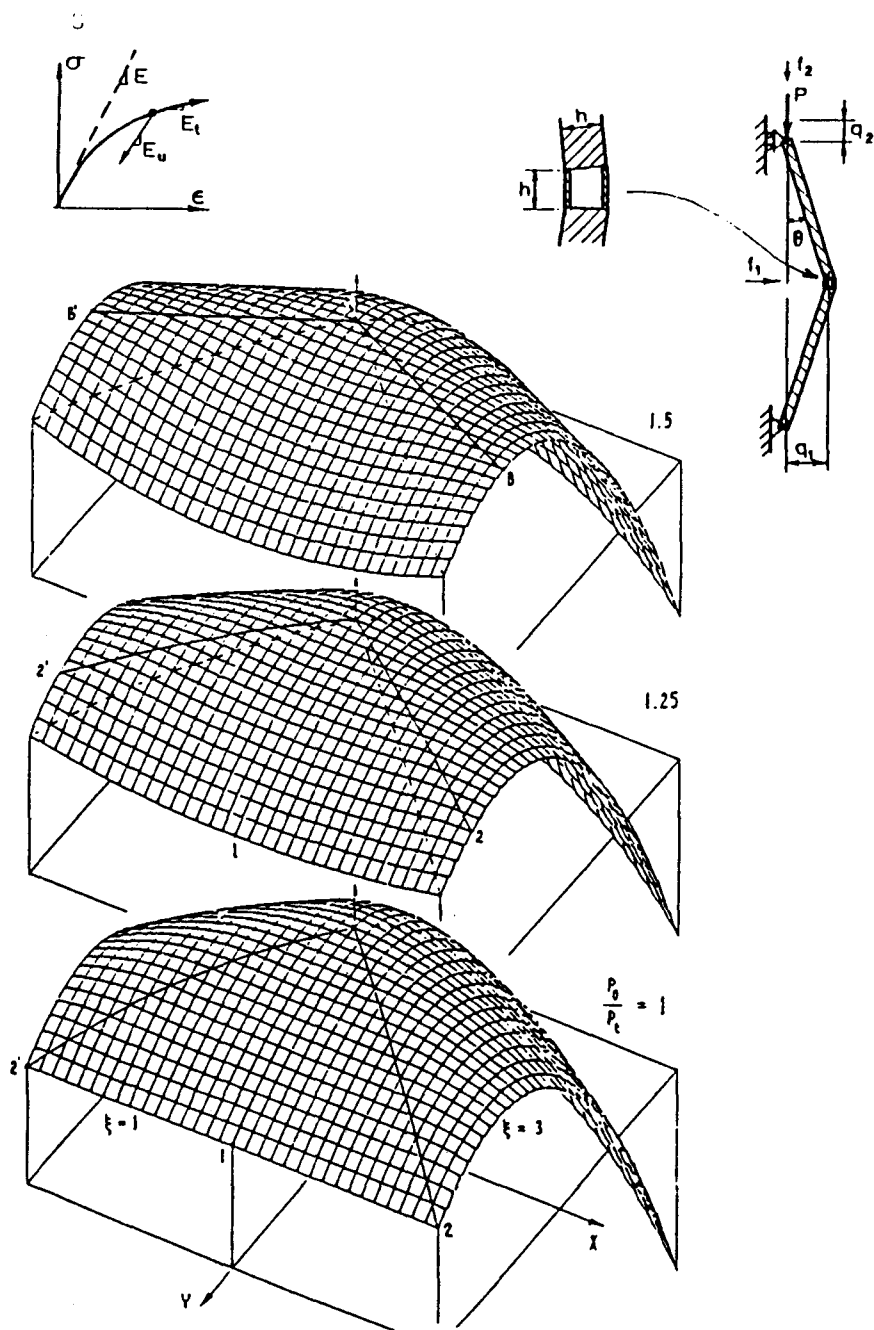


Fig. 1 Shanley's elastoplastic column and surfaces of second-work (minus the internally produced entropy increment) at tangent modulus load  $P_t$ , intermediate load, and reduced modulus load  $P_r$ .

a portion of the surface is cylindrical. The same is true, under conditions of load control, for all the values of axial load up to the reduced modulus load  $P_0 = P_t$ , for which a portion of the surface is a hyperbolic paraboloid. Only at the reduced modulus load, a horizontal path from the origin, indicating a state of neutral equilibrium, becomes available, which represents the critical state of stability limit. Above the reduced modulus load, ascending paths exist, which indicates instability. The column parameters for which the surface is plotted and other details are given in [1].

For  $P_0 \geq P_t$ , there are bifurcations and the deflected equilibrium paths are shown in Fig. 1 by the solid descending curves. Evaluating the second-order works along the deflected and undeflected paths, and taking the difference, one obtains [1]:

for displacement control:

$$\delta^2W(1) - \delta^2W(2) = \frac{(P_0 - P_t)(\xi_u - 1)P_t \ell}{\left[1 + \frac{(\xi_u - 1)^2}{4\xi_u}\right] P_r - P_0} Y^2 \quad (8)$$

for load control:

$$\delta^2W(1) - \delta^2W(2) = - \frac{(P_0 - P_t)(\xi_u - 1)h^2}{2(P_r + P_0)(\xi_u + 1)P_t \ell} \delta f_2^2 \quad (9)$$

in which  $P_r$  is the reduced modulus load of Engesser.

From these expressions, as well as from the comparison of points 1 and 2 on the deflected and undeflected equilibrium paths in Fig. 1, it follows that the second-order work along the equilibrium path is minimized for the deflected path if the axial displacement is controlled, and is maximized for the deflected path if the axial load is controlled. In either case, it means that the deflected path is the path which must occur in reality, i.e., is the stable path. This of course agrees with what has been well known on the basis of experiments as well as analysis of imperfection influence. Nevertheless, this simple example convincingly illustrates that minimization of the second-order work along the equilibrium path, which has been in the past tacitly assumed for all the situations, is the correct condition of stable path only for the case of displacement control, while under load control the opposite condition, namely the maximization of the second-order work, is applicable.

### 3.2 Propagation of Interacting Shear Bands

As another example, we now briefly summarize the analysis of stable propagation of shear bands in an elastoplastic material with a degrading yield limit, recently presented by Droz and Bažant [3]. We consider a concrete block loaded in compression, and assume the material to be

governed by the Mohr-Coulomb yield criterion, which permits the strength limits in tension and compression to be different and can describe internal friction. The flow rule of the material is assumed to be associated, i.e., obeying the normality condition. Strain-softening is introduced by changing the value of the plastic hardening modulus  $H'$ , which is normally positive, to a negative value. So  $H'$  represents the softening parameter. The plasticity model is characterized by the following parameters: Young's elastic modulus,  $E = 21 \times 10^6$  MPa; Poisson ratio,  $\nu_0 = 0.15$ ; cohesion,  $C = 5 \times 10^6$  MPa; internal friction angle  $\varphi = 45^\circ$ , and softening modulus  $H' = -4.45 \times 10^6$  MPa. We analyze by finite elements the response of a two-dimensional rectangular block of width 30cm and height 55cm, compressed uniaxially in the direction parallel to the longest side.

The constitutive relation, which could be used to describe concrete, exhibits strain-softening. This is however an unacceptable feature of the local continuum, as indicated by stability analyses which show that strain-softening tends to localize into a zone of vanishing volume if the finite element mesh is refined to zero size (cf. [10]). As a consequence, the structure is predicted to fail with a zero energy dissipation, an unacceptable behavior. Therefore, strain-softening must be restricted to a zone of a certain size which is a material property. This can be achieved by some form of a nonlocal approach. A rather effective nonlocal model, which is simple to implement in finite element programs, has recently been presented in [18-19]. In this latest nonlocal formulation, only those variables which cause nonlocal strain-softening are subjected to a nonlocal description, while the other variables are nonlocal. In particular the total strains remain local variables. In that case, the variational derivation of the differential equations of motion or equilibrium and of the boundary conditions from the principle of virtual work leads to field equations and boundary conditions which are of the standard form, whereas in the totally nonlocal formulations additional terms arise in the field equations and boundary conditions.

The present strain-softening plasticity model is identical to that introduced by Bažant and Lin [20]. The variable which introduces strain-softening is the plastic strain increment,  $\epsilon^P$ , which is defined as the plastic shear strain increment that is equivalent to the plastic increment tensor in terms of work. The strain  $\epsilon^P$  is processed through a spatial averaging operator using a bell-shaped weighting function. The size of the averaging zone, corresponding to the so-called representative volume of a statistically heterogeneous medium, is determined by the characteristic length of the material,  $l$ , a material property related to the size of inhomogeneities in the microstructure. For the present computations, the characteristic length was assumed to be  $l = 12$ cm.

Calculations have been made with two regular meshes, mesh 1 with the finite element size  $h_1 = 3$ cm, and mesh 2 with  $h_2 = 6$ cm. Two kinds of boundary conditions, namely lubricated (laterally sliding) ends and bonded fixed ends, have been considered. The loading platten on top as well as the base platten were assumed to be perfectly rigid. The loading in compression was implemented by prescribing small increments of vertical displacements at the top end, which were equal at all the boundary nodes. Four-node isoparametric finite elements and a constant stiffness (initial stiffness) algorithm [21] have been used in computations. The specimen was assumed to be in the state of plane strain. To initiate the strain-softening plastic zones, an imperfection was introduced, assuming that there is a weak zone in the middle of the specimen in which the cohesion value is 1% less than in the remaining finite elements. The weak zone is indicated by the shaded finite elements in Fig. 2a, 3a, 4a, and 5a.

Mesh 1 - Sliding Boundary. The calculations of Droz and Bažant [3] indicate that failure occurs through the formation of two symmetric inclined shear bands intersecting each other. Fig. 2b represents the force-displacement curve (dashed line). Fig. 2c shows the pattern of the symmetric shear bands just after the peak load. The stars represent the points in which there is active plastic loading in the strain-softening range, and the circles represent the points which are elastically unloading or reloading after previously experiencing plasticity. An interesting point to note is that the width of the two shear bands is obtained approximately equal to the characteristic length,  $l$ .

Another calculation was carried out assuming that all the parameters were the same except that two interior nodes have been slightly displaced laterally so as to introduce asymmetry. The results reveal that just after the peak load one of the two diagonal shear bands unloads while the other remains active (Fig. 2d). The force-displacement curve (solid line in Fig. 2b) is the same for the two calculations after the peak-load point, at which a bifurcation takes place. The bifurcation is caused by a breakdown of symmetry in the specimen response, and represents a consequence of instability due to strain-softening of the material.

According to the criterion of stable path, the path that occurs in reality after the bifurcation point must minimize, for the present conditions of displacement control, the second-order work  $\delta^2 W = \delta f \delta u / 2$  where  $f$  is the applied force on top of the specimen and  $u$  is the prescribed displacement on top. Computations indicate that a smaller value of the second-order work is obtained for the asymmetric response mode (see Fig. 2b). This shows that the symmetric response mode represents a spurious, unstable path, even though it consists of a succession of equilibrium states which are stable, as can be checked according to the second-order work criterion of stability.

The symmetric response path is found to correspond to the formation of two inclined shear bands as shown in Fig. 2c. The asymmetric response path is found to correspond to a behavior in which first two symmetric shear bands initiate, but then, after the bifurcation state, only one shear band continues loading with strain-softening while the points of the other shear band reverse to elastic unloading.

Mesh 1 - Fixed Boundary. The same analysis has been performed also for a specimen in which the nodes at the top and bottom ends are not allowed to displace laterally. A regular square mesh of elements is again used. The calculations indicate asymmetric response which consists of two intersecting inclined shear bands, of which one exhibits continued strain-softening and the other undergoes elastic loading after the peak-load point (Fig. 3b,c). In this case the asymmetric response has been obtained even without assuming any asymmetric initial imperfection, merely as a consequence of the numerical round-off errors, which occur with a slight asymmetry.

Another calculation has been performed under conditions of enforced symmetry. Only one half of the finite element mesh with the boundary conditions of symmetry on the vertical symmetry axis has been used in this case. The computations indicated a symmetric response mode with two intersecting inclined shear bands which are both active even after the peak-load point. It may be noted that in implementing the spatial averaging in the finite element program, one must first create an image field with respect to the axis of symmetry, and then let the averaging zone protrude to the other side of the axis of symmetry and take for averaging the image values as if no symmetry assumption were made.

Again the condition of minimum second-order work along the path indicates that the stable response path is the asymmetric one. Note that this path leads, after the bifurcation point, to a steeper descent.

Mesh 2 - Sliding Boundary. The results obtained by the coarse mesh appear to be different from those described previously. After the spread of plasticity from the assumed weak zone, the specimen fails with active plasticity only, i.e., strain-softening. No unloading is observed in this case, even if the loading steps are extremely refined. The force-displacement curve is similar to that obtained previously (Fig. 4b) but the failure pattern is totally different as no shear bands are found (Fig. 4c). The same results are obtained if the two internal nodes mentioned previously are considered to be initially slightly displaced in the lateral direction.

Mesh 2 - Fixed Boundary. The same calculations have further been performed with a regular mesh for the fixed boundary condition, and a very stiff response of the specimen has been found in this case. The peak load is much higher (see the dashed line in Fig. 5b), and no tendency to the development

of shear bands is observed (Fig. 5c). When the two aforementioned internal nodes are assumed to be initially slightly displaced in the lateral direction, the same type of response is found (Fig. 5d), except that a few more points which undergo unloading are present and the load-deflection curves descends, consequently, slightly steeper.

It may be noted that, from the results obtained with the coarse mesh (mesh 2), one may conclude that a shear band cannot be modelled with a nonlocal plasticity theory if the element size  $h$  is greater than a certain fraction of the characteristic length,  $l$ . In particular, computational experience indicates that for  $h > l/n$ , modelling of shear bands is impossible, and for  $h \leq l/n$ , modelling is possible, in which  $n$  is an empirical coefficient found to be equal to approximately 3.5. Further numerical experience indicated that this conclusion about shear band width remains valid for different choices of the characteristic length  $l$ . It should also be mentioned that further refinements of the mesh indicate that the solution converges correctly.

In the case that a coarse finite element mesh has to be used in calculations, a different approach to the modelling of shear bands is needed. A shear band, represented as a discontinuity must be embedded in the finite elements by enriching the distribution functions of displacements with a discontinuous displacement field. Such an approach was recently proposed by Ortiz, Leroy and Needleman (1986).

It may also be mentioned that the problem with the use of coarse finite elements is formally identical to that encountered in the modelling of tensile fractures. Such fractures, too, can be modelled with large finite elements in which a discontinuous shape function is embedded, as by shown Droz [23].

### 3.3 Propagation of Interacting Damage Bands or Cracks

Fig. 7 shows an example of tensioned rectangular panel calculated at Northwestern University by F. B. Lin. The material undergoes strain-softening described by the Tresca plasticity criterion whose yield limit decreases as the function of the effective plastic strain. A square mesh of finite elements is used, in this case without a nonlocal approach, since the element size  $h$  is assumed to be equal to the characteristic length  $l$  of the material. The initial yield limit in the two central elements on each side of the panel is assumed to be slightly less than in the other elements. At each stage of softening, one can find two solution paths for the load increment: (1) the softening band grows symmetrically, being two elements wide, and (2) the softening extends asymmetrically in only one row of elements while the adjacent elements start unloading. The states obtained in each of these solutions are stable if the loading is displacement controlled. This has been checked on the basis on the condition of minimum second-order work along the path.

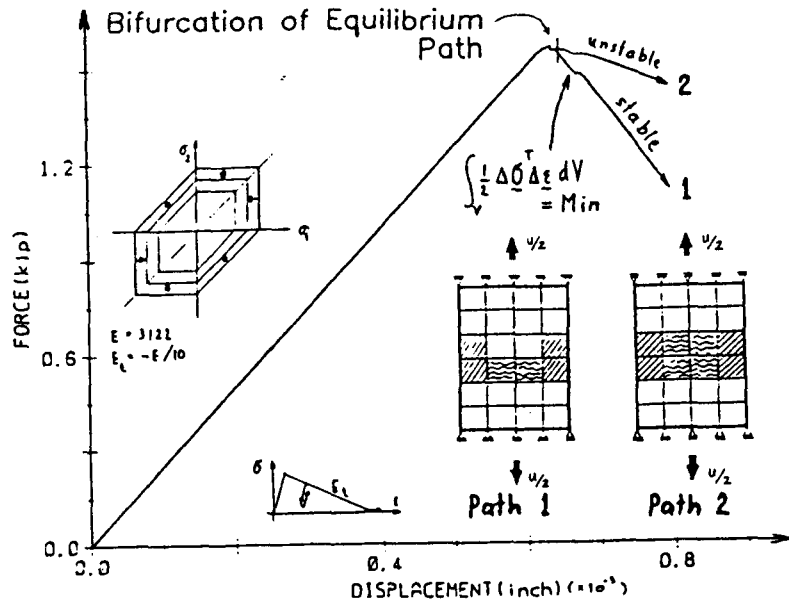


Fig. 6 Finite element results of F.-B. Lin and Z. P. Bažant (1985) on bifurcation due to propagation of a crack band with nonlocalized and localizing fronts.

Only path 2, however, i.e. the localized path of a single-element width of the cracking front, is stable. So the response becomes asymmetric. This has been concluded from the values of the second-order work along the two paths. However, if the present conditions of path stability are not included in the finite element program, and no disturbance to symmetry is introduced, the result of calculation is path 1, i.e. the symmetric path, with crack front of two-element width. This occurs despite the fact that the iterations in each loading step converge well. Yet such a solution is incorrect, since it does not maximize the changes of internal entropy.

In damage propagation problems, frequently there may be interaction between two or more damage fronts. This problem is entirely analogous to the interaction between the tips of two or more cracks. There are typically bifurcations, due to the fact that only one damage band (or one crack) may extend or several of them may extend simultaneously.

Fig. 7 shows some recent results for problems with interacting damage bands (or crack tips) [4,5]. Initially the damage bands in the rectangular panel are symmetric, and it is possible that either both fronts extend or only one extends while the material at the other front loads. These two possibilities cause that a bifurcation of the equilibrium response path in the stress-displacement diagram is possible at each stage of loading. The response for two propagating damage fronts is shown in Fig. 6 and 7 by the dashed lines, and the response for the case where a single damage front propagates is shown by the solid response curves. Calculation of the second-order work along the loading path, under conditions of prescribed displacements at the ends, indicates that invariably the localized path is the stable path. The symmetric cracks in the panels shown in Fig. 6 and 7, therefore, become asymmetric, propagating only on one side. A symmetric propagation is in fact impossible under a continuous loading process. Thus, a symmetric crack can occur only due to other actions, for example when it is cut artificially. It may be noted that a similar conclusion about the propagation of symmetric cracks or damage bands in a tensile specimen have been obtained by Rots [24,25] although without a proof that the localized path (a single cracking band extending) is the only possible (i.e. stable) solution.

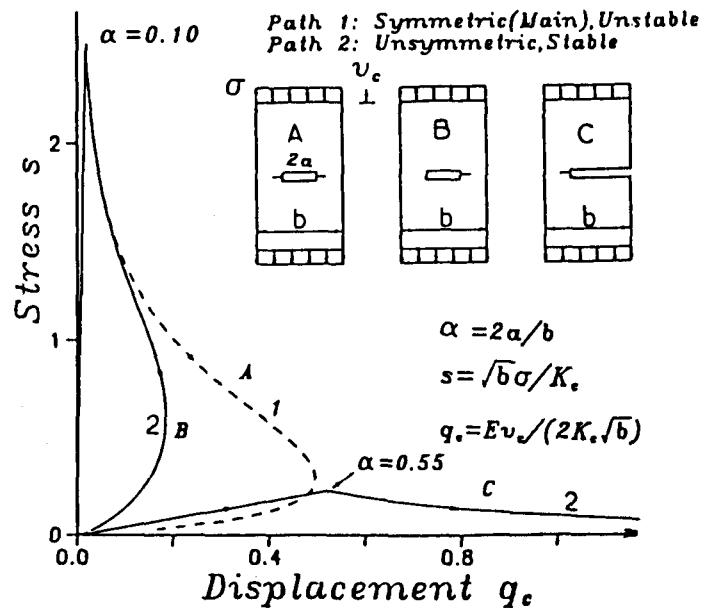


Fig. 7 Stable (solid curve) and unstable (dashed) paths of a panel with a centric damage band (after Bažant and Tabbara, 1989).

Fig. 8 also shows the surface of second-order work as a function of the relative axial displacement at the ends of the panel,  $\delta u/b$ , and relative rotation of the end cross sections,  $\delta \theta$ . The equilibrium response paths of the panel are labelled on the surfaces as 1, 2, 3. It is found that while the symmetric damage bands are stable, under displacement control conditions, the loading path which leads to these paths is unstable because it leads to smaller entropy (this is apparent from the heights of points 1, 2 and 3 seen from the surfaces).



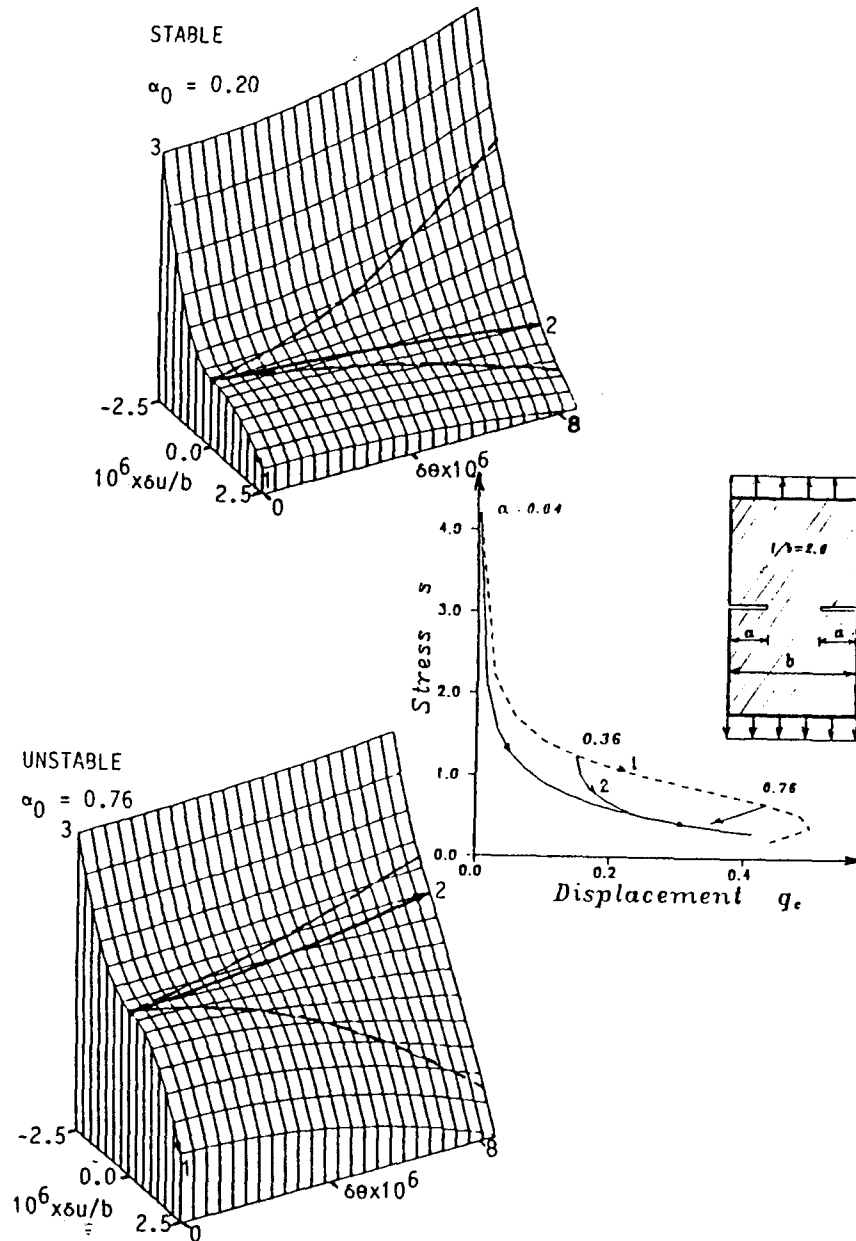


Fig. 8 Stable (solid) and unstable (dashed) paths of a panel with two interacting damage bands, and surfaces of second-order work for various relative lengths  $\alpha_0 = 2a/b$  of the damage bands.

### 3.4 Localization of Strain-Softening in Ellipsoidal Domains

After showing some finite element solutions, it may be refreshing to also present the results of a localization problem where exact analytical solution is possible, and can serve as a yardstick for measuring the correctness of finite element programs. It is the problem of localization of an initially uniform state of strain in an infinite space into an ellipsoidal region. Solutions of bifurcation states leading to localization states have been obtained by Rudnicki and Rice [26], Rice [27] and Rudnicki [28]. But they were limited to Drucker-Prager elastoplastic material and did not analyze stability as such. The solution for a general material and the condition of stability has been obtained in [29,30].

A uniform state of strain in an infinite solid will undergo localization into an ellipsoidal region when the following  $6 \times 6$  matrix ceases to be positive definite.

$$\underline{Z} = \underline{D}_t + \underline{D}_u (\underline{Q}_u^{-1} - \underline{I}) \quad (10)$$

Here  $\underline{I}$  = unit matrix,  $\underline{D}_t$  = matrix of tangential moduli for further loading from the given initial state of uniform strain,  $\underline{D}_u$  = matrix of tangential moduli for initial unloading from the same initial state, and  $\underline{Q}_u = 6 \times 6$  matrix of Eshelby coefficients for an elastic ellipsoidal inclusion. Matrix  $\underline{Q}_u$  depends on the elastic moduli  $\underline{D}_u$  as well as the aspect ratios of the ellipsoid. The coefficients of matrix  $\underline{Q}_u$  were expressed by Eshelby [32] in terms of elliptic integrals, and Kinoshita and Mura [33] generalized this solution to anisotropic material properties. The condition of instability for localization into an infinite layer in an infinite space can be obtained as a limiting case of Eq. 9.

The positive definiteness of matrix  $\underline{Z}$  is a condition of stability of state. The implications of this condition have been numerically studied for a broad range of material properties. Some such results, obtained by Bazant and Lin (1988) are plotted in Fig. 9 which pertains to a Drucker-Prager softening material with elastic shear modulus  $G$ , plastic hardening-softening modulus  $H$ , dilatancy ratio  $\beta$ , and internal friction coefficient  $\beta'$ ;  $a_1$  and  $a_2$  are the principal axes of elliptical localization region. As seen from Fig. 9, as the elliptical region is getting flatter, instability occurs at a smaller downward softening slope  $H$  ( $< 0$ ). It is interesting to note that unless the ellipsoid is very elongated, the value of  $H$  at the stability limit is rather high.

In the light of the foregoing discussion, one must suspect that the localization of strain may first occur as a stable bifurcation before it can occur as an instability. It has been proven that this is indeed so [31]. While the instability occurs only after the peak load, and occurs at rather far after the peak load unless the ellipsoidal region

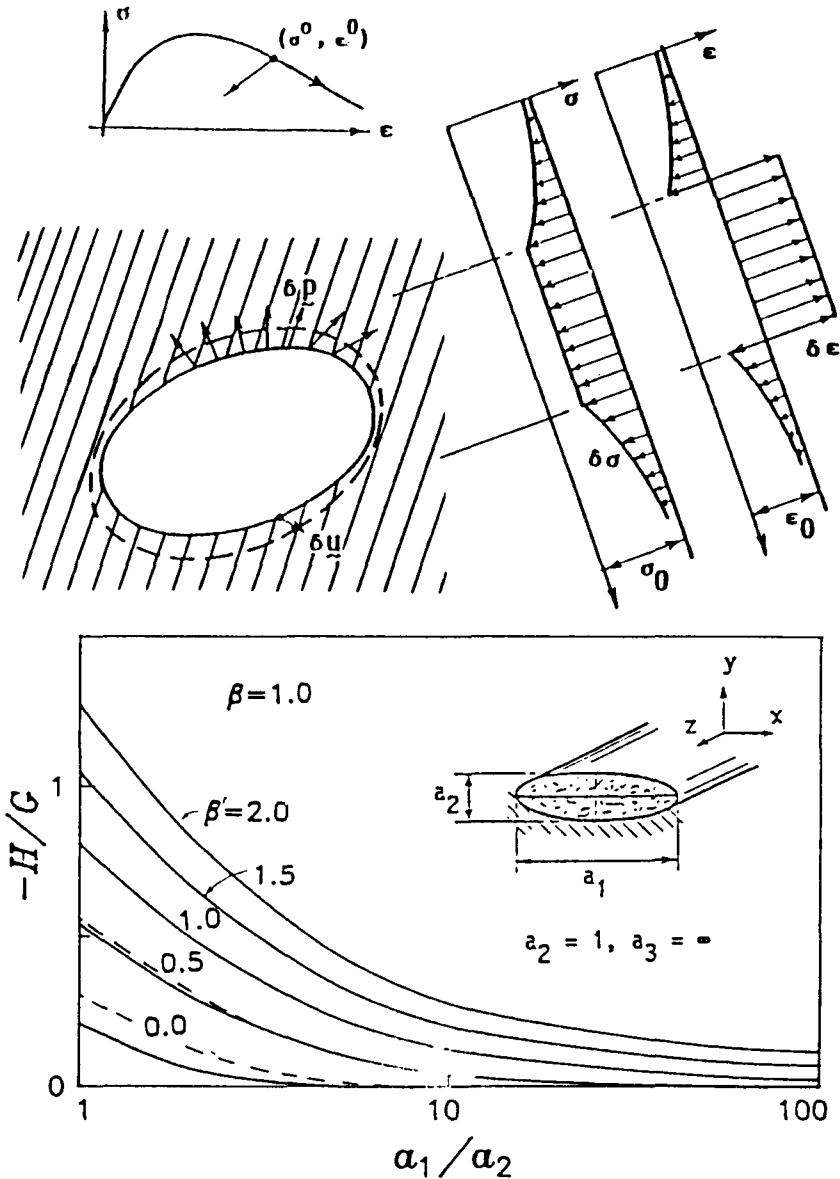


Fig. 9 Localization of a uniform strain-softening in an infinite solid into an ellipsoidal region, and diagrams of strain-softening slope  $H$  vs. aspect ratio of the ellipsoid, for Drucker-Prager material with friction angle  $\beta'$  and dilatancy ratio  $\beta$ .

is very elongated, the bifurcation of equilibrium response path at increasing strain occurs in the form of ellipsoidal localization already when the peak load is reached. The thermodynamic criterion of stable path can further be used to verify that the localization must occur right at the peak load, provided the infinite solid is loaded with a continuous increase of the stresses.

However, there exist certain types of loading for which stable states of uniform strain in the softening range beyond the peak stress point can be reached. This can happen, for example, when the value of  $H$  is suddenly changed by heating or other external environmental circumstances.

#### 4. CONCLUSIONS

1. To be able to pose the question of stability of a structure exhibiting plasticity or damage, and to answer this question, it is necessary to introduce the concept of a tangentially equivalent elastic structure, which makes it possible to apply thermodynamics to the analysis of stability.

2. The structure is stable if the internally produced increment of the tangentially equivalent elastic structure is obtained to be negative for all kinematically possible deviations from the initial state.

3. In the case of bifurcations of the loading path, which are ubiquitous in structures with plasticity or damage, one must distinguish the concepts of stable state and stable path because it is possible that all the immediate post-bifurcation states on all the post-bifurcation branches are stable, but only one branch can be followed in practice. The stable path is that for which the internally produced increment of entropy is maximized. This reduces to the condition that the second-work of prescribed displacement increments on the response force increments minus the second-order work of the prescribed force increments on the response displacement increments must be minimized.

4. Stability of iterations of a loading step depends only on the stability of state, and gives no information on the stability of path.

5. The predictions are in agreement with the well known Shanley's theory of plastic buckling.

6. Applications to various structures with damage due to either degradation of the yield limit or to degradation of material stiffness show that generally the stable path is that which produces localization of damage.

7. In the case of interactive damage fronts, examples show that their simultaneous propagation is usually not a stable response path.

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## REFERENCES

1. BAŽANT, Z. P. - "Stable States and Paths of Structures with Plasticity or Damage". ASCE Journal of Engineering Mechanics **114** No.2, pp.2013-2034, 1988.
2. BAŽANT, Z. P. - "Stable States and Stable Paths of Propagation of Damage Zones and Interactive Fracture", Preprints, France-U.S. Workshop on Strain Localization and Size Effect Due to Cracking and Damage, ed. J. Mazars and Z. P. Bažant, held at E.N.S., Cachan, France, September 1988; see also Cracking and Damage, ed. by J. Mazars and Z. P. Bažant, Elsevier Applied Science, London and New York, pp.183-206, 1989.
3. DROZ, P., and BAŽANT Z. P., - "Nonlocal Analysis of Stable States and Stable Paths of Propagation of Damage Shear Bands", Preprints, France-U.S. Workshop on Strain Localization and Size Effect Due to Cracking and Damage, ed. J. Mazars and Z. P. Bažant, held at E.N.S., Cachan, France, September 1988; see also Cracking and Damage, ed. J. Mazars and Z. P. Bažant, Elsevier Applied Science, London and New York, pp.415-425, 1989.
4. BAŽANT, Z. P., TABBARA, M., and KAZEMI, M. T. - "Stable Path of Interacting Crack Systems and Micromechanics of Damage", Advances in Fracture Research, PROC., 7th Inter. Conf. on Fracture, ICF 7, held in Houston, Texas, March, ed. by K. Salama et al., Pergamon Press, Oxford and New York, pp.2141-2152, 1989.
5. BAŽANT, Z. P., and TABBARA, M. - "Stable Path of Propagation of Interacting Fractures". Report in preparation, Center for Advanced Cement-Based Materials, Northwestern University, Evanston, 1989.
6. BAŽANT, Z. P., and CEDOLIN, L. - "Stability of Structures: Elastic, Inelastic and Fracture Theories", (in press), Oxford University Press, New York.
7. GUGGENHEIM, E. A. - "Thermodynamics, Classical and Statistical", Encyclopedia of Physics, ed. by S. Flugge, V: III/2, Springer Verlag, Berlin, pp.1-118, 1959.
8. HILL, R., - "A General Theory of Uniqueness and Stability in Elastic-plastic Solids". J. of the Mech. Phys. of Solids, **6**, pp.236-249, 1958.
9. MEIER, G., - "Incremental Plastic Analysis in the Presence of Large Displacements and Physical Instabilizing Effects". Int. J. of Solids and Structures, **7**, pp.345-372, 1971.
10. BAŽANT, Z. P. - "Instability, Ductility and Size Effect Strain-Softening Concrete". ASCE Journal of Engineering Mechanics **102**, 3, pp.331-344, 1976; with closure **103**, **8**, pp. 775-777.
11. PETRYK, H. - "On Stability and Symmetry Conditions in Time-Independent Plasticity". Archiwum Mechaniki Stosowanej **37**, 4-5, pp. 503-520, 1985.
12. PETRYK, H. - "On the Onset of Instability in Elastic-Plastic Solids", Plasticity Today: Modelling, Methods and Applications, ed. A. Sawczuk and G. Bianchi, Elsevier, London, pp.429-447, 1985.
13. NGUYEN, Q. S. - "Bifurcation and Post-Bifurcation Analysis in Plasticity and Brittle Fracture". Journal of Mechanics and Physics of Solids **35**, 3, pp.303-324, 1987.
14. NGUYEN, Q. S. - "Bifurcation et stabilite des systems irreversibles obeissant au principe de dissipation maximale". Journal de Mecanique Theorique et Appliquee **3**, 1, pp.41-61, 1984.
15. NGUYEN, Q. S., and STOLZ, C. - "Energy Methods in Fracture Mechanics: Bifurcation and Second Variations", IUTAM Symp. on "Applications of Multiple Scaling in Mechanics", E.N.S., Paris, France, 1986.
16. STOLZ, C. - "On Some Aspect of Stability and Bifurcation in Fracture and Damage Mechanics", Preprints, France-U.S. Workshop on Strain Localization and Size Effect due to Cracking and Damage, ed. J. Mazars and Z. P. Bažant, held at E.N.S., Cachan, France, September 1988; see also Cracking and Damage, ed. J. Mazars and Z. P. Bažant, Elsevier Applied Science, London and New York, pp.207-216.
17. PETRYK, H. - "On Energy Criteria of Plastic Instability", in Plastic Instability, Proc. of Considere Memorial, Paris, Ecole Nationale des Ponts et Chaussee Press, Paris, pp.215-226, 1985.
18. BAŽANT, Z. P. and PIJAUDIER-CABOT G. - "Modelling of Distributed Damage and Nonlocal Continuum with Local Strain", Numerical Methods in Fracture Mechanics, Preprints of 4th Int. Conf. held in San Antonio, Texas, March 1987, ed. A. R. Luxmoore et al., Pineridge Press, Swansea, U.K., pp.411-432, 1987.
19. PIJAUDIER-CABOT, G., and BAŽANT, Z. P. - "Nonlocal Damage Theory". ASCE Journal of Engineering Mechanics **113**, 10, pp.1512-1533, 1987.

32. ESHELBY, J. D. - "The Determination of the Elastic Field of an Ellipsoidal Inclusion and Related Problems", Proc. of the Royal Society of London, A241, p.376, 1957.
33. KINOSHITA, N., and MURA, T. - "Elastic Fields of Inclusions in Anisotropic Media". Physica Status Solidi, 2, pp.759-768, 1971.

20. BAŽANT, Z. P., and LIN F.-B. - "Non-local Yield Limit Degradation". International Journal for Numerical Methods in Engineering, 26, pp.1805-1823, 1988.
21. OWEN, D. R. J., and HINTON, E. - "Finite Elements in Plasticity: Theory and Practice", Pineridge Press, Swansea, U.K., 1980.
22. ORTIZ, M., LEROY, Y., and NEEDLEMAN, A. - "A Finite Element Method for Localized Failure Analysis". Computer Methods for Applied Mechanics and Engineering, pp.124-189, 1961.
23. DROZ, P. - "Numerical Modelling of Nonlinear Behavior of Massive Structures of Unreinforced Concrete", Doctoral Thesis No. 682, Ecole Polytechnique Federale de Lausanne, Switzerland, 1987.
24. ROTS, J. G. - "Computational Modelling of Concrete Fracture"; Doctoral Thesis, Delft University of Technology, September 1988.
25. ROTS, J. G., HORDIJK, D. A., AND DeBORST, R. - "Numerical Simulation of Concrete Fracture in Direct Tension", Proc. 4th Int. Conf. on Numerical Methods in Fracture Mechanics, ed. by A. R. Luxmoore et al., Pineridge Press, Swansea, pp.457-471, 1987.
26. RUDNICKI, J., and RICE, J. R., - "Conditions for the Localization of Deformation in Pressure Sensitive Dilatant Materials". J. of the Mech. and Physics of Solids, 23, pp.371-394, 1975.
27. RICE, J. R., - "The Localization of Plastic Deformation", Preprints, 14th Int. Congress of Int. Union of Theor. & Appl. Mech. (IUTAM), Delft, University of Technology, Netherlands, ed. by W. Koiter, 1976.
28. RUDNICKI, J., - "The Inception of Faulting in a Rock Mass with a Weakened Zone". J. of Geographical Research 82 (No. 5), pp.844-854, 1977.
29. BAŽANT, Z. P. - "Softening Instability: Part II - Localization into Ellipsoidal Regions". ASME Journal of Applied Mechanics 55 (Sept.), pp.523-529, 1988.
30. BAŽANT, Z. P., and LIN, F.-B. - "Localization of Softening in Ellipsoids and Bands: Parameter Study". Report No. 87-7/498s, Center for Concrete and Geomaterials, Northwestern University, Evanston, IL, 1989; also in the Int. Journal of Solids and Structures, in press.
31. BAŽANT, Z. P. - "Equilibrium Path Bifurcation Due to Strain-Softening Localization in Ellipsoidal Region". ASME Journal of Applied Mechanics, submitted to, 1989.