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Microplane constitutive model for inelastic behavior of soils

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ABSTRACT: A three-dimensional mathematical model for cohesive soils is presented. It is based on the microplane concept, which consists of characterizing the constitutive material properties on planes (called microplanes) of various orientations, and then superimposing the contributions of all possible spatial directions to obtain the macroscopic behavior. The model uses a kinematic constraint (the strains on a microplane are the resolved components of the macroscopic strain tensor) and an uncoupled formulation between the strain-stress and pore water pressure equations. Creep is introduced in the model by means of the principle of activation of energy (rate process theory). Numerical results are given which show stress-strain relations, pore water pressure or volume change variations, creep phenomena, etc. These results show a good qualitative agreement with laboratory tests results taken from the literature. The numerical implementation and applicability of the model are also discussed.

1. INTRODUCTION

The development during the past two decades of ever faster and larger computers, has provided the means to develop mathematical models to describe more accurately the materials used in engineering applications. The work in this field is not necessarily gratuitous: it spawns from the need of reducing material costs whilst increasing safety. The classic approach to material modeling has been of the phenomenological type, which for soils involve the theory of plasticity with various modifications and hardening and softening rules. The critical state theory (Roscoe & Burland, 1968) and the endochronic theory (Valanis, 1971; Bažant & Krizek, 1976; Cuellar et al., 1977) are two good examples of realistic and quite general representation of material behavior.

Other modeling techniques based on microscopic material characteristics have been also used to describe and predict material behavior; these include (a) the classical slip theory of plasticity (Batdorf & Budiansky, 1949) and its variants such as the multilaminar and microplane models (Zienkiewicz & Pande, 1977; Pande & Sharma, 1983; Bažant, 1984; Bažant & Kim, 1986; Bažant & Prat, 1987, 1988; Prat & Bažant, 1990; Carol et al., 1990); and (b) particle simulation (Cundall et al., 1979; Trent et al., 1987; Gili, 1988)

This paper attempts constitutive modeling of cohesive soils by a simplified micromechanics approach, that consists of specifying the stress-strain relation on planes of various orientations in the material (as suggested by Taylor, 1938) assuming that the strains on the planes, now called the microplanes, are the resolved components of the macroscopic strain tensor (kinematic constraint). The contribution of the planes of various orientations are then related to the macroscopic response simply by superposition or by means of the principle of virtual work (Bažant, 1984; Carol et al., 1990)

For granular materials (Fig. 1a) such as sand or concrete, the microplanes may be imagined to characterize the planes of grain contact within the microstructure, while for clays (Fig. 1b) the microplanes may be imagined to represent the slip plane between the clay platelets (Bažant & Kim, 1986) or alternatively the planes normal to the clay platelets (Bažant & Prat, 1987) in which slip is manifested by normal strain. A true micromechanics model would describe the material behavior by means of the laws governing the interactions of the particles on the contact points on such planes. In our simplified approach, however, the planes have no inmediate physical interpretation: they are simply a tool to formulate the model.

Before attempting to describe the present model, mention must be made to the hypotheses and assumptions needed both in regard of the material behavior, and in order to simplify the model and reduce the computational costs. These are:

1. The strains on a microplane are the resolved components of the macroscopic strain tensor, ε_{ij}:

$$\begin{aligned} \varepsilon_N &= n_j \varepsilon_j^n = n_j n_k \varepsilon_{jk} \\ \varepsilon_{T_i} &= (\delta_{ij} - n_i n_j) n_k \varepsilon_{jk} \\ \varepsilon_T &= \|\boldsymbol{\varepsilon}_T\| = \sqrt{\varepsilon_{T_i}\varepsilon_{T_i}} \end{aligned} \tag{1}$$

in which δ_{ij} is the Kronecker's delta tensor.

2. The response on each microplane depends explicitly on the volumetric strain (ε_V), in the sense that the microplane equations must include a separate treatment of the volumetric (spherical) and deviatoric normal components of strain.

The hypothesis means that we need to consider on a microplane three components of strain: $\varepsilon_V = \varepsilon_{kk}/3$, $\varepsilon_D = \varepsilon_N - \varepsilon_V$, and $\boldsymbol{\varepsilon}_T$, the last being a vector. The volumetric component (ε_V) is the same for all the microplanes. The deviatoric normal component (ε_D) for a given microplane is a vector whose direction is fixed once the microplane has been chosen. The shear strain component ($\boldsymbol{\varepsilon}_T$), however, is a vector lying on the microplane, whose direction is not immediately determined by the orientation of the microplane.

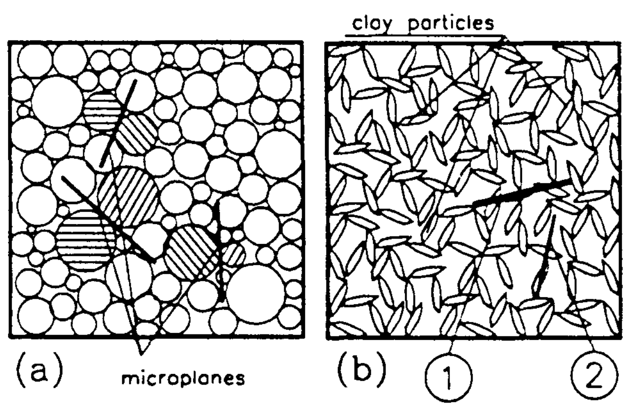

Fig. 1 Physical interpretation of the microplanes

3. The volumetric response, deviatoric normal response and the shear response on each microplane are mutually independent, i.e. decoupled.

4. The vector of shear stress ($\boldsymbol{\sigma}_T$) and the vector of shear strain ($\boldsymbol{\varepsilon}_T$) acting on a microplane are parallel, i.e. $\sigma_{T_i} \sim \varepsilon_{T_i}$. This hypothesis is introduced in the interest of reducing the number of unknowns.

5. The microplane stress–strain relations for monotonic loading histories (i.e. histories with no unloading) are path-independent, that is, they can be written as total rather than incremental stress–strain relations.

2. MICROPLANE EQUATIONS

We assume the existence of a functional relation between the strain and effective stress components on each microplane. Because of hypothesis 3, the three components (volumetric, deviatoric normal, shear) are mutually independent. These components can be further divided into their time-independent and creep contributions:

$$\varepsilon_V = \varepsilon_V^{\mathrm{i}} + \varepsilon_V^{\mathrm{cr}}, \quad \varepsilon_D = \varepsilon_D^{\mathrm{i}} + \varepsilon_D^{\mathrm{cr}}, \quad \varepsilon_T = \varepsilon_T^{\mathrm{i}} + \varepsilon_T^{\mathrm{cr}} \tag{2}$$

Thus we can write

$$\begin{aligned} \sigma'_V &= \mathcal{F}_V(\varepsilon_V^{\mathrm{i}}) = \mathcal{F}_V(\varepsilon_V - \varepsilon_V^{\mathrm{cr}}) \\ \sigma'_D &= \mathcal{F}_D(\varepsilon_D^{\mathrm{i}}) = \mathcal{F}_D(\varepsilon_D - \varepsilon_D^{\mathrm{cr}}) \\ \sigma'_T &= \mathcal{F}_T(\varepsilon_T^{\mathrm{i}}) = \mathcal{F}_T(\varepsilon_T - \varepsilon_T^{\mathrm{cr}}) \end{aligned} \tag{3}$$

$\mathcal{F}_V(\varepsilon_V^{\mathrm{i}})$, $\mathcal{F}_D(\varepsilon_D^{\mathrm{i}})$, and $\mathcal{F}_T(\varepsilon_T^{\mathrm{i}})$ are material functions which define the constitutive relation and are determined empirically. For simplicity we will omit the superindex 'i' in the following equations.

2.1 Volumetric stress–strain relation

We need to distinguish between hydrostatic compression and tension. For compression we assume a bilinear relation between ε_V and $\log \sigma'_V$, as shown in Fig. 2. For virgin (initial) loading, the relation is described by the following equations:

$$\begin{aligned} \varepsilon_V &= C_S^{*} \log\left(\frac{\sigma'_V}{\sigma_V^{\mathrm{o}}}\right) \quad \text{if } p_c > \sigma_{\mathrm{vert}} \\ \varepsilon_V &= \varepsilon_{\mathrm{m}} + C_C^{*} \log\left(\frac{\sigma'_V}{\sigma_{\mathrm{m}}}\right) \quad \text{if } p_c = \sigma_{\mathrm{vert}} \end{aligned} \tag{4}$$

where σ_V^{o} is the initial effective volumetric stress in situ, σ_{m} and ε_{m} are the maximum effective volumetric stress and strain ever reached, p_c the preconsolidation pressure, σ_{vert} the vertical stress, and C_S^{*} and C_C^{*} ara model parameters (i.e. the slopes of the two branches).

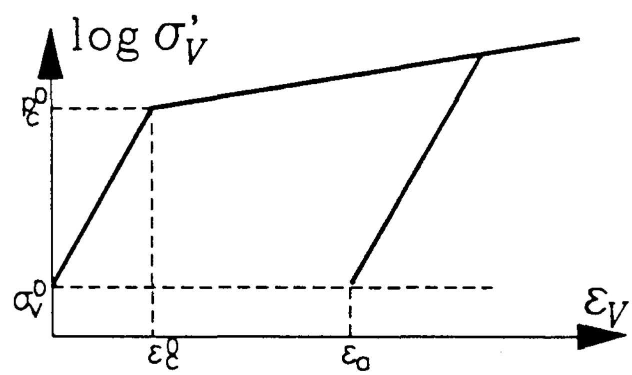

Fig. 2 Microplane volumetric stress–strain relation

For unloading, the equation is:

$$\varepsilon_V = \varepsilon_{\mathrm{a}} + C_S^{*} \log\left(\frac{\sigma'_V}{\sigma_V^{\mathrm{o}}}\right) \tag{5}$$

where ε_{a} is the value of ε_V corresponding to the point on the unloading branch at which $\sigma_V = \sigma_V^{\mathrm{o}}$.

For loading in "tension," i.e. when the current $\sigma'_V < \sigma^\circ_V$, we assume a stress–strain curve with a peak and a descending branch asymptotically approaching zero:

$$\sigma'_V = \sigma^\circ_V + E^\circ_V(\varepsilon_V - \varepsilon_a)e^{-\frac{1}{p}\left|\frac{\varepsilon_V - \varepsilon_a}{\varepsilon_p}\right|^p} \qquad (6)$$

where $E^\circ_V = \sigma^\circ_V / C^*_S$, and p and ε_p are material parameters.

Finally, for "tension" unloading, we assume a linear branch with slope E°_V such that

$$\Delta\sigma'_V = E^\circ_V \Delta\varepsilon_V \qquad (7)$$

2.2 Deviatoric stress–strain relation

We assume the following relations with a horizontal (plastic) plateau (Fig. 3):

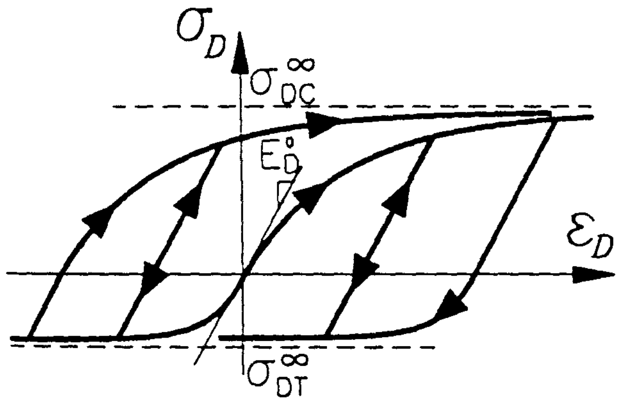

Fig. 3 Microplane deviatoric stress–strain relation

$$\begin{aligned} \sigma_D &= \sigma^\infty_{DC}[1 - e^{-k_{DC}|\varepsilon_D|}] \quad \text{if } \sigma_D \geq 0 \\ \sigma_D &= \sigma^\infty_{DT}[1 - e^{-k_{DT}|\varepsilon_D|}] \quad \text{if } \sigma_D < 0 \end{aligned} \qquad (8)$$

where the sign of σ_D (positive in compression, negative in tension) is chosen according to the usual convention in soil mechanics. In Eqs. 8 σ^∞_{DC}, σ^∞_{DT}, k_{DC}, and k_{DT} are empirical material constants, not entirely independent if we enforce continuity of slopes at the origin. In that case, this relation must hold: $|\sigma^\infty_{DC} k_{DC}| = |\sigma^\infty_{DT} k_{DT}| = E^\circ_D$, where E°_D is the initial elastic modulus. Eqs. 8 apply only for loading on the microplane. For unloading, we assume on each microplane linear elastic behavior with elastic modulus E°_D. It must be noted that the relationships defined by Eqs. 8 act as the envelopes for future loading–unloading–reloading cycles.

2.3 Shear stress–strain relation

The shear stress–strain relation must show a dependency on the overconsolidation ratio $r_{OCR} = p_c/\sigma^\circ_{vert}$, to take into account its effect on the macroscopic behavior (Fig. 4):

$$\begin{aligned} \sigma_T &= \sigma^\infty_T[1 + (a\varepsilon_T - 1)e^{-k_T\varepsilon_T}] \\ a &= a_0(r_{OCR} - 1) \end{aligned} \qquad (9)$$

σ^∞_T, k_T, and a_0 are empirical parameters. Parameter a depends on r_{OCR}: for normally consolidated clays, $a = 0$ and the curve has always a positive slope with an asymptotic horizontal plateau. For overconsolidated clays, $a > 0$ and the stress–strain curve exhibits a peak followed by a descending branch asymptotically approaching a residual value σ^∞_T. Note that if the initial shear modulus E°_T is known, then the exponent k_T in Eq. 9 can be computed as $k_T = E^\circ_T/\sigma^\infty_T - a$.

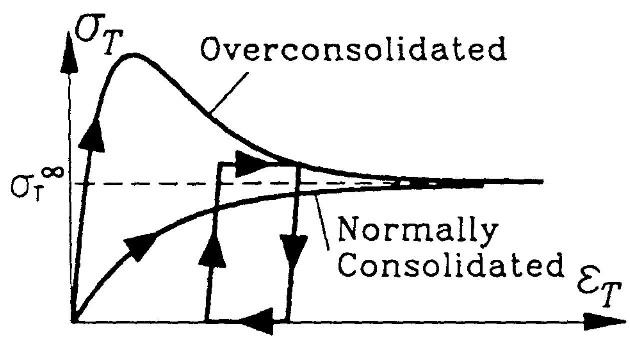

Fig. 4 Microplane shear stress–strain relation

Eq. 9 represents only a relation between the norms of the shear stress and shear strain vectors. However, since we have assumed (Hypothesis 4) that these vectors are parallel, we can easily obtain the components of the stress vector as

$$\sigma_{T_i} = \sigma_T \frac{\varepsilon_{T_i}}{\varepsilon_T} \qquad (10)$$

where $\sigma_T = \|\boldsymbol{\sigma}_T\| = \mathcal{F}_T(\varepsilon_T)$, and $\varepsilon_T = \|\boldsymbol{\varepsilon}_T\|$.

3. PORE WATER PRESSURE

Since the deviatoric effective and total stress tensor are identical ($s'_{ij} \equiv s_{ij}$), we conclude that the pore pressure needs to be introduced only into the volumetric equations and therefore, the effective stress-strain model (for which we use the microplane model) can be uncoupled from the pore water pressure model, which is scalar and does not necessitate the microplane approach. For drained tests, in which no pore pressure is allowed to develop, the effective and total stress tensors coincide and therefore $\sigma'_V = \sigma_V$. Thus, the total volumetric stress can be simply written as

$$\sigma_V = \sigma'_V = \mathcal{F}_V(\varepsilon_V) \qquad (11)$$

However, in undrained tests the total volumetric stress has to be computed as

$$\sigma_V = \sigma'_V + p_w = \mathcal{F}_V(\varepsilon_V) + p_w \qquad (12)$$

To analyze the undrained behavior of a saturated cohesive soil, we will use the model developed by Ansal et al. (1979) as an extension of Bažant and Krizek's (1975) formulation for a two-phase medium. The basic assumptions are: *(a)* both free and bound water behave elastically; *(b)* their compressibilities are the same; and *(c)* both can carry only volumetric stress. According to these hypotheses, the pore water pressure can be computed as

$$p_w = \frac{C_w K}{nK + C_w}\left[\frac{\sigma_V}{K} + 3\varepsilon''\right] \tag{13}$$

where $K = 2G(1+\nu)/3(1-2\nu)$ = bulk modulus, G is the shear modulus (non-constant along the stress path), ε'' the accumulated inelastic strain, C_w the water compressibility, and n the porosity.

4. CREEP LAWS

According to the rate process theory (Glasstone et al., 1941), which is generally accepted for the creep of clays, the strain rates on the microplanes may be expressed as

$$\begin{aligned} \dot{\varepsilon}_V^{cr} &= k_1^V \sinh(k_2^V \sigma_V) \\ \dot{\varepsilon}_D^{cr} &= k_1^D \sinh(k_2^D \sigma_D) \\ \dot{\varepsilon}_T^{cr} &= k_1^T \sinh(k_2^T \sigma_T) \end{aligned} \tag{14}$$

where the k_1's and k_2's are constants depending on temperature, activation energy and time. The current creep component of each strain (volumetric, deviatoric or shear) may be computed after each time increment Δt as

$$[\varepsilon^{cr}]_{t_1} = [\varepsilon^{cr}]_{t_0} + \dot{\varepsilon}^{cr}\Delta t \tag{15}$$

These values are then introduced into Eqs. 3, from which the stresses on each microplane are calculated.

5. MACROSCOPIC CONSTITUTIVE LAW

The basic structure we will use for the effective stress-strain law has been developed recently by Carol et al. (1990). The constitutive law is written in terms of the current effective stresses and strains (and not in terms of their increments), which allows the model to be explicit. The macroscopic effective stress tensor can be expressed as:

$$\sigma'_{ij} = \sigma'_V \delta_{ij} + \int_\Omega n_i n_j \sigma'_D \Psi_N d\Omega + \int_\Omega \frac{(n_i\delta_{rj} + n_j\delta_{ri} - 2n_i n_j n_r)}{2}\sigma'_{T_r}\Psi_N d\Omega \tag{16}$$

where Ψ_N is a weighing function of the orientations $\mathbf{n}$ which in general can introduce anisotropy of the material in its initial state. The macroscopic total stress tensor, σ_{ij}, then results from the principle of effective stresses of soil mechanics: $\sigma_{ij} = \sigma'_{ij} + p_w\delta_{ij}$ where σ'_{ij} is obtained from Eq. 16 and the pore water pressure p_w from Eq. 13. The latter equation can be rewritten as

$$p_w = \alpha\sigma_{kk} + \beta\varepsilon'' \tag{17}$$

with $\alpha = C_w/3(nK + C_w)$ and $\beta = 9K\alpha$. Thus,

$$\sigma_{ij} = \sigma'_{ij} + \alpha\sigma_{kk}\delta_{ij} + \beta\varepsilon''\delta_{ij} \tag{18}$$

Calling $B_{ij} = \sigma'_{ij} + \beta\varepsilon''\delta_{ij}$ (which is a known tensor), we obtain the following system of equations in the unknowns σ_{ij}:

$$\sigma_{ij} - \alpha\sigma_{kk}\delta_{ij} = B_{ij} \tag{19}$$

The solution of this system of equations gives the values of the macroscopic total stress tensor and, therefore, the value of the pore water pressure as well.

6. NUMERICAL IMPLEMENTATION

The model presented has been developed for constitutive equations in finite element computations. The computer code can be used without modification as part of the finite element code, or as standalone for model verification. A detailed process of the method can be seen in Carol et al. (1990).

The computer subroutine procedure shows a single loop over the number of microplanes, 28 in our case. After computing the strain components on each microplane, the corresponding constitutive laws are used to calculate the microplane stresses, which are finally assembled into the macroscopic stress tensor using Eq. 16. Since these equations are written in terms of the total stresses (not of their increments), no numerical integration is necessary within the load steps. This feature is particularly important when using finite element analysis because, since the microplane model involves numerical integration on the surface of a hemisphere at each Gauss point of each finite element, it reduces the necessary computation time by more than one order of magnitude, making the algorithm suitable for this kind of applications.

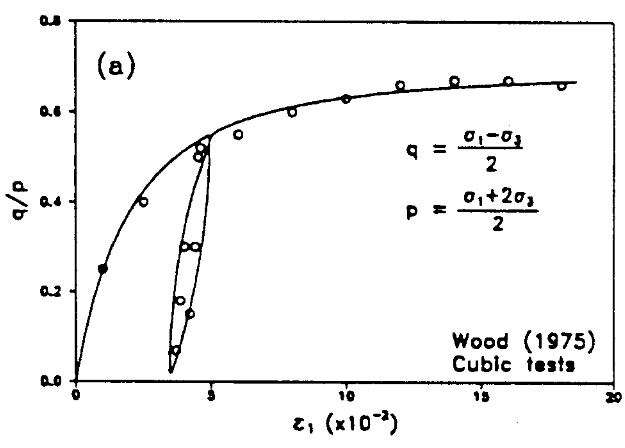

Fig. 5 Comparison with data from Wood (1975)

Fits of several typical test data from the literature are exhibited in Figs. 5–8. The measurements are shown as the data points while the results of the model are plotted as solid curves.

Fig. 5 exhibits the results obtained by Wood (1975) on cubic tests of kaolin clay, with one unloading–reloading cycle. The microplane model, it is shown, can reproduce the experimental data rather well, including the unloading–reloading phenomena.

Fig. 6 shows the test data on overconsolidated clays (r_{OCR}=24) obtained by Henkel (1956). The results from the microplane model can reproduce the main features of the behavior of the material, such as slight strain-softening and dilatancy in the case of the drained samples, and the variation of the pore water pressure in the case of undrained samples.

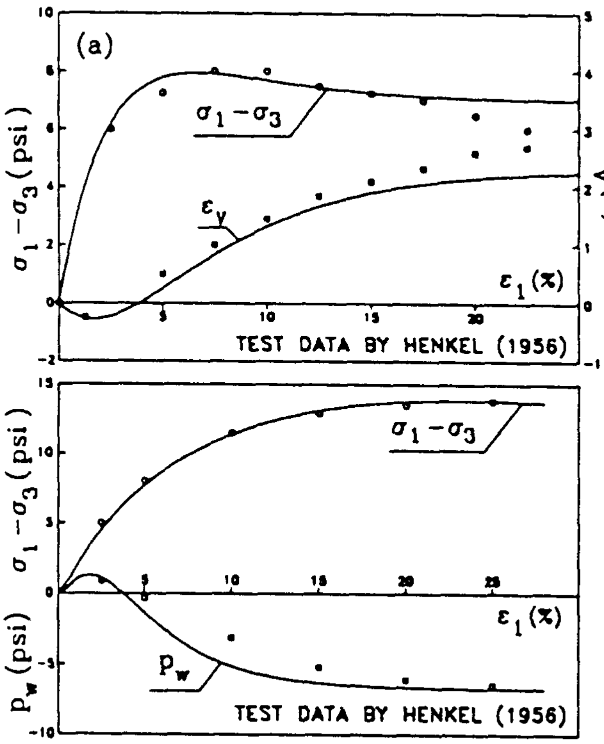

Fig. 6 Comparison with drained and undrained tests from Henkel (1956)

Fig. 7 exhibits several data from plane-strain tests carried out by Hambly (1972). The data correspond to different states of stress and strain, with different proportion between stresses or strains, and with undrained as well as drained conditions. The theoretical curves show a good qualitative agreement with the experimental data in all cases.

Finally, it has been found (Fig. 8) that the present model can provide a reasonable qualitative agreement with creep tests performed on Batiscan clay by Leroueil et al. (1985). This shows that the hypotheses and techniques used may be presumed to be sufficiently accurate to describe the real behavior of the material.

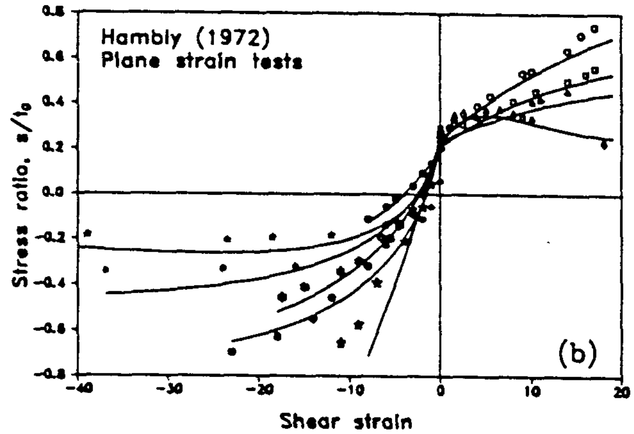

Fig. 7 Comparison with data from Hambly (1972)

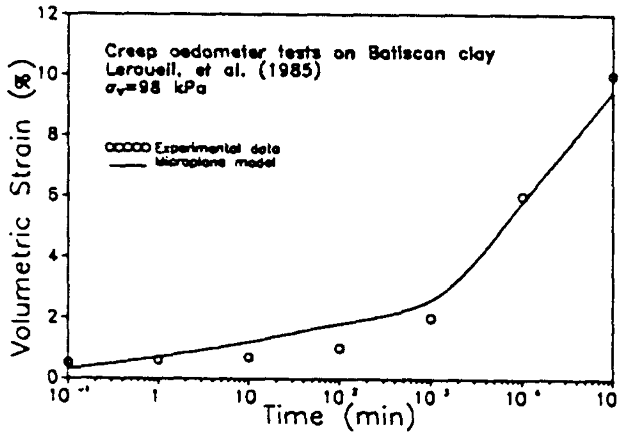

Fig. 8 Comparison with data from Leroueil et al. (1985)

8. SUMMARY AND CONCLUSIONS

The microplane formulation has been found to describe qualitatively well the behavior of cohesive soils, including strain-softening of overconsolidated samples, plain strain and stress and other multiaxial complex loading histories, creep, etc.

A pore water pressure term is included in a manner that gives undrained as well as drained behavior as special cases. The model seems capable to reproduce well the main trends of both types of behavior, including the pore pressure variation (for undrained tests) and the volume change (for drained tests).

Since the model is explicit (in the sense that no iterations are required to obtain a stress state from a given initial strain state and strain increment) and the number of parameters that need to be adjusted to fit complex test data is small enough for practical purposes, its use in a finite element code is easy and efficient despite the need to integrate over a hemisphere, typical of the microplane formulations.

ACKNOWLEDGMENTS

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