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38 CURRENT STATUS AND ADVANCES IN THE THEORY OF CREEP AND INTERACTION WITH FRACTURE
(*Keynote Lecture*)

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Abstract
The lecture has a two-fold purpose—to give an appraisal of the current status on the theory of creep and shrinkage of concrete, and to present some recent advances at Northwestern University in the modeling of interaction of creep with fracture. After pointing out that the deterministic linear aging viscoelastic analysis of fracture has become an essentially complete theory, with no significant further advances likely to occur, attention is focused on four research directions which seem to be most promising at present: (1) Interaction of creep with fracture, in the context of fracture mechanics, (2) micromechanics of the processes causing and influencing creep and shrinkage, in the sense of quantitative mathematical models correlating the micro- and macro-levels and contrasting with mere microscopic observations, (3) integration of diffusion models for moisture and heat transport with the analysis of creep and shrinkage in structures, and (4) introduction of statistical data and probabilistic theories into the practical creep and shrinkage analysis of structures and extrapolation of short-time measurements. Two advances which concern the first aforementioned research direction are then described in detail—a generalization of the cohesive (fictitious) crack model and a generalization of the R-curve model for an equivalent LEFM crack, in which the rate-dependence of fracture growth and the creep in the bulk of the structure are taken into account. Numerical results verifying these rate and creep type generalizations are also cited in the lecture. Furthermore, regarding the second aforementioned direction, it is shown that application of the age-adjusted effective modulus method to the creep of the mortar constituent in concrete can yield simple quasi-elastic predictions of the compliance function of concrete as an aggregate-mortar composite.
Keywords: Concrete, Creep, Shrinkage, State-of-Art, Fracture Mechanics, Rate Effect, Aging Viscoelasticity, Composites.

1 Introduction

During the last quarter of century, the theory of creep and shrinkage of concrete has undergone a tremendous development. Compared to the status 25 years ago, the formulations are now much more realistic, much better supported by experimental results, and from the mathematical viewpoint much more rational. The main impetus for progress was initially provided by the needs of nuclear power industry, which justified large infusion of research funds. This source of funding drastically diminished and in many countries almost totally evaporated by 1985. However, motivation for further

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progress has nevertheless been provided by the challenge of maintaining and rebuilding the civil engineering infrastructure of bridges and roads, which at present provides the main driving force for new developments.

The objective of the present lecture is, first, to reflect on the current status of the theory, and second, outline some recent developments at the author's home institution, focusing especially on the interaction of creep and fracture growth in structures, which represents one important new trend of research. A new model for the rate effect in fracture growth, which is based on micromechanics considerations and is suitable for incorporation into creep and cracking analysis of structures, will be presented. No claims for exhaustiveness of the coverage of the literature are made. A systematic review of the vast literature that exists would take too many pages and has in fact been presented not too long ago [1].

2 Reflections on Current Status

2.1 Deterministic Linear Aging Viscoelastic Analysis of Structures—A Closed Subject?

In contrast to creep of many other materials, for example metals at high temperature or clay, concrete creep is approximately linear within the service stress range. This greatly simplifies the analysis, but there are other complicating factors. In contrast to metals and clay, concrete creep is a hereditary phenomenon with a much longer memory, that is, a very broad relaxation spectrum, and exhibits the phenomenon of aging, which in the case of concrete means the change of properties with the age of concrete, which is caused by the chemical reaction of hydration of cement.

Beginning with the pioneering studies of Glanville, Dischinger, Arutyunyan, Maslov, McHenry and others before and after the Second World War [see 1], the linear aging viscoelastic analysis of concrete structures has been studied systematically and it seems that by 1990 the development of this subject has become essentially complete, with apparently little chance for significant further advance. This view of the writer, though, has not prevailed in a large part of the research community. Curiously, many researchers still continue studies of linear aging viscoelastic analysis which for the most part can lead to hardly more than miniscule refinements or rehash of previous solutions.

One popular stream of papers that began shortly after the publication in 1952 of the famous treatise by Arutyunyan [2] has consisted of the exercise of deriving the integral or differential equations generalizing the elastic solutions of various structural analysis problems. The reason that such endeavors are no more than exercises, mostly fruitless, is that, for a realistic form of the compliance function of concrete, the resulting systems of integral or differential equations in time are not ammenable to an analytical solution and must therefore be solved numerically by step-by-step methods. But an equivalent numerical solution of the problem can in general always be obtained without undertaking this tedious exercise—simply by converting immediately to an incremental form the viscoelastic stress-strain relation. This incremental form for each loading step has the form of Hooke's elastic stress-strain relation, which means that the solution of any problem can be obtained by solving a succession of elasticity problems. The result of the analysis is the same as that obtained when the differential or integral equations

of the problem are solved numerically in a succession of time steps, but the solution is simpler. Thus, no gain is achieved by going through the exercise of deriving and writing out the lengthy systems of integral or differential equations of the problem in time.

There are a few minor exceptions to the foregoing appraisal. One is a certain type of *probabilistic* analysis of the effect of random environmental history in which the spectral method is used. Another is the analysis of *periodic* solutions for the case of sequential construction with a great number of repetitions of like situations. An example of such a span-by-span construction sequence is the building of a continuous bridge with many identical spans, or the progress from one floor to another in a high-rise building. The solutions of these problems can of course be obtained by the standard step-by-step method through a succession quasielastic solutions, however, such an approach cannot reveal the asymptotic periodic solution in which the situation is exactly repeated after each cycle [3,4]. Imposing the conditions of periodicity for a construction cycle generally requires writing out the integral or differential equations of the problem before introducing the step-by-step discretization.

A more serious criticism of the wide-spread penchant for research papers presenting linear aging viscoelastic solutions of various structures can be made. These solutions are in most cases unrealistic and incapable of agreement with a broad range of observations on structures (they can of course always be adjusted to fit very limited test data). These solutions neglect much more serious problems, such as

- the effects of moisture diffusion and heat transport,
- the cracking and fracture produced by internal stresses that result from nonuniform shrinkage, nonuniform creep and nonuniform heating, and
- the statistical errors caused by uncertainty in the material properties and the environment.

In view of these very important influences, a high degree of sophistication and complexity in linear aging viscoelastic analysis of structures makes no sense. The complexity of integral or differential equations is unjustifiable. One can justify only very simple linear elastic analysis, such as that based on the age-adjusted effective modulus, the error of which is not larger than that of the complicated solutions of integral equations, when compared to the behavior of real structures affected by humidity, temperature, cracking and various random phenomena.

In conclusion: A closed subject?—Yes, with minor reservations.

It must be emphasized, however, that the foregoing conclusion does not apply to the formulation of linearly viscoelastic stress-strain relations, and especially not to the micromechanical aspects to linear aging viscoelasticity, such as the micromechanical modeling of the solidification process. This is still very much a basic research problem. So is the formulation of new more efficient computer algorithm for large-scale structural analysis.

2.2 Which Research Trends Are Promising?

One can identify four pregnant topics which may be expected to bear fruit, or in which practical applications are not yet taking place even though the discrepancies with respect to experimental observations are large.

1. **Interface of creep and shrinkage theory with fracture mechanics.**— With the exception of creep buckling, misprediction of creep or shrinkage usually does not cause structural collapse. Rather, it merely puts the structure out of service, that is, the structure does not live out its projected life span. The wide-spread occurrence of such lack of long-term serviceability inflicts a tremendous economic damage on many nations. The direct cause of damage that puts a structure out of service is typically cracking, which may cause major fractures. The analysis of cracking and fracture produced by creep, as well as the effect of cracking and fracture on the subsequent creep, is not merely a question of a nonlinear generalization of the viscoelastic stress-strain relation.

   Such nonlinear generalizations have been studied widely in the past, but they can generally describe only the departures from linearity at the beginning of distributed cracking. They are inherently incapable of describing the localization of cracking damage into major fractures and the propagation of such fractures. They are not even sufficiently realistic to describe the growth of microcracks, since all cracking calls for analysis based on the principles of fracture mechanics. The main characteristic of fracture mechanics, distinguishing it from elasticity, plasticity, and viscoplasticity, is that the spread of cracking and failure is characterized not only in terms of stress and strain, that is, the strength limit, but also, and mainly, by energetic criteria for the propagation of cracks.

   Furthermore, it is known that the crack propagation rate is an important parameter in the fracture propagation criterion, and that the creep in the material surrounding the fracture tip has a great effect on the energy flux into the fracture tip. The study of these phenomena in the context of long-term creep of concrete structures is in its infancy. Extensive studies of the rate effect have been made in the context of dynamic fracture, in which the effect of creep is minor [5–9]. Experimental and analytical studies of the interaction of creep and rate effects with fracture mechanics and the associated size effect in structures have been begun [10, 11], however, much more remains to be learned.

2. **Micromechanics of creep and shrinkage in concrete as a composite.** There have been extensive studies of the microstructure of cement and concrete in relation to creep and shrinkage, however, in the vast majority of them the correlations to the macroscopic behavior have been intuitive and non-quantitative. The macroscopic constitutive relations have been conjectured by intuition on the basis of various phenomena observed or postulated on the microstructural or even molecular level. Such studies generally have not, and could not have, borne much fruit. What is needed is a quantitative analysis, that is, micromechanics of the phenomena involved in creep, in shrinkage and in the related processes of diffusion and fracture. Only such an approach can bring significant progress in predicting

the effect of concrete composition or humidity condictions on its creep properties. The modeling of this effect at present relies on purely empirical relations, with no micromechanics models involved. Yet the uncertainty in the predictions of long-term creep associated with the variations of concrete composition is enormous, actually much larger than any uncertainty except that due to the randomness of environment. Attempts for mathematical micromechanical modeling of some phenomena have already begun, for example the solidification theory as a basis for the modeling of aging [12], however, many fruitful opportunities no doubt lie ahead.

3. **Influence of diffusion phenomena on creep and shrinkage.**—In this topic, extensive theoretical results have already been obtained but what is largely lacking is their incorporation into practice. With the powerful computers available today, this approach has become quite feasible. In concrete structures exposed to the environment or subjected to variable temperatures, there is no hope of obtaining realistic stresses, and thus also predictions of cracking and fracture, without actually solving the associated problems of moisture and heat transport, at least in an approximate manner. It has been shown that creep and shrinkage analysis based on diffusion analysis of a box girder bridge segment yields enormous stresses which are routinely neglected in practice [13]. The cracks produced by nonuniform drying affect the rate of drying itself [14], etc. In addition, there are phenomena in this topic that are not sufficiently understood—for example the modeling and micromechanics of the increase of creep due to changes of temperature or water content.

4. **Stochastic phenomena and probabilistic models.**—Similar to the preceding topic, extensive researches have been made in this topic, too, but they are not being introduced into practice. Although the statistical variability of concrete creep under controlled laboratory conditions is quite small, very large statistical fluctuations are caused by the environment as well as the uncertainties in the effect of concrete composition. In most practical situations, sophisticated deterministic mathematical analysis makes in fact little sense because the uncertainties of stochastic origin are much larger than the errors of simple effective modulus solutions compared to sophisticated deterministic analytical solutions of differential or integral equations [15, 16].

   Creep-sensitive structures are still being designed for the mean creep properties. However, the only meaningful approach is to design them for a certain suitable probability (such as 95%) that a specified strength limit or deflection limit will not be exceeded [17, 18]. The main reason is that a moderate coefficient of variation of material and environmental characteristics can translate in a very large (or very small) coefficient of variation of deflection, crack length, etc.

   Despite numerous contributions to the literature over the last 15 years, introduction of probabilistic models will require further research, especially on the practical side, as well as education. An important probabilistic problem in the prediction of creep effects is the updating of long-time predictions based on short-time measurements. Such updating can enormously reduce the coefficient of variation of long-time predictions. Various statistical regression approaches as well as Bayesian approaches have been studied, but introduction to practice is needed.

Another updating which could greatly reduce the uncertainties is that which exploits the knowledge of creep data for similar concretes, which means concretes from the same geographical region. For example, the concretes used in the Chicago region are generally quite similar, and instead of repeating measurements for each new major structure one can greatly improve predictions on the basis of previously obtained data for a similar concrete from the same region.

In the rest of this lecture, three recent developments made at Northwestern University will be outlined.

3 Time-Dependent Generalization of Fracture Models and Interaction with Creep

3.1 Cohesive Crack Model

Although the rate sensitivity of concrete and other structural materials has been studied extensively, both the experimental and computational aspects [5–9, 19–24], a generalization of the crack band or nonlocal smeared cracking model, or the discrete cohesive (fictitious) crack model, to long-time fracture with creep and rate effects is not available. A new development, whose application is presented in a parallel paper by Wu and Bažant (1993) in this volume, will now be presented.

The rate effect in fracture stems from two sources. One source is obviously the creep, which occurs, in the case of concrete, in the entire volume of the structure. For most of the structural volume, one may consider the creep to be linear, but within the fracture process zone the creep is certain to be nonlinear. However, it is not clear whether a separate consideration must be given to the creep in the fracture process zone because time-dependent effects arising in the process zone may be included in a time dependent generalization of the fracture model itself.

The second source is the time dependence of the rupture of bonds. Although the classical fracture mechanics is a time-independent theory, the breakage of bonds, which is the origin of fracture, cannot happen instantly when the bond strength is exceeded, but occurs at a certain finite rate. This rate is governed by the statistics of the thermal vibrations of atoms or molecules. The law governing the rate of bond breakages responsible for the fracture can be derived in the following manner which is analogous to that used in material science models for creep or plastic flow (which also involves bond breakages) and is based on the same principle as the activation energy theory for the rate of chemical reactions [25–27].

The thermal vibrations of atoms or molecules, which have a typical frequency $\nu \approx 10^{13}\text{s}^{-1}$, are random and the frequency at which any specified energy level U is exceeded is given by the Maxwell-Boltzmann distribution, $f = k_b e^{-U/RT}$, in which T = absolute temperature, R = gas constant, k_b = constant = $\alpha\nu$, and α = entropy factor. The potential of the bond forces, U, which is schematically plotted in Fig. 1 as a function of the distance x from the equilibrium position, has a certain maximum, Q, near the equilibrium position. This maximum is called the activation energy Q. It must be overcome for the bond to rupture. Therefore, the rate of bond breakages is given by $f_b = k_b e^{-Q/RT}$.

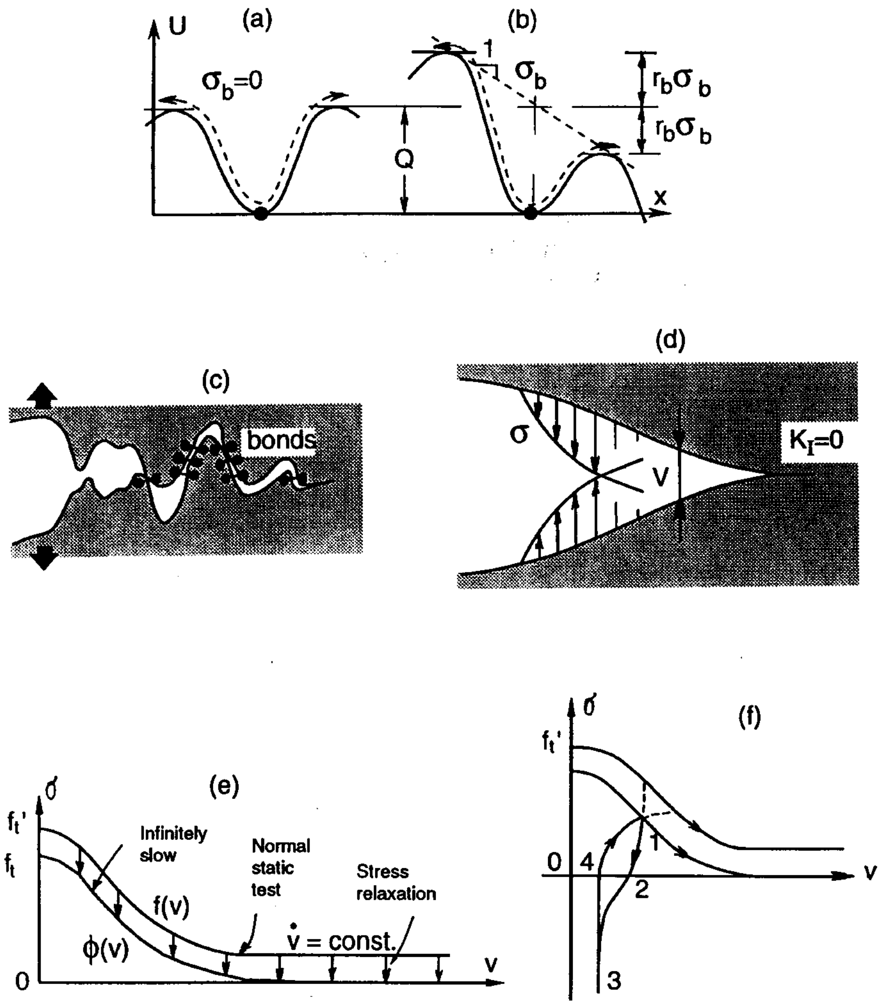

Fig. 1 Activation energy concept (a, b), bond ruptures at fracture front (c), crack bridging stresses in the cohesive crack model (d), stress–displacment relation for the rate dependent model (e), and behavior at unloading and reloading (f).

Now, if stress σ_b is applied to the bond, the bond potential U is modified as schematically shown in Fig. 1b, with the result that the potential energy barriers for a particle (atom or molecule) escape to the right or left are changed to $U_2 = Q - r_b\sigma_b$ and $U_1 = Q + r_b\sigma_b$; here r_b represents the distance from the equilibrium position (minimum potential energy) to the maximum potential energy (the reason is that $r_b\sigma_b$ represents the work of the applied stress on displacement r_b). In absence of applied stress σ_b, the potential energy barriers for movements to the left and right are equal, and so, even though the bonds rupture and the particles move, there is no net overall movement either to the right or to the left. However, when stress σ_b is applied, the frequency f_2 of the jumps over the potential energy barrier U_2 to the right exceeds the frequency f_1 of the jumps over the potential energy barrier U_1 to the left. Obviously, the rate of crack opening must be proportional to the difference of these frequencies. Therefore, substituting the foregoing expressions for U_2 and U_1, we obtain

$$\dot{v} = k_f(f_2 - f_1) = k_b k_f \left[e^{-(Q-r_b\sigma_b)/RT} - e^{-(Q+r_b\sigma_b)/RT}\right] = 2k_b k_f \sinh\left(\frac{r_b\sigma_b}{RT}\right) e^{-Q/RT} \tag{1}$$

It is now convenient to introduce the reference temperature T_0 and denote $C_0 = 2k_b k_f e^{-Q/RT_0}$ = constant. Furthermore, we need to relate the stress σ_b in the bonds that undergo fracturing to the overall crack bridging stress σ. To do this precisely, we would need to know the number of the resisting bonds, which is changing with the crack opening, as well as the deformation of crack surface which is influenced by the local stiffness of the microstructure at various parts of the crack surface. This aspect of modeling is obviously complicated, and a simplification must be introduced.

We will simply assume that σ_b is proportional to $\sigma - \phi(v)$ where $\phi(v)$ is a function that has the shape shown in Fig. 1f and is supposed to describe the stress-displacement relation $\sigma = \phi(v)$ of the cohesive crack model for an infinitely slow rate $\dot{v}$ of the crack opening. Thus we may set $(r_b/RT_0)\sigma_b = k_0[\sigma - \phi(v)]$, in which r_b, R, T_0 and k_0 are constants. With the aforementioned notations, Eq. (1) at reference temperature becomes

$$\dot{v} = C_0 \sinh\{k_0[\sigma - \phi(v)]\} \qquad \text{for} \qquad T = T_0 \tag{2}$$

and Eq. (1) at any temperature T becomes

$$\dot{v} = C_0 \sinh\left(\frac{T_0}{T} k_0 [\sigma - \phi(v)]\right) \exp\left(\frac{Q}{RT_0} - \frac{Q}{RT}\right) \tag{3}$$

Function $\phi(v)$ for an infinitely small rate of crack opening cannot be measured directly, and it is convenient to relate it somehow to the stress-displacement relation $\sigma = f(v)$ for fracture in the normal static tests in the laboratory. It seems that a simple but realistic assumption might be $\phi(v) = k_1 f(v)$, in which $\sigma = f(v)$ is the stress-displacement relation of the cohesive crack model (Fig. 1f) for the normal loading rate in a static test, that is, the loading rate that leads to the maximum load within about 5 min., and k_1 = constant that is close to 1 but less than 1 (roughly $k_1 = 0.8$). Here it is assumed that the normal static test does not reduce the crack bridging stress to zero but terminates with a plateau $\sigma = \sigma_\infty$ limiting displacement $v = v_1$; σ_∞ is assumed to be a small positive constant roughly equal to $0.1 f_t'$ (f_t' = direct tensile strength).

The foregoing assumption might explain why in static tests the load–deflection diagrams have a curiously long tail and why normally the load is not seen to get reduced

to zero even at very large deflections. This long tail might well be a consequence of the rate effect. To reduce the stress at very large v to 0 it is necessary to extend the test duration by several orders of magnitude, which causes stress relaxation, as indicated by the downward arrows in Fig. 1e. Fitting of experimental data is needed to determine whether such a simple assumption would be adequate. As for the behavior at unloading and reloading, it seems reasonable to delete the rate-dependence of fracture as long as the crack bridging stress is below $\phi(v)$.

The present model for rate-dependent crack opening either can be implemented directly in the form of a rate-dependent cohesive (fictitious) crack model, generalizing the model of Hillerborg et al. [28–30], or it can be converted to a stress–strain relation for the crack band model or a nonlocal model for continuum damage (cracking) that leads to fracture. This is accomplished simply by setting $v = h\epsilon_f$ where ϵ_f is the fracturing strain in the stress-strain relation with softening and h is a characteristic length of the material which represents either the width of the crack band or a length over which the spatial averaging in the nonlocal continuum model is carried out.

The latter approach has been adopted by Wu and Bažant [31] in another paper in this volume. In that paper, the rate-dependent fracture model is combined with a creep model for the material in the bulk of the specimen. The creep model is based on the solidification theory, with the creep of the cement constituent represented by the Kelvin chain model. It is shown in that paper that such a combined model for creep and fracture rate dependence compares favorably with the existing experimental data for various rates of loading and for various specimens sizes, and also reproduces the observed size effect reasonably well. It is also noted that inclusion of both creep in the bulk of specimen and the rate-dependence of the crack opening is important; if only one of these aspects is modeled, good agreement with the experimental data cannot be obtained. The creep in the bulk of the specimen has of course considerable effect on the stresses at the fracture tip. For slow loading or arrest of the opening displacement increase, the creep causes significant stress relaxation around the fracture process zone, leading to its unloading.

3.2 R-Curve Model

The simplest approach to the analysis of nonlinear fracture with a finite size fracture process zone is an equivalent linear model characterized by the so-called R-curve, that is, a curve treated as a material property which gives the energy R required for the propagation of an equivalent LEFM crack (fracture resistance) as a function of the crack length measured from the notch or the tip of the traction-free actual crack [e.g. 32]. We will now briefly summarize the formulation given in Bažant and Jirásek [33].

To analyze the response of structures or specimens under a controlled rate r of the crack mouth opening displacement (CMOD) Δ, one needs the following relations from linear elastic fracture mechanics (LEFM): $\Delta = P\delta(\alpha)/Eb$ and $K = Pk(\alpha)/b\sqrt{d}$, in which P = applied load, E = elastic modulus, b = specimen thickness, d = characteristic dimension (taken here as the beam depth), $\alpha = a/d, a$ = crack length and δ, k = functions that are known from LEFM. The classical rate-independent R-curve concept of fracture propagation is based on the assumption that the critical energy release rate R depends on the crack propagation length $c = a - a_0$, where a_0 = initial crack (notch) length. The corresponding critical stress intensity factor is $K_R(c) = \sqrt{ER(c)}$. The R-

curve can be completely determined from the effect of size on the measured maximum loads of geometrically similar specimens of different sizes. A well-known procedure yields a function ρ such that $R(c) = G_f\rho(c/c_f), K_R(c) = K_f\sqrt{\rho(c/c_f)}$, where G_f, K_f and c_f is the fracture energy, fracture toughness and effective process zone length for an infinitely large specimen, which are geometry (shape) independent constants.

One process causing the rate effect is the process of bond ruptures at fracture tip, which depends on the stress intensity factor K. As already explained, this process is described by the rate process theory for interatomic or intermolecular bond ruptures based on the concept of activation energy and Maxwell-Boltzmann distribution of thermal energies. Based partly on such considerations, the crack growth rate is expressed as $\dot{a} = \kappa_0 K^n \exp(-Q/RT)$, in which κ_0, n = constants, Q = activation energy, R = gas constant and T = absolute temperature. The factor K^n, however, is an empirical replacement of the sinh function. Since we do not study the temperature effect, this reduces to $\dot{a} = \kappa_1 K^n$. This relation may be applicable only to materials with a very small fracture process zone, and must therefore be generalized, which can be done using the R-curve concept. A simple generalization is given by the power law

$$\dot{a} = \kappa(K/K_R)^n, \tag{4}$$

in which κ and n are constants to be found empirically. The exponent n is expected to be very large so that for small K/K_R, the crack growth rates are negligible.

Replacing the elastic modulus E, according to the elastic viscoelastic analogy with the corresponding compliance operator for creep, we obtain from the elastic formula for CMOD the following expression for the creep effect on CMOD:

$$\Delta(t) = b^{-1}\int_{t_0}^{t} J(t,t')\,d[P(t')\delta(t')] \tag{5}$$

in which t = current time, t_0 = time at the first loading and $J(t, t')$ = compliance function for creep in the bulk of the specimen, which is well known.

One interesting observation from testing of the size effect in concrete at various rates [34] is that, in the plot of the logarithm of the nominal stress versus the logarithm of size d, the response shifts to the right (i.e. toward the LEFM) as the loading rate is getting slower. This suggests that c_f should decrease with a decreasing rate of loading, which must be described in terms of $\dot{a}$ because the material properties cannot directly depend on the loading rate. Therefore, it has been further assumed that $c_f = c_{f0}(\dot{a}/\dot{a}_0)^{1/m}$, in which $\dot{a}_0$ is a constant chosen for convenience and c_{f0}, m are constants to be determined empirically. This effect is, however, not seen in the tests of limestone [34], which is probably related to the fact that limestone does not exhibit any creep.

The problem can now be reduced to the solution of two coupled nonlinear integral and differential equations, which in the case of a constant CMOD rate r read

$$\int_{t_0}^{t} J(t,t')\,d[P(t')\delta(t')] = br(t-t_0) \qquad \text{(creep)} \tag{6}$$

$$\dot{\alpha}(t) = \frac{\kappa c_{f0}^{n/2}}{b^n d^{n+1-n/2m} K_f^n \dot{a}_0^{n/2m}}\left[\frac{P(t)k(\alpha(t))\dot{\alpha}^{1/2m}(t)}{\rho^{1/2}(\alpha(t)-\alpha_0)}\right]^n \qquad \text{(crack growth)} \tag{7}$$

and are to be solved for the proper initial conditions. The solution of these two equations has been carried out in [33] numerically in small time steps. The values of the material parameters κ, n, K_f, c_{f0}, and m have been varied first in a trial-and-error fashion and ultimately optimized. Fig. 2a (left), taken from [33], shows the best fit of experimental data reported in [35], which has been obtained using Eq. (6)–(7).

To model the rate and size effects for concrete as reported in [34] (Fig. 2b), the rate equation (1) has been used with $\kappa = 8\cdot10^{-6}\,\mathrm{m\,s^{-1}}$, $n = 29$ and $K_f = 900\,\mathrm{kN\,m^{-3/2}}$. The parameter c_f has been allowed to vary with varying crack propagation rate according to (5) with $c_{f0} = 14\,\mathrm{mm}$, $\dot{a}_0 = 0.01\,\mathrm{m\,s^{-1}}$ and $m = 17$, in order to get a shift of the size effect plot towards LEFM for slow loading rates (Fig. 2b). However, the experimentally observed shift of brittleness (Fig. 2a) seems to be too strong to be exactly reproduced by the present numerical model. A deeper experimental study of the physical nature of this interesting phenomenon is needed.

The effect of a sudden change of loading rate was experimentally studied by Bažant, Gu and Faber (submitted to ACI Mat. Journal, 1993). The initial CMOD rate was held constant until the load passed the peak value P_1 and dropped to some lower value P_c; then the rate was suddenly increased or decreased by several orders of magnitude. For a sufficiently large increase of the loading rate, the load started increasing again and a second peak P_2 could be observed (Fig. 2c). On the other hand, a decrease of the loading rate was followed by a fast drop of the load-CMOD curve. The rate-dependent R-curve model exhibits qualitatively the same behavior (Fig. 2e). The tests suggest that, after a rate change, the curve for the new rate asymptotically approaches the curve for a constant rate test with a rate equal to the new rate. The theory agrees with this behavior also (Fig. 2e).

It may be concluded that the R-curve concept, if generalized for the crack propagation rate effect and creep in the specimen bulk, can give a good description of the measurements of both the size effect and the rate effect in concrete (as well as limestone). It is a simple model of this complex phenomenon, which lends itself easily to practical applications.

4 Creep Analysis of Concrete as Aggregate-Matrix Composite

Prediction of the elastic constants of a particulate composite from the elastic constants of the aggregate and the matrix has been studied extensively [35–41], and various formulations for concrete in which the matrix exhibits aging linear creep have also been proposed [43–47]. However, a good, realistic prediction method that agrees with experimental results over a broad range of situations still does not exist.

There are two difficulties in the analysis: (1) How to model the interaction between the aggregate and the matrix in a composite, and (2) how to incorporate linear aging creep, which is in general defined by Volterra integral equations in time. Some attempts to describe creep of the matrix of the composite model by means of a first-order differential equation have been made [46], but such simplifications as well as the resulting predictions are not realistic. The use of the effective modulus also does not appear quite adequate.

There exists, however, another quasi-elastic method, the age-adjusted effective modulus method [see e.g. 32] which has been shown to give excellent predictions compared

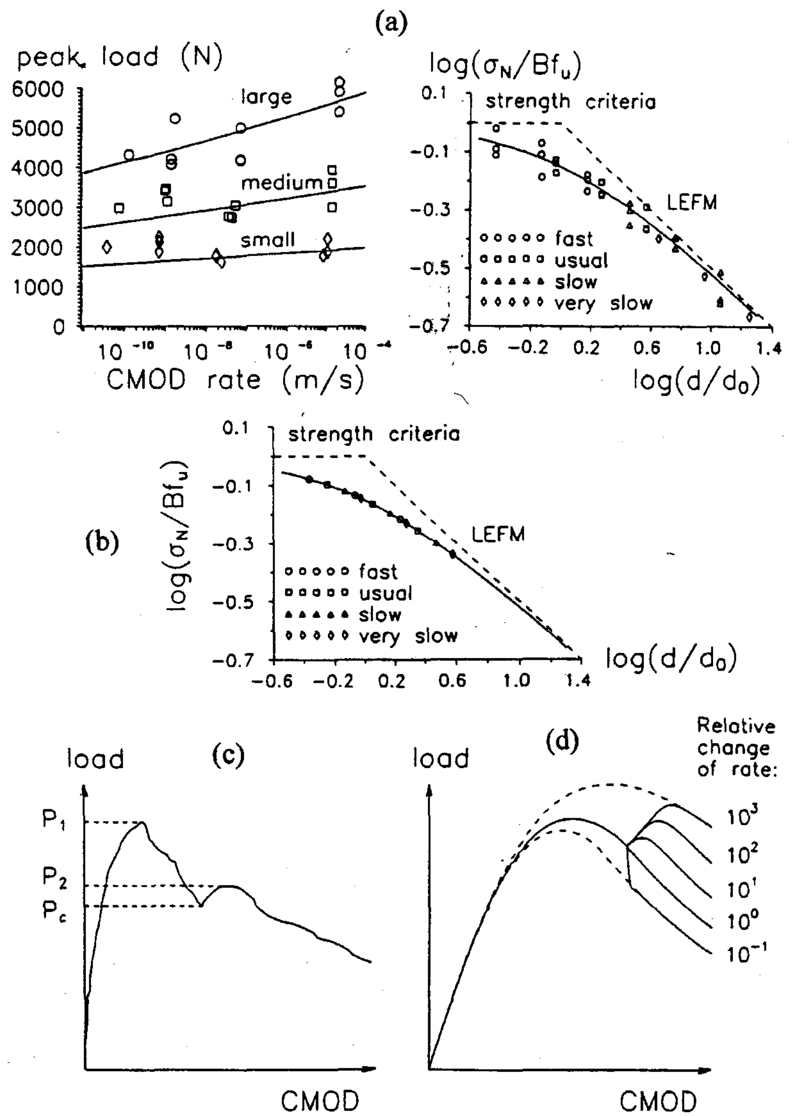

Fig. 2 Tests of Bažant and Gettu (1992) of the rate effect in notched concrete specimens of various sizes (a), size effect plot of normalized nominal strength σ_N vs. relative size d/d_0 measured by for different specimen sizes and different loading rates, (b) measurements of Bažant, Gu and Faber of the reversal of softening to hardening due to a sudden 1,000–fold increase of the loading rate (c), and simulation of the effects of rate increase or decrease by the R-curve model generalized for rate effect (d) (after Bažant and Jirásek, 1993) (d).

to the exact solutions of integral equations for various problems of structural analysis with steady state loading, and may be expected to perform with equal accuracy for the modeling of composites. Therefore it is proposed to use this method in a quasi-elastic composite analysis. In an ongoing study at Northwestern University conducted jointly with L. Granger, S. Baweja and P. Simeonov, various types of composite models, including the self-consistent model of Hill [39–40] and the Hashin-Shtrickman [36–37] variational bounds have been used, but so far it has transpired that these sophisticated models do not offer better predictions than a simple combination of parallel and linear coupling of the elements representing the aggregate and the matrix (mortar).

While the age-adjusted effective modulus method can be with all the preceding types of composite models, for the aforementioned reasons it may be best to use it in conjunction with the combined series-parallel model (Fig. 3). We assume that the aggregate represents the volume fraction V_a, and the mortar (matrix) the volume fraction $V_m = 1 - V_a$. First we calculate the compliance function for the parallel coupling of V_a and βV_m. Then we assume that the volume βV_m acts in parallel with V_a, and the rest of V_m, that is $(1-\beta)V_m$, acts in series with the parallel coupling of V_a and βV_m (Fig. 3).

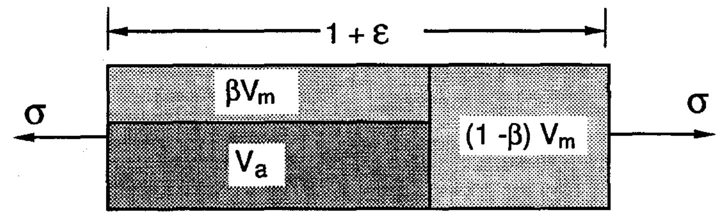

Fig. 3 Combined series–parallel model for creep interaction of aggregate and mortar matrix.

The age-adjusted effective modulus method requires writing two types of elastic relations: one for the initial instantaneous elastic loading at time t_0, and the second for the time increment from t_0 to any current time t. The creep of the mortar (matrix) is characterized by compliance $J_m(t,t_0)$ and elastic modulus $E_m = 1/J_m(t_0,t_0)$. The aggregate is characterized by elastic modulus E_a. For the case of parallel coupling one has the following compatibility and equilibrium relations for the strain and stress increments: $\Delta\epsilon_a = \Delta\epsilon_m = \Delta\epsilon$, $V_a\Delta\sigma_a + \beta(1-V_a)\Delta\sigma_m = \Delta\sigma$. We have $\Delta\sigma_a = E_a\Delta\epsilon$, and for the mortar (increments from t_0 to t), we have $\Delta\sigma_m = E''_m(\Delta\epsilon_m - \Delta\epsilon''_m)$ and $\Delta\epsilon''_m = (\sigma_0/E_m)\phi_m$; here $\sigma_0 = \sigma$–value at t_0 $\phi_m = E_m J_m - 1$ = creep coefficient of the mortar and $E''_m = (E_m - R_m)/\phi_m$, $\Delta\epsilon = \epsilon(t) - \epsilon(t_0)$, etc., and R_m = relaxation function of the mortar corresponding to the compliance function, which may be evaluated from the approximate but quite accurate one-line formula of Bažant and Kim [see e.g. 32]. Now the equilibrium relations for the composite according to the age-adjusted effective modulus method may be written as

$$\text{At } t: \quad V_a E_a \Delta\epsilon + \beta(1-V_a)E''_m\left(\Delta\epsilon - \frac{\sigma_{m0}}{E_m}\phi_m\right) = \Delta\sigma = 0 \tag{8}$$

$$\text{At } t_0: \quad V_a E_a \epsilon_0 + \beta(1-V_a)E_m\epsilon_0 = \sigma \tag{9}$$

where we imposed the condition that stress σ is constant from t_0 to t, and $\epsilon_0 = \sigma/E_p$. Subsequently we solve these equations for ϵ_0 and $\Delta\epsilon$, denote $E_p = V_a E_a + \beta(1 - V_a)E_m$, $E'' = V_a E_a + \beta(1 - V_a)E''_m$ and calculate $J(t, t_0) = (\epsilon_0 + \Delta\epsilon)/\sigma$. This yields for the parallel coupling of aggregate and mortar (shown in Fig. 3) the partial compliance function

$$J_p(t, t_0) = \frac{1}{E_p} + \frac{\beta(1 - V_a)E''_m \phi_m}{E'' E_m} \tag{10}$$

In the second step of calculation we assume that volume $(1 - \beta)V_m$ of the matrix, with β being an empirical constant, is coupled in series with the foregoing parallel coupling model of aggregate and mortar. This yields the result:

$$J(t, t_0) = \frac{1}{E} + \frac{\beta(1 - V_a)E''_m \phi_m}{E'' E_m} + (1 - \beta)J_m(t, t_0) \tag{11}$$

From prelimary results is seems that this expression for the compliance function of the composite can give very good approximations over a broad range of situations, although parameter β must be determined empirically. The question is whether this parameter can be fixed once for all as a constant or characterized as a certain simple function.

Studies of the use of Hashin-Shtrikman variational bounds are continuing, and it is expected that the conference presentation would include numerical results and evaluations.

5 Concluding Remarks

To sum up, in spite of great progress in the theory of creep and shrinkage of concrete during the last quarter century, significant problems remain before the theory can yield completely realistic predictions of many structural analysis and design problems. While deterministic linear aging viscoelastic analysis of structures seems to be in essence known and the opportunity for further significant advances seems very small, significant progress is needed in four areas: (1) Interaction of fracture with creep and rate effects stemming from fracture growth; (2) micromechanics of creep and shrinkage, expressed in quantitative terms and providing mathematical relations of macroscopic behavior to the phenomena on the microstructural level; (3) integration of diffusions models for moisture and heat transport into the analysis and design of structures for creep and shrinkage; and (4) introduction of statistical and probabilistic analysis into the long-time predictions of creep and shrinkage in structures, and updating of these predictions on the basis of extrapolation from short-time measurements or measurements on similar concretes. Some examples of recent developments at Northwestern University in the aforementioned research areas have been given, indicating how the fracture models can be generalized for their integration with creep structural analysis and incorporation in finite element programs for concrete creep, and how one micromechanics aspect, namely the interaction of aggregate and cement mortar matrix in concrete considered as a composite can be in a simple manner modeled mathematically. Much further research, however, will be needed in the aforementioned research areas in order to make the theory of concrete creep and shrinkage serve adequately the needs of design of concrete structures, especially the more daring or more efficient structures which are often super-sensitive to creep.

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