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# Nonlocal Integral Formulations of Plasticity and Damage: Survey of Progress

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**Abstract:** Modeling of the evolution of distributed damage such as microcracking, void formation, and softening frictional slip necessitates strain-softening constitutive models. The nonlocal continuum concept has emerged as an effective means for regularizing the boundary value problems with strain softening, capturing the size effects and avoiding spurious localization that gives rise to pathological mesh sensitivity in numerical computations. A great variety of nonlocal models have appeared during the last two decades. This paper reviews the progress in the nonlocal models of integral type, and discusses their physical justifications, advantages, and numerical applications.

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## Introduction

### *Historical Beginnings*

Most standard constitutive models for the mechanical behavior of solids used in engineering applications fall within the category of *simple nonpolar materials* (Noll 1972), for which the stress at a given point uniquely depends on the current values and possibly also the previous history of deformation and temperature *at that point only*. Deformation is in this context characterized by the deformation gradient or by an appropriate strain tensor, i.e., it is fully determined by the first gradient of the displacement field. Intuitively, it seems to be clear that the history of observable variables (strain and temperature) defines the "excitation" of the material point and that the corresponding "response" in terms of stress and entropy evolution should be a unique functional of the local excitation at that point. However, this intuitive feeling tacitly relies on the assumption that the material can be treated as a continuum at an arbitrarily small scale. Only then the finite body can be decomposed into a set of idealized, infinitesimal material volumes, each of which can be described independently as far as the constitutive behavior is concerned. Of course, this does not mean that the individual material points are completely isolated, but their interaction can take place only on the level of balance equations, through the exchange of mass, momentum, energy, and entropy.

In reality, however, no material is an ideal continuum. Both natural and man-made materials have a complicated internal structure, characterized by microstructural details whose size ranges over many orders of magnitude. Some of these details can be described explicitly by spatial variation of the material properties. But this can never be done simultaneously over the entire range of scales. One reason is that such a model would be prohibitively expensive for practical applications. Another, more fundamental reason is that on a small enough scale, the continuum description per se is no longer adequate and needs to be replaced by a discrete mass-point model (or, ultimately, by interatomic potentials based on quantum mechanics).

Constructing a material model, one must select a certain resolution level below which the microstructural details are not explicitly "visible" to the model and need to be taken into account approximately and indirectly, by an appropriate definition of "effective" material properties. Also, one should specify the characteristic wave length of the imposed deformation fields that can be expected for the given type of geometry and loading. Here, the term "wave length" applies not only to dynamics, where its meaning is clear, but also to statics, where it characterizes to the minimum size of the region into which the strain can localize.

If the characteristic wave length of the deformation field remains above the resolution level of the material model, a conventional continuum description can be adequate. On the other hand, if the deformation field is expected to have important components with wave lengths below the resolution level, the model needs to be enriched so as to capture the real processes more adequately. Instead of refining the explicit resolution level, it is often more effective to use various forms of generalized continuum formulations, dealing with materials that are nonsimple or polar, or both.

Some early attempts can be traced back to the 19th century (Voigt 1887, 1894), but the first effective formulation of this kind was proposed by Cosserat and Cosserat (1909). They considered

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material particles as objects having not only translational but also rotational degrees of freedom, described by the rotation of a rigid frame consisting of three mutually orthogonal unit vectors. A somewhat simpler concept was used by Oseen (1933) and Ericksen (1960) in their work on liquid crystals—they enriched the kinematic description by the rotation of a single vector, characterizing the orientation of the elongated axis of each crystal. After Günther (1958) had reopened the question of an *oriented continuum* and pointed out its relation to the theory of dislocations, the old idea of the Cosserat brothers inspired a rapid development leading to the couple-stress elasticity (Mindlin and Tiersten 1962; Toupin 1962, 1964; Koiter 1964), theory of elasticity with microstructure (Mindlin 1964), micropolar and micromorphic theories (Eringen and Suhubi 1964; Eringen 1964, 1966a,b), and multipolar theory (Green and Rivlin 1964a; Green 1965). Nonlinear extensions were proposed, e.g., by Lippmann (1969) and Besdo (1974).

All these generalized Cosserat theories characterize the motion of a solid body by additional fields that are independent of the displacement field and provide supplementary information on the small-scale kinematics. For example, the continuum with microstructure (Mindlin 1964) uses for this purpose a second-order tensor field that has the meaning of a "microscopic deformation gradient," in general different from the macroscopic deformation gradient evaluated from the displacement field. If the microscopic deformation gradient is restricted to orthogonal tensors, the Cosserat or micropolar continuum is recovered as a special case. On the other hand, a further generalization of the continuum with microstructure leads to the micromorphic continuum (Eringen 1966b). In fact, the continuum with microstructure can be identified with the so-called micromorphic continuum of grade 1 and degree 1.

The micropolar continuum model can be conceived as a continuum approximation of elastic lattices whose members possess a finite bending stiffness. For instance, an orthotropic micropolar model was developed for large regular elastic frames and applied to buckling of tall buildings (Bažant 1971; Bažant and Christensen 1972a,b).

Another important family of enriched continua retains the displacement field as the only independent kinematic field and improves the resolution by incorporating the gradients of strain (i.e., higher gradients of displacement) into the constitutive equations. Interest in such *higher-grade materials* or *gradient theories* was stimulated by Aero and Kuvshinskii (1960), Grioli (1960), Rajagopal (1960), and Truesdell and Toupin (1960). These pioneers took into account only those components of the strain gradient that correspond to curvatures, i.e., to gradients of rotations. This is equivalent to the Cosserat theory with constrained rotations, in which the rotations of the rigid frame associated with each material particle are not independent but are identified with the rotation tensor resulting from the polar decomposition of the macroscopic deformation gradient. Subsequently, the gradient theory was extended by including the effects of the stretch gradients (Toupin 1962), second strain gradients (Mindlin 1965), and gradients of all orders (Green and Rivlin 1964b). Krumhansl (1965) discussed the need for higher-order displacement gradients in continuum-based approximations of discrete lattices.

The last broad family of enriched continuum models is the family to be reviewed here. It consists of *nonlocal models of the integral type*. As early as 1893, Duhem noted that stress at a point should, in principle, depend on the state of the whole body. Nonlocal approaches were exploited in various branches of physical sciences, e.g., in optimization of slider bearings (Rayleigh 1918), or in modeling of liquid crystals (Oseen 1933), radiative transfer (Chandrasekhar 1950), and electric wave phenomena in the cortex (Hodgkin 1964). Rogula (1965) proposed a nonlocal form of the constitutive law for elastic materials.

Nonlocal elasticity was subsequently refined by Eringen (1966c), Kröner and Datta (1966), Kröner (1966, 1967), Kunin (1966a,b, 1968), Edelen (1969), Edelen and Laws (1971), Edelen (1971), Eringen et al. (1972), Eringen and Edelen (1972), and others. These early studies, frequently motivated by homogenization of the atomic theory of Bravais lattices, aimed at a better description of phenomena taking place in crystals on a scale comparable to the range of interatomic forces. They showed that nonlocal continuum models can approximate the dispersion of short elastic waves and improve the description of interactions between crystal defects such as vacancies, interstitial atoms, and dislocations.

During the last quarter of a century, it has become clear that neither distributed damage in materials nor transitions to discrete microstructural models can be adequately characterized by local constitutive relations between stress and strain tensors. A great variety of nonlocal models, involving either spatial integrals or gradients of strain or internal variables, have been developed. The present paper attempts to review the main existing models, classify them, and compare their properties. Before we focus our attention on models of the integral type, let us discuss nonlocality in a more general context.

## Strong and Weak Nonlocality

In solid mechanics, an integral-type nonlocal material model is a model in which the constitutive law at a point of a continuum involves weighted averages of a state variable (or of a thermodynamic force) over a certain neighborhood of that point. Clearly, nonlocality is tantamount to an abandonment of the principle of local action of the classical continuum mechanics. A gradient-type nonlocal model, while adhering to this principle mathematically, takes the field in the immediate vicinity of the point into account by enriching the local constitutive relations with the first or higher gradients of some state variables or thermodynamic forces. A salient characteristic of both the integral- and gradient-type nonlocal models is the presence of a characteristic length (or material length) in the constitutive relation.

The term "nonlocal" has in the past been used with two senses, one narrow and one broad. In the narrow sense, it refers strictly to the models with an averaging integral. In the broad sense, it refers to all the constitutive models that involve a characteristic length (material length), which also includes the gradient models. This broad sense stems from the realization that some gradient models are derived as approximations to the nonlocal averaging integrals, and that for all the gradient models the gradient, in fact, includes a dependence on the immediate (infinitely close) neighborhood of the point under consideration.

A mathematical definition of nonlocality has been given, e.g., by Rogula (1982). The fundamental equations of any physical theory can be written in the abstract form

$$Au = f \tag{1}$$

where $f$ = given excitation, $u$ = unknown response, and $A$ = corresponding operator (possibly nonlinear) characterizing the system. Typically, $u$ and $f$ are functions or distributions defined over a certain spatial domain $V$. Operator $A$ is called *local* if it has the following property:

1. If two functions $u$ and $v$ are identical in an open set $O$, then their images $Au$ and $Av$ are also identical in $O$.

Equivalently, one could say that whenever $u(\mathbf{x})=v(\mathbf{x})$ for all $\mathbf{x}$ in a neighborhood of point $\mathbf{x}_0$, then $Au(\mathbf{x}_0)=Av(\mathbf{x}_0)$. It is easily seen that differential operators satisfy this condition, because the derivatives of an arbitrary order do not change if the differentiated function changes only outside a small neighborhood of the point at which the derivatives are taken. For example, standard one-dimensional elasticity is described by the ordinary differential equation

$$-[E(x)u'(x)]'=f(x) \tag{2}$$

where $E$ = modulus of elasticity, $u$ = displacement, $f$ = body force, and the prime denotes differentiation with respect to the spatial variable $x$. It is easily verified that the locality condition is satisfied. Equation (2) combines the strain-displacement equation, $\epsilon(x)=u'(x)$, equilibrium equation, $\sigma'(x)+f(x)=0$, and the (local) elastic constitutive equation, $\sigma(x)=E(x)\epsilon(x)$, where $\epsilon$ is the strain and $\sigma$ is the stress. In nonlocal elasticity, the constitutive equation has the form

$$\sigma(x)=\int_{-\infty}^{\infty}E(x,\xi)\epsilon(\xi)\;\mathrm{d}\xi \tag{3}$$

where $E(x,\xi)$ = kernel of the elastic integral operator, generalizing the notion of the elastic modulus. The corresponding generalization of Eq. (2) then reads

$$-\left[\int_{-\infty}^{\infty}E(x,\xi)u'(\xi)\;\mathrm{d}\xi\right]'=f(x) \tag{4}$$

Due to the presence of a spatial integral, the locality condition is violated [unless the elastic kernel has the degenerate form $E(x,\xi)=E(x)\delta(x-\xi)$, where $\delta$ is the Dirac distribution, in which case the local elasticity is recovered].

According to the foregoing definition, one could say that the local theories are those described by differential equations and nonlocal theories are those described by integrodifferential equations. But this refers to nonlocality in the narrow sense. There is another important aspect, related to the presence or absence of a characteristic length. From the mathematical point of view, the absence of a characteristic length is manifested by the invariance of the fundamental equations with respect to scaling of the spatial coordinates [for a precise definition, see Rogula (1982)]. For example, in standard linear elasticity, fundamental Eq. (2) remains valid if $x$ is replaced by $\tilde{x}=sx$, $u$ is replaced by $\tilde{u}=su$, and $f$ is replaced by $\tilde{f}=f/s$, where $s$ is a positive scaling parameter (the prime is then interpreted as the derivative with respect to $\tilde{x}$). This indicates that the theory does not possess any characteristic length. A local theory invariant with respect to spatial scaling is called *strictly local*, while a local theory not invariant with respect to spatial scaling is called *weakly nonlocal*. Weakly nonlocal theories are typically described by differential equations that contain derivatives of different orders. The coefficients multiplying the terms of different orders have different physical dimensions, and from their ratios it is possible to deduce a characteristic length.

Typical examples of weakly nonlocal theories are the Navier–Bernoulli beam on an elastic (Winkler) foundation, or a Timoshenko beam. In the former case, the characteristic length is proportional to the flexural wave length (of spatial oscillations produced by a concentrated force), which is itself proportional to $\sqrt[4]{EI/c}$ where $EI$ = bending cross-sectional stiffness and $c$ = foundation modulus (elastic constant of the foundation).

A Timoshenko beam can be considered as a specific one-dimensional version of a Cosserat continuum. The characteristic length is dictated by the square root of the ratio between the bending stiffness and the shear stiffness of the cross section. For a fixed shape of the cross section, the characteristic length is proportional to the beam depth. Note that, in this one-dimensional description, one ''point'' of the generalized continuum corresponds to a cross section of the beam. Therefore, only the span, but not the depth, of the beam is an actual geometric dimension of the model in the physical space. The beam depth is a part of the generalized material model, represented by the moment-curvature relation and by the relation between the shear force and the shear distortion. The presence of a characteristic length means that the solutions for different spans cannot be obtained by simple scaling of a reference solution for a given span. Such a scaling would be possible only if the beam depth was also scaled, but this corresponds to a change of the ''material.''

The foregoing example clearly shows that the solution of a problem can be governed by the ratio of the physical dimensions of a structure to an intrinsic material length. In the present case, this material length arises from the dimensional reduction and has its origin in the geometrical dimension that is no longer explicitly resolved by the model. In analogy to that, the material lengths that are present in various forms of generalized continuum theories arise from the homogenization procedure and have their origin in the characteristics of the heterogeneous microstructure that are no longer explicitly resolved.

To summarize the suggested classification, continuum models for the mechanical behavior of solids (same as other continuum theories) can be divided into

1. strictly local models, which encompass nonpolar simple materials;
2. weakly nonlocal models, exemplified by polar theories and gradient theories (higher-grade materials); and
3. strongly nonlocal models, such as models of the integral type.

It is worth noting that the recently emerged implicit gradient models (Peerlings et al. 1996; Geers et al. 2001; Engelen et al. 2002) are classified as strongly nonlocal, because they are equivalent to integral-type models with special weight functions used for weighted averaging. In the present survey, we restrict our attention exclusively to strongly nonlocal models of the integral type.

## Integral-Type Nonlocal Models

### *Nonlocal Elasticity*

The theories of nonlocal elasticity advanced by Eringen and Edelen in the early 1970s (Edelen et al. 1971; Eringen 1972; Eringen and Edelen 1972) attributed a nonlocal character to many fields, e.g., to the body forces, mass, entropy, or internal energy. They were too complicated to be calibrated and experimentally verified, let alone to be applied to any real problems. Later simplifications finally led to a practical formulation in which only the stress–strain relations are treated as nonlocal, while the equilibrium and kinematic equations and the corresponding boundary conditions retain their standard form (Eringen and Kim 1974; Eringen et al. 1977). The related variational principles have recently been developed by Polizzotto (2001). This formulation will now be briefly presented, in order to introduce the basic concepts and prepare a basis for extensions to the inelastic behavior, especially to nonlocal plasticity.

A linear, small-strain, nonlocal theory of elasticity can be derived from the assumption that the elastic energy of a body $V$ is given by the quadratic functional

$$W=\frac{1}{2}\int_V\int_V \boldsymbol{\epsilon}^T(\mathbf{x})\mathbf{D}_e(\mathbf{x},\boldsymbol{\xi})\boldsymbol{\epsilon}(\boldsymbol{\xi})\mathrm{d}\mathbf{x}\mathrm{d}\boldsymbol{\xi} \tag{5}$$

where $\boldsymbol{\epsilon}(\mathbf{x})$ = strain field and $\mathbf{D}_e(\mathbf{x},\boldsymbol{\xi})$ = generalized form of the elastic stiffness. The difference from the standard local theory consists in the fact that, in general, it is impossible to express the global energy as a spatial integral of an energy density that would depend only on the local value of strain. Only if $\mathbf{D}_e(\mathbf{x},\boldsymbol{\xi}) = \mathbf{D}_e(\mathbf{x})\delta(\mathbf{x}-\boldsymbol{\xi})$ Eq. (5) reduces to

$$W=\frac{1}{2}\int_V \boldsymbol{\epsilon}^T(\mathbf{x})\mathbf{D}_e(\mathbf{x})\boldsymbol{\epsilon}(\mathbf{x})\mathrm{d}\mathbf{x}=\int_V w[\boldsymbol{\epsilon}(\mathbf{x}),\mathbf{x}]\mathrm{d}\mathbf{x} \tag{6}$$

where $w(\boldsymbol{\epsilon},\mathbf{x})=\frac{1}{2}\boldsymbol{\epsilon}^T\mathbf{D}_e(\mathbf{x})\boldsymbol{\epsilon}$ is the elastic energy density. Physically, the generalized energy expression (5) is needed if the body cannot be decomposed into infinitely small cells that interact only through tractions on their boundaries. This is the case, for instance, in the presence of long-range interactions among material particles, such as atoms or molecules.

Differentiating Eq. (5) with respect to time, we obtain the rate of change of elastic energy

$$\dot{W}=\frac{1}{2}\int_V\int_V \dot{\boldsymbol{\epsilon}}^T(\mathbf{x})\mathbf{D}_e(\mathbf{x},\boldsymbol{\xi})\boldsymbol{\epsilon}(\boldsymbol{\xi})\mathrm{d}\mathbf{x}\mathrm{d}\boldsymbol{\xi} + \frac{1}{2}\int_V\int_V \boldsymbol{\epsilon}^T(\mathbf{x})\mathbf{D}_e(\mathbf{x},\boldsymbol{\xi})\dot{\boldsymbol{\epsilon}}(\boldsymbol{\xi})\mathrm{d}\mathbf{x}\mathrm{d}\boldsymbol{\xi}=\frac{1}{2}\int_V\int_V \boldsymbol{\epsilon}^T(\mathbf{x})[\mathbf{D}_e(\mathbf{x},\boldsymbol{\xi}) + \mathbf{D}_e^T(\boldsymbol{\xi},\mathbf{x})]\dot{\boldsymbol{\epsilon}}(\boldsymbol{\xi})\mathrm{d}\mathbf{x}\mathrm{d}\boldsymbol{\xi}=\int_V\int_V \dot{\boldsymbol{\epsilon}}^T(\mathbf{x})\mathbf{D}_e^{\mathrm{sym}}(\mathbf{x},\boldsymbol{\xi})\boldsymbol{\epsilon}(\boldsymbol{\xi})\mathrm{d}\mathbf{x}\mathrm{d}\boldsymbol{\xi} \tag{7}$$

where

$$\mathbf{D}_e^{\mathrm{sym}}(\mathbf{x},\boldsymbol{\xi})=\frac{1}{2}[\mathbf{D}_e(\mathbf{x},\boldsymbol{\xi})+\mathbf{D}_e^T(\boldsymbol{\xi},\mathbf{x})] \tag{8}$$

is the symmetric part of the generalized stiffness. Note that the symmetrization is carried out simultaneously with respect to the components of the matrix $\mathbf{D}_e$ as well as to the arguments $\mathbf{x}$ and $\boldsymbol{\xi}$. We will assume that the generalized stiffness is right away defined such that it satisfies the symmetry conditions $\mathbf{D}_e=\mathbf{D}_e^T$ and $\mathbf{D}_e(\mathbf{x},\boldsymbol{\xi})=\mathbf{D}_e(\boldsymbol{\xi},\mathbf{x})$, and we will drop the superscript "sym." The last expression in (7) can be written in the form

$$\dot{W}=\int_V \dot{\boldsymbol{\epsilon}}^T(\mathbf{x})\boldsymbol{\sigma}(\mathbf{x})\mathrm{d}\mathbf{x} \tag{9}$$

where

$$\boldsymbol{\sigma}(\mathbf{x})=\int_V \mathbf{D}_e(\mathbf{x},\boldsymbol{\xi})\boldsymbol{\epsilon}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} \tag{10}$$

Eq. (9) is the standard expression for the internal power delivered by stress $\boldsymbol{\sigma}$ at strain rate $\dot{\boldsymbol{\epsilon}}$. Consequently, $\boldsymbol{\sigma}$ is identified as the stress, and Eq. (10) is the constitutive equation of nonlocal elasticity. Since the internal power expression has the standard form, the principle of virtual power leads to exactly the same equilibrium equations and traction boundary conditions as it does in standard (local) elasticity.

For simplicity, we will restrict our attention to macroscopically homogeneous bodies. It is reasonable to assume that the interaction effects decay with distance between the two points $\mathbf{x}$ and $\boldsymbol{\xi}$ (this is sometimes called the "attenuating neighborhood hypothesis") and, unless there is experimental evidence to the contrary, that all stiffness coefficients decay in the same manner. This motivates the commonly assumed form of the generalized stiffness

$$\mathbf{D}_e(\mathbf{x},\boldsymbol{\xi})=\mathbf{D}_e\alpha(\mathbf{x},\boldsymbol{\xi}) \tag{11}$$

where $\alpha$ = certain attenuation function. The stress–strain law then reads

$$\boldsymbol{\sigma}(\mathbf{x})=\int_V \mathbf{D}_e\alpha(\mathbf{x},\boldsymbol{\xi})\boldsymbol{\epsilon}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}=\mathbf{D}_e\int_V \alpha(\mathbf{x},\boldsymbol{\xi})\boldsymbol{\epsilon}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}=\mathbf{D}_e\overline{\boldsymbol{\epsilon}}(\mathbf{x}) \tag{12}$$

where

$$\overline{\boldsymbol{\epsilon}}(\mathbf{x})=\int_V \alpha(\mathbf{x},\boldsymbol{\xi})\boldsymbol{\epsilon}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi} \tag{13}$$

is the nonlocal strain.

In an infinite isotropic body, the attenuation function depends only on the distance between points $\mathbf{x}$ and $\boldsymbol{\xi}$, and we can write

$$\alpha(\mathbf{x},\boldsymbol{\xi})=\alpha_\infty(\|\mathbf{x}-\boldsymbol{\xi}\|) \tag{14}$$

To remove ambiguity, $\alpha_\infty$ is scaled so as to satisfy the normalizing condition

$$\int_{V_\infty}\alpha_\infty(\|\boldsymbol{\xi}\|)\mathrm{d}\boldsymbol{\xi}=1 \tag{15}$$

where $V_\infty$ = entire (one-, two-, or three-dimensional) Euclidean space in which the problem is formulated. Consequently a uniform "local" strain field $\boldsymbol{\epsilon}(\mathbf{x})=\boldsymbol{\epsilon}_0$ is transformed into a uniform nonlocal strain field

$$\overline{\boldsymbol{\epsilon}}(\mathbf{x})=\int_{V_\infty}\alpha(\mathbf{x},\boldsymbol{\xi})\boldsymbol{\epsilon}(\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}=\boldsymbol{\epsilon}_0\int_{V_\infty}\alpha(\mathbf{x},\boldsymbol{\xi})\mathrm{d}\boldsymbol{\xi}=\boldsymbol{\epsilon}_0 \tag{16}$$

and the corresponding stress field is given by $\boldsymbol{\sigma}(\mathbf{x})=\mathbf{D}_e\boldsymbol{\epsilon}(\mathbf{x}) = \mathbf{D}_e\boldsymbol{\epsilon}_0=\boldsymbol{\sigma}_0$. This gives to $\mathbf{D}_e$ the physical meaning of elastic stiffness under uniform straining.

The attenuation function, also called the nonlocal weight function or the nonlocal averaging function, is often assumed to have the form of the Gauss distribution function

$$\alpha_\infty(r)=(\ell\sqrt{2\pi})^{-N_{\mathrm{dim}}}\exp\left(-\frac{r^2}{2\ell^2}\right) \tag{17}$$

where $\ell$ = parameter with the dimension of length and $N_{\mathrm{dim}}$ = number of spatial dimensions. Function (17) has an infinite support, which means that nonlocal interaction takes place between any two points, no matter how far from each other they are. For reasons of computational efficiency, it is more advantageous to use attenuation functions with a finite support, e.g., the polynomial bell-shaped function (Bažant and Ožbolt 1990)

$$\alpha_\infty(r)=c\left\langle 1-\frac{r^2}{R^2}\right\rangle^2 \tag{18}$$

where the Macauley brackets $\langle\cdots\rangle$ denote the positive part, defined as $\langle x\rangle=\max(0,x)$. Definition (18) contains, again, a parameter with the dimension of length $R$, which in this case plays the role of the interaction radius, because $\alpha_\infty(r)$ vanishes for $r\geq R$. The scaling factor $c$ is determined from condition (15) and is equal to $15/(16R)$ in one dimension, $3/(\pi R^2)$ in two dimensions, and $105/(32\pi R^3)$ in three dimensions.

There is no unique way of defining the exact form of the attenuation function in a finite body. In nonlocal elasticity, it is usually assumed that $\alpha(\mathbf{x},\boldsymbol{\xi})$ is still given by Eq. (14), regardless of the presence of boundaries. The integral $\int_V\alpha(\mathbf{x},\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi}$ taken over the finite body $V$ is then smaller than 1 for $\mathbf{x}$ close to the boundary $\partial V$, and uniform straining of the finite body does not generate a uniform stress. For materials with long-range elastic forces, this phenomenon has a clear physical explanation. For instance, consider a regular atomic lattice under constant ''strain,'' i.e., under a uniform relative increase of the interatomic distances. For atoms that are sufficiently far from the boundary, the forces generated by the displacement of the surrounding atoms within the interaction distance cancel out due to symmetry. However, if a part of the neighborhood is cut off by the boundary, some of these forces disappear and equilibrium gets disturbed. In a local continuum, it is sufficient to replace the effect of the missing material by tractions on the boundary. However, in a lattice with long-range interactions, there is a boundary layer of thickness $R$ that ''feels'' the absence of the material behind the boundary. To restore equilibrium, it is not sufficient to apply traction on the boundary, but additional forces are needed in the entire boundary layer.

### *Motivations of Nonlocality*

The aforementioned initial advances were motivated by deviations from the local constitutive models at small scales, caused solely by microstructural *heterogeneity* on the scale of the characteristic length. Recently, a sophisticated explanation of the need for nonlocal terms in homogenized elastic models of random composites has been given by Drugan and Willis (1996) and Luciano and Willis (2001).

An entirely different motivation of nonlocality—the strain-softening character of distributed damage—came to light during the 1970s. It happened as a result of entering the computer era. Finite-element programs made it suddenly feasible to simulate the distributed cracking observed in failure tests of concrete structures. The need to develop concrete vessels and containments for nuclear reactors led to lavish research funding. To approximate the distributed cracking by a continuum, damage models with *strain softening* had to be introduced into finite-element codes. The first among these models was probably the smeared cracking model of Rashid (1968).

The concept of strain softening violated the basic tenets of continuum mechanics as understood at that time, particularly the conditions of stability of material (Drucker's stability) and well posedness of the boundary value problem. Many theoreticians took the firm position that the concept of strain softening in any form was unsound and dismissed its proponents contemptuously as diletants. The controversy, amusing in retrospect but deadly serious at that time, created passionate polemics at conferences, arguments with reviewers, and fights for money at funding agencies (Bažant 2002a). A proposal for a model with strain softening was sure to be rejected if sent for review to these dogmatic theoreticians. But, eventually, it all had a positive effect—a compelling motivation for nonlocal models of a new kind.

The finite-element simulations of failures with distributed (smeared) cracking demonstrated, and the analysis of stability and bifurcation confirmed, that a local inelastic constitutive law with strain-softening damage inevitably leads to spurious localization of damage into a zone of zero volume (Bažant 1976). This causes the numerical solution to become unobjective with respect to the choice of mesh and, upon the mesh refinement, to converge to a solution with a vanishing energy dissipation during structural failure.

Such physically absurd computational results were linked to two problematic features, pointed out already by Hadamard (1903), discussed by Thomas (1961), and emphasized by Sandler (1984), Read and Hegemier (1984), and others. For a dynamic problem in one spatial dimension, they can be described as follows:

1. A material whose tangential stiffness becomes negative has an imaginary wave speed, and thus cannot propagate waves. (Today we know that strain-softening concrete, in fact, can propagate unloading waves and, due to a rate effect on crack growth, also loading waves of a sufficiently steep front.)
2. The dynamic initial-boundary-value problem then changes its type from hyperbolic to elliptic and becomes ill posed, which means that an infinitely small change in the initial conditions can lead to a finite change in the dynamic solution. This was analytically documented for wave propagation in a strain-softening bar by Bažant and Belytschko (1985); see also Bažant and Cedolin (1991), Sec. 13.1.

In multiple spatial dimensions, some wave speeds can remain real even when the tangential stiffness tensor ceases being positive definite, which means that stress waves can still propagate but not in an arbitrary direction. The initial-boundary-value problem is, again, ill posed even though, in general, it does not become elliptic.

Introduction of a characteristic length into the constitutive model (Bažant 1976; Bažant and Cedolin 1979; Cedolin and Bažant 1980; Pietruszczak and Mróz 1981; Bažant and Oh 1983), and formulation of a nonlocal strain-softening model (Bažant et al. 1984) and its second-gradient approximation (Bažant 1984a), were then shown to prevent the spurious localization of strain-softening damage (i.e., to serve as a localization limiter), to regularize the boundary value problem (i.e., make it well posed), and to ensure numerical convergence to physically meaningful solutions. In relation to nonlinear fracture mechanics, the characteristic length in quasibrittle materials with distributed cracking may be physically interpreted as (or related to) the effective size of the fracture process zone at the tip of a macroscopic crack. With respect to homogenization theory, the characteristic length may be taken as equal to (or related to) the size of the representative volume of the material.

The third, practically most compelling, motivation of nonlocality was the *size effect* (in the present context understood as the dependence of the nominal strength on the structure size). The existence of a nonstatistical size effect was brought to light by fracture experiments on concrete (Walsh 1972, 1976; Bažant and Pfeiffer 1987; Bažant and Planas 1998; Bažant 2002b) and discrete numerical simulations using, e.g., the random particle and lattice models (Bažant et al. 1990; Schlangen and van Mier 1992; Schlangen 1993; Jirásek and Bažant 1995; van Mier 1997). In the absence of a characteristic length, the size effect must have the form of a power law. This is the case, for example, in linear elastic fracture mechanics. The incorporation of a characteristic length is needed to describe a transitional type of size effect, in which the scaling according to one power law at scales much smaller than the characteristic length transits to scaling according to another power law at scales much larger than the characteristic length (Bažant 1984b; Bažant and Planas 1998; Bažant 2002b).

Clearly pronounced size effects were also observed in tests of metals on the millimeter and micrometer scales. The results of experiments with bending of thin beams (metallic films) (Richards 1958; Stolken and Evans 1998), torsion of thin wires (Morrison 1939; Fleck et al. 1994), and microindentation (Nix 1989; Ma and Clarke 1995; Poole et al. 1996) cannot be described by the standard plasticity theory that lacks a characteristic length. Even in the elastic range, size effects on the torsional and bending

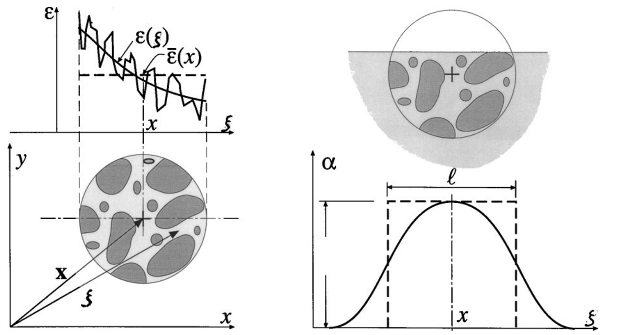

**Fig. 1.** Left: representative volume of material used for nonlocal strain averaging; the scatter of microstresses and smoothed macrostress profile demonstrate how the stress value at the center of representative volume differs from the stress corresponding to average strain over this volume. Top right: averaging zone when the theoretical nonlocal neighborhood of a point protrudes through the boundary of solid. Bottom right: bell-shaped weight function for nonlocal averaging integral and its relation to characteristic length $\ell$.

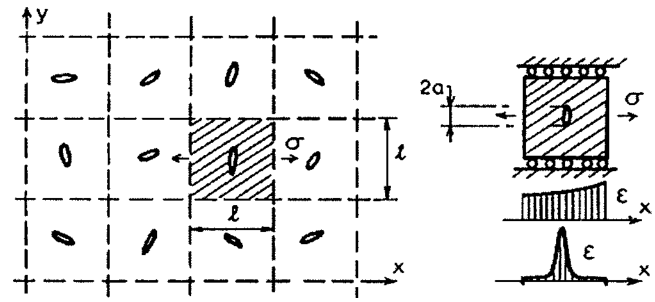

**Fig. 2.** Idealized crack array and energy release zone of one crack, analyzed in Bažant (1991)

stiffness contradicting the standard elasticity theory were measured for porous materials such as foams and human bones (Lakes 1986).

To sum up, the development of nonlocal models was historically motivated by

1. the need to capture small-scale deviations from local continuum models caused by material heterogeneity;
2. the need to achieve objective and properly convergent numerical solutions for localized damage;
3. the need to regularize the boundary value problem (prevent ill posedness); and
4. the need to capture size effects observed in experiments and in discrete simulations.

### Causes of Nonlocality

The physical causes of nonlocality, some of which have already been mentioned among the historical motivations, may be summarized as follows:

1. Heterogeneity of microstructure and its homogenization on a small scale on which the smoothed strain field cannot be considered as uniform. What matters for the macroscopic stress (averaged over the representative volume) is not the strain value at the center point but the average strain value within the representative volume, which can be very different (Fig. 1, left). The heterogeneity may, for instance, be caused by, and the characteristic length governed by, the grain (inclusion) size, the pore size, or the size of crystals in a metal. In concrete, what is decisive is the maximum aggregate size and spacing.
2. Homogenization of regular or statistically regular lattices or frames, for example, those to be used for large planned space structures or for very tall buildings (this may, of course, be regarded as a special case of heterogeneity).
3. The fact that distributed cracking is physically observed yet impossible to simulate numerically with local continuum models, as already elaborated on.
4. The fact that the growth of a microcrack is not decided by the local stress or strain tensor at the continuum point corresponding to the microcrack center but by the overall energy release from a finite volume surrounding the whole microcrack (Fig. 2). The growth depends on the average deformation of that volume (Bažant 1987, 1991). The size of this volume is determined by the size of the microcracks and their average spacing, which may, but need not, be related to the size of the inhomogeneities in the material.
5. Microcrack interaction, particularly the fact that one microcrack may either amplify the stress intensity factor of another adjacent microcrack or shield that crack, depending on the orientations of the microcracks, the orientation of the vector joining their centers, and the microcrack sizes. This leads to a different kind of nonlocality that is not described explicitly by a spatial averaging integral but implicitly by an integral equation with a kernel of zero average (Bažant 1994; Bažant and Jirásek 1994; Jirásek and Bažant 1994).
6. In the case of metal plasticity, the density of geometrically necessary dislocations in metals, whose effect, after continuum smoothing, naturally leads to a first-gradient model (Fleck and Hutchinson 1993; Gao et al. 1999; Huang et al. 2000; Fleck and Hutchinson 2001).
7. Weibull-type extreme value statistics of quasibrittle failure. As recently realized, without assuming the failure probability at a point of the material to depend on the average strain from a finite neighborhood of the point rather on the continuum stress at that point, a Weibull-type weakest link theory of quasibrittle structural failure runs into some paradoxical situations or incorrect predictions (Bažant and Xi 1991; Bažant and Novák 2000a,b,c), which cannot be avoided without resorting to a nonlocal probabilistic model (see the section on nonlocal probabilistic models of failure).

### Nonlocal Averaging Operator

Generally speaking, the nonlocal integral approach consists in replacing a certain variable by its nonlocal counterpart obtained by weighted averaging over a spatial neighborhood of each point under consideration. If $f(\mathbf{x})$ is some "local" field in a solid body occupying a domain $V$, the corresponding nonlocal field, labeled by an overbar, is defined by

$$\overline{f}(\mathbf{x})=\int_V \alpha(\mathbf{x},\boldsymbol{\xi})f(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi} \tag{19}$$

where $\alpha(\mathbf{x},\boldsymbol{\xi})=$ chosen nonlocal weight function. In applications to softening materials, it is often required that the nonlocal operator should not alter a uniform field, which means that the weight function must satisfy the normalizing condition

$$\int_V \alpha(\mathbf{x},\boldsymbol{\xi})\quad \mathrm{d}\boldsymbol{\xi}=1 \qquad \forall \mathbf{x}\in V \tag{20}$$

In an infinite, isotropic, and homogeneous medium, the weight function depends only on the distance $r=\|\mathbf{x}-\boldsymbol{\xi}\|$ between the "source" point, $\boldsymbol{\xi}$, and the "receiver" point, $\mathbf{x}$. So, we may write $\alpha(\mathbf{x},\boldsymbol{\xi})=\alpha_\infty(\|\mathbf{x}-\boldsymbol{\xi}\|)$ where $\alpha_\infty(r)$ is typically chosen as a nonnegative bell-shaped function (17) or (18), monotonically decreasing for $r\geqslant 0$ (Fig. 1, bottom right). The smallest distance between points $\mathbf{x}$ and $\boldsymbol{\xi}$ at which the interaction weight $\alpha_\infty(\|\mathbf{x}-\boldsymbol{\xi}\|)$ vanishes (for weight functions with a bounded support) or becomes negligible (for weight functions with an unbounded support) is called the nonlocal interaction radius $R$. The interval, circle, or sphere of radius $R$, centered at $\mathbf{x}$, is called the domain of influence of point $\mathbf{x}$.

In the vicinity of the boundary of a finite body, it is simply assumed (without any deep theoretical support) that the averaging is performed only on the part of the domain of influence that lies within the solid (Fig. 1, top right). To satisfy condition (20), the weight function is usually defined as

$$\alpha(\mathbf{x},\boldsymbol{\xi})=\frac{\alpha_\infty(\|\mathbf{x}-\boldsymbol{\xi}\|)}{\int_V \alpha_\infty(\|\mathbf{x}-\boldsymbol{\zeta}\|)\mathrm{d}\boldsymbol{\zeta}} \tag{21}$$

However, this modification breaks the symmetry of the weight function with respect to the arguments $\mathbf{x}$ and $\boldsymbol{\xi}$. In certain types of nonlocal theories it is desirable to work with a symmetric weight function. Polizzotto (2002) and Borino et al. (2002) proposed another modified weight function,

$$\alpha(\mathbf{x},\boldsymbol{\xi})=\alpha_\infty(\|\mathbf{x}-\boldsymbol{\xi}\|)+\left[1-\int_V \alpha_\infty(\|\mathbf{x}-\boldsymbol{\zeta}\|)\mathrm{d}\boldsymbol{\zeta}\right]\delta(\mathbf{x}-\boldsymbol{\xi}) \tag{22}$$

which preserves symmetry and satisfies condition (20).

In computer programs, the nonlocal average at $\mathbf{x}$ is calculated as a weighted sum over the values at all the finite-element integration points $\boldsymbol{\xi}$ lying within the nonlocal interaction radius $R$. One inevitable penalty of nonlocal averaging is that the bandwidth of the stiffness matrix gets increased. This increases the relative attractiveness of explicit finite-element schemes, which do not necessitate the assembly and decomposition of the stiffness matrix. For nonlocal damage models, it is nevertheless possible to construct the consistent tangent stiffness matrix (Huerta and Pijaudier-Cabot 1994) and use it efficiently in implicit finite-element computations (Jirásek and Patzák 2002).

### *Original Nonlocal Model for Strain Softening and Its Limitations*

In the 1980s, nonlocal models were extended to inelastic materials. Eringen developed his nonlocal formulation of isotropically hardening plasticity in strain space (Eringen 1981), perfect plasticity with associated flow, and deformation theory of plasticity (Eringen 1983). Subsequently, Eringen and Ari (1983) applied these models to simulation of the yielding zone at the fracture front.

The nonlocal averaging concept was also applied to a strain-softening damage model by Bažant et al. (1984) in order to regularize the boundary value problem and prevent spurious dependence of the process zone width and of the energy dissipation on mesh refinement. Initially, the averaging operator was applied to the total strain tensor $\boldsymbol{\epsilon}$. With a uniform weight function, this model could be easily implemented in a finite-element code by imbricating (i.e., overlapping) the finite elements in the manner of roof tiles ("imbrex" in Latin), as shown in Fig. 3. With the condition that the element size be kept constant regardless of mesh refinement, the elements themselves performed the strain averaging. The characteristic length $\ell$ was then proportional to the fixed element size.

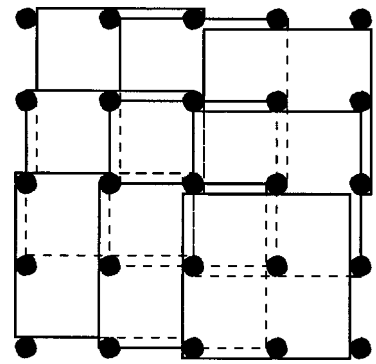

**Fig. 3.** Imbrication (overlapping) of finite elements, approximately equivalent to averaging of total strain with uniform weight function (Bažant et al. 1984)

However, this "imbricate" concept caused unwanted problems. A nonlocal continuum in which the *total* strain is nonlocal can exhibit zero energy modes of instability (it appears that similar problems could have afflicted also various previous nonlocal elastic models and went undetected, apparently for lack of finite-element implementation). That such instabilities exist becomes clear upon noting that the one-dimensional integral equation $\overline{\epsilon(x)}\equiv\int_{x-R}^{x+R}\epsilon(s)\ \mathrm{d}s=0$ is satisfied by any function

$$\epsilon(s)=A\sin\frac{\pi n(s+c)}{R} \tag{23}$$

where $n=$arbitrary positive integer, and $A$ and $c=$arbitrary real constants. In general, it was shown that such instabilities are avoided if and only if the Fourier transform of the weight function $\alpha_\infty(r)$ is everywhere positive (Bažant and Chang 1984). This is not the case for a uniform weight function (and thus not for the "imbricate" continuum model). Neither is this the case for a triangular weight function or a truncated polynomial function (18).

The Gaussian function (17) has a positive Fourier transform. But, since the tail of its Fourier transform approaches zero, the situation is close to instability. A robust remedy is to add a multiple of the Dirac distribution to the weight function, which is equivalent to an "overlay" (parallel coupling) with an elastic local continuum. But, then, full strain softening down to a zero stress becomes impossible.

So, this original nonlocal model for regularizing strain-softening problems was found usable only for a material that suffers merely partial softening damage followed by rehardening, but not in general situations. Therefore, other remedies were sought, as described next.

## Nonlocal Damage and Smeared Cracking

### *Nonlocal Damage Models*

Understanding the source of the aforementioned problems suggested the remedy—the instability modes obviously cannot arise if the nonlocal averaging is applied to variables that can never decrease. Such variables, for example, include the damage variable $\Omega$ or the maximum level of damage energy release rate $Y_{\max}$ in continuum damage mechanics, or the cumulative plastic strain $\kappa$ in strain-softening plasticity.

This idea proved successful. It was first applied to continuum damage mechanics, exemplified by the simple isotropic damage model with one scalar damage variable (e.g., Lemaitre and Chaboche 1990). In its local form, this model can be described by the following set of equations:

$$\boldsymbol{\sigma}=(1-\Omega)\mathbf{D}_e\boldsymbol{\epsilon} \tag{24}$$

$$\Omega=\omega(Y_{\max}) \tag{25}$$

$$Y_{\max}(t)=\max_{\tau\leq t} Y(\tau) \tag{26}$$

$$Y=\tfrac{1}{2}\boldsymbol{\epsilon}^T\mathbf{D}_e\boldsymbol{\epsilon} \tag{27}$$

In the above, $\boldsymbol{\sigma}$ = column matrix of six stress components, $\boldsymbol{\epsilon}$ = column matrix of six engineering strain components, $\mathbf{D}_e$ = elastic material stiffness matrix, and $\Omega$ = damage variable that grows from zero (virgin state) to one (fully damaged state) depending on $Y_{\max}$, which is the maximum value of the damage energy release rate $Y$ ever attained in the previous history of the material up to the current state. Of course, $Y$ is not a rate in the sense of a derivative with respect to time. It equals minus the derivative of the free-energy density $\rho\psi(\boldsymbol{\epsilon},\Omega)=(1-\Omega)\boldsymbol{\epsilon}^T\mathbf{D}_e\boldsymbol{\epsilon}/2$ with respect to the damage variable $\Omega$, and so it represents the "rate" at which energy would be released during (artificially induced) damage growth at constant strain and temperature.

The monotonically increasing function $\omega$ in Eq. (25) controls the evolution of damage and thus affects the shape of the stress–strain curve. It is usually designed such that $\Omega=0$ as long as $Y_{\max}$ remains below a certain threshold value, $Y_0$. In view of the aforementioned conclusion about the proper nonlocal approach, this local damage evolution law was adapted to a nonlocal form in either of the two following ways (Pijaudier-Cabot and Bažant 1987; Bažant and Pijaudier-Cabot 1988), which represent averaging of the damage energy release rate or of the damage variable:

$$\Omega=\omega(\overline{Y}_{\max}), \quad \text{or} \quad \Omega=\overline{\omega(Y_{\max})} \tag{28}$$

The convergence and stability of this formulation was verified in various ways; e.g., by refining the mesh for a uniaxially stretched softening bar. The numerical solutions for a progressively increasing number of finite elements are plotted in Fig. 4. As seen, the profiles of strain increment throughout the softening zone converge as the number of elements along the bar is increased from 12 to 100. Also, the averaging of damage and of energy release rate give, in this case, very similar results.

The latter, though, is not true for large postpeak deformations, as demonstrated by Jirásek (1998b). At very large extensions of the bar, a complete fracture must be simulated, which means that the stress must be reduced to zero. Jirásek (1998b) showed that this is not true for some types of nonlocal averaging, and found that averaging of different variables gives rather different responses. The nonlocal damage formulations that he considered are summarized in Table 1. The central column presents in a compact form the nonlocal stress–strain law for the isotropic damage model whose local version is described by Eqs. (24)–(27). The right column shows a possible generalization to anisotropic damage (top part) or to a completely general inelastic model (bottom part). Beside the symbols already defined, the following notations are used: $\gamma=\Omega/(1-\Omega)$ = compliance variable, $\mathbf{D}_s$ = damaged (secant) stiffness matrix, $\mathbf{D}_u$ = unloading stiffness matrix, $\boldsymbol{\Omega}$ = damage tensor, $\mathbf{Y}$ = tensor of damage energy release rates work-conjugate to $\boldsymbol{\Omega}$, $\mathbf{C}_e=\mathbf{D}_e^{-1}$ = elastic compliance matrix, $\mathbf{C}_i$ = inelastic compliance matrix, and $\mathbf{s}=\mathbf{D}_e\boldsymbol{\epsilon}-\boldsymbol{\sigma}$ = inelastic stress. An overdot denotes differentiation with respect to time.

The load-displacement diagrams generated in a uniaxial tensile test by different nonlocal damage formulations are shown in Fig. 5. The (local) damage evolution law $\Omega=\omega(Y_{\max})$ is constructed such that the response remains linear up to the peak stress and the softening curve is exponential, asymptotically approaching the horizontal axis. The dashed curve in Fig. 5 corresponds to the unstable solution for which the strain remains uniform. The actual stable solutions are characterized by a nonuniform strain distribution, with strain increments localized into a finite interval, the size of which is controlled by the characteristic length (see Fig. 4). The initial response, right after the onset of localization, is about the same for all the models considered here. At later stages of softening, only the formulations labeled $\omega(\overline{\epsilon})$, $\overline{Y}$, and $\overline{\gamma}$ give a reasonable behavior, with the residual strength approaching zero as the applied elongation is increased [Fig. 5(a)]. The other formulations lead to locking effects and sometimes fail to converge [Fig. 5(b)]. Thus, it can be concluded that the complete fracture is correctly reproduced by models that average the equivalent strain,

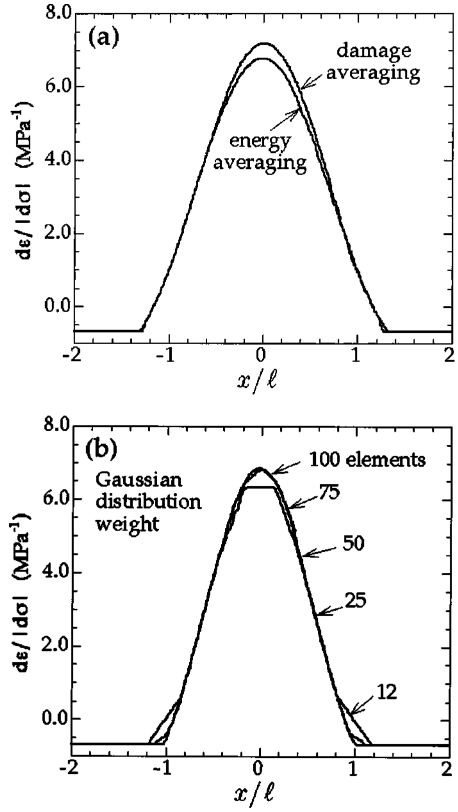


**Fig. 4.** Profiles of incremental strain per unit stress increment obtained in nonlocal numerical simulation of a tensioned bar: (a) comparison of two types of averaging; (b) convergence as the number of elements is increased (after Bažant and Pijaudier-Cabot 1988).

**Table 1.** Overview of Nonlocal Damage Formulations

| Formation | Isotropic damage model | General model |
|---|---|---|
| $\omega(\overline{\varepsilon})$ | $\boldsymbol{\sigma}=[1-\omega(Y(\overline{\varepsilon}))]\boldsymbol{D}_e\varepsilon$ | $\boldsymbol{\sigma}=\boldsymbol{D}_s(\overline{\varepsilon})\varepsilon$ |
| $\overline{Y}$ | $\boldsymbol{\sigma}=[1-\omega(\overline{Y(\varepsilon)})]\boldsymbol{D}_e\varepsilon$ | $\boldsymbol{\sigma}=\boldsymbol{D}_s(\Omega(\overline{Y(\varepsilon)}))\varepsilon$ |
| $\overline{\varepsilon}$ | $\boldsymbol{\sigma}=[1-\overline{\omega(Y(\varepsilon))}]\boldsymbol{D}_e\varepsilon$ | $\boldsymbol{\sigma}=\overline{\boldsymbol{D}_s(\varepsilon)}\varepsilon$ |
| $\overline{\gamma}$ | $\boldsymbol{\sigma}=[1+\overline{\gamma(Y(\varepsilon))}]^{-1}\boldsymbol{D}_e\varepsilon$ | $\boldsymbol{\sigma}=[\boldsymbol{C}_e+\overline{\boldsymbol{C}_e(\varepsilon)}]^{-1}\varepsilon$ |
| $\overline{s}$ | $\boldsymbol{\sigma}=\boldsymbol{D}_e\boldsymbol{\varepsilon}-\overline{\omega(\varepsilon)\boldsymbol{D}_e\varepsilon}$ | $\boldsymbol{\sigma}=\boldsymbol{D}_e\varepsilon-\overline{s(\varepsilon)}$ |
| $\overline{\Delta s}$ | $\dot{\boldsymbol{\sigma}}=(1-\omega)\boldsymbol{D}_e\dot{\varepsilon}-\overline{\dot{\omega}\boldsymbol{D}_e\varepsilon}$ | $\boldsymbol{\sigma}=\boldsymbol{D}_u\dot{\varepsilon}-\overline{\dot{s}(\varepsilon,\dot{\varepsilon})}$ |
| $s(\overline{\varepsilon})$ | $\boldsymbol{\sigma}=\boldsymbol{D}_e\boldsymbol{\varepsilon}-\omega(\overline{\varepsilon})\boldsymbol{D}_e\overline{\varepsilon}$ | $\boldsymbol{\sigma}=\boldsymbol{D}_e\varepsilon-s(\overline{\varepsilon})$ |

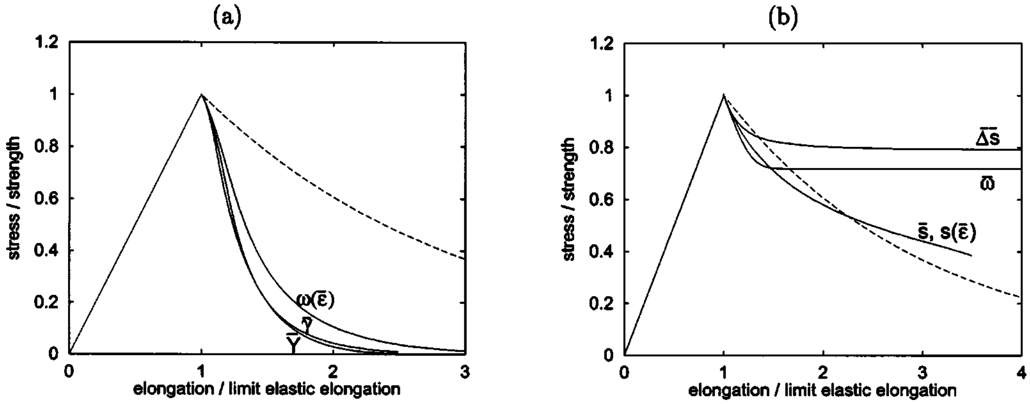


**Fig. 5.** Comparison of postpeak load-deflection curves of a tensioned bar calculated for nonlocal damage with averaging applied to different variables: (a) nonlocking formulations $\omega(\bar{\epsilon})$, $\bar{Y}$ and $\bar{\gamma}$; and (b) locking formulations $\bar{\omega}$, $\bar{s}$, $\overline{\Delta s}$, and $s(\bar{\epsilon})$.

the energy release rate, or the compliance variable. The evaluation of the inelastic stress from the nonlocal strain, the same as the averaging of the damage variable, inelastic stress, or inelastic stress increment, leads to spurious residual stresses and to an expansion of the softening zone across the entire bar. Models based on averaging of inelastic stress will be addressed in more detail in the section on nonlocal adaptation of general constitutive models.

The basic model with damage evolution driven by the damage energy release rate (27) is simple and appealing from the theoretical point of view, since it can be formulated within the framework of generalized standard materials (Halphen and Nguyen 1975). However, it is not suitable for quasibrittle materials, because it gives the same response in tension and in compression. To emphasize the effect of tension on the propagation of cracks, Mazars (1984) proposed to link the damage to the so-called equivalent strain, defined as the norm of the positive part of the strain tensor. He developed an isotropic damage model for concrete with two damage parameters, $\omega_t$ and $\omega_c$, which correspond to uniaxial tension and uniaxial compression, respectively, and both depend on the maximum previously reached value of the equivalent strain. For a general stress state, the actual damage parameter is interpolated according to the current values of principal stresses. The nonlocal formulation of Mazars' model was refined by Saouridis (1988) and Saouridis and Mazars (1992), following the basic idea of Pijaudier-Cabot and Bažant (1987). The averaged quantity was the equivalent strain, which corresponds to a natural generalization of formulation $\bar{Y}$.

A number of nonlocal damage formulations appeared in the literature during the last decade. For instance, di Prisco and Mazars (1996) improved the nonlocal version of Mazars' model by introducing irreversible strains and an additional internal variable controlling the volumetric expansion of concrete in compression. The model was further refined by Ferrara (1998), who considered the characteristic length as a variable parameter that depends on the current level of damage. Valanis started from his anisotropic damage theory based on the integrity tensor (Valanis 1990) and reformulated it as nonlocal, with the damage rate dependent on the current local damage, strain, and positive part of the nonlocal strain rate (Valanis 1991). Kennedy and Nahan (1996) proposed a nonlocal anisotropic damage model (with fixed axes of material orthotropy) for composites and applied it to the failure analysis of laminated shells (Kennedy and Nahan 1997). Their approach is an anisotropic extension of the formulation $\omega(\bar{\epsilon})$. A different extension of this formulation was used by Comi (2001), who defined two internal lengths leading to two nonlocal averages of the strain tensor, driving the damage evolution in tension and in compression. Jirásek (1999) started from the microplane damage framework established by Carol and Bažant (1997), which exploits the principle of energy equivalence (Cordebois and Sidoroff 1979). He developed an anisotropic damage model for tensile failure of concrete and regularized it by averaging the compliance parameters related to microplanes of different orientations, which corresponds to an anisotropic extension of formulation $\bar{\gamma}$. Fish and Yu (2001) derived a nonlocal damage model for composites by a multiscale asymptotic analysis of the damage phenomena occurring at the micro-, meso-, and macrolevel. Huerta et al. (1998) developed a mesh-adaptive technique for nonlocal damage models, and Rodríguez-Ferran and Huerta (2000) performed adaptive simulations of concrete failure using a residual-type error estimator (Díez et al. 1998).

### *Nonlocal Smeared Cracking*

The nonlocal concept has further been applied to the smeared cracking models widely used for concrete structures and rocks. Such models transform the cumulative effect of microcrack growth and coalescence into additional inelastic strain due to cracking. Adding the smeared cracking strain to the elastic strain, one obtains the total strain. A secant or tangential compliance tensor of the cracked material, which is orthotropic, can be easily obtained. If the damage remains distributed, the cracking strain corresponds to the average of crack opening divided by the spacing between parallel cracks. But, the model can be used even after localization of damage into large macroscopic cracks. To obtain an objective description, it is necessary either to relate the softening part of the stress–strain law to the numerically resolved width of the fracture process zone (which depends on the finite-element size and tends to zero as the mesh is refined), or to reformulate the model as nonlocal and enforce a mesh-independent width of the process zone by introducing a characteristic length.

The smeared cracking model has two variants: (1) the fixed crack model, in which the crack orientation is fixed when the maximum principal stress first attains the strength limit (Rashid 1968; de Borst 1986); and (2) the rotating crack model (Cope et al. 1980; Gupta and Akbar 1984; Rots 1988), in which the crack orientation is rotated so as to always remain perpendicular to the maximum principal strain direction. The latter concept, which usually gives more realistic results, does not mean that the

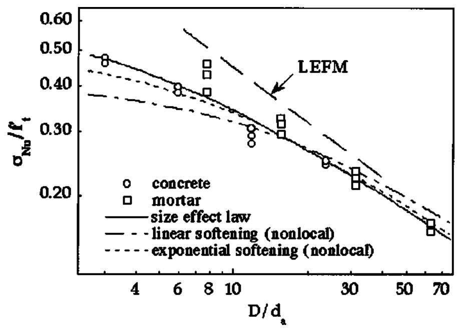


**Fig. 6.** Size effect obtained with the nonlocal smeared cracking model, for two types of strain-softening law (after Bažant and Lin 1988a)

cracks would actually rotate. Rather, it means that cracks of many orientations exist; cracks of some orientation close and cracks of another orientation open, with the effect that the orientation of the dominant cracks rotates. In the case of the fixed crack model, one needs to cope with the problem of a secondary crack system crossing at some angle the system formed previously (de Borst 1986).

Since smeared cracking models take into account the orientation of cracks, they automatically reflect the crack-induced anisotropy of the material. It is possible to interpret them formally as special anisotropic damage models.

A nonlocal generalization of the rotating crack model, in which the cracking strain is averaged using the nonlocal operator defined in Eq. (19), was investigated by Bažant and Lin (1988a). It was shown that the size effect obtained by finite-element simulations approximately follows Bažant's size effect law (Bažant 1984b). Fig. 6 illustrates the differences in size effect for two different types of the strain-softening law (linear and exponential).

An alternative nonlocal formulation of the rotating crack model can be based on the evaluation of the damaged material stiffness matrix from the nonlocal average of the total strain (Jirásek and Zimmermann 1998b). The stress is then computed as the product of the nonlocally evaluated stiffness with the local strain (this ensures that the response in the elastic range is local). In terms of the nonlocal damage formulations summarized in Table 1, the Bažant–Lin approach corresponds to formulation $\overline{\gamma}$ while the Jirásek–Zimmermann approach corresponds to formulation $\omega(\overline{\epsilon})$. When applied directly to the standard rotating crack model, the latter formulation leads to instabilities due to negative shear stiffness coefficients. A stable behavior is guaranteed for a modified rotating crack model with transition to scalar damage (Jirásek and Zimmermann 1998b).

The following example, taken from Jirásek and Zimmermann (1997), demonstrates the ability of the nonlocal rotating crack model to reproduce a relatively complex curved fracture pattern. In a series of experiments, Nooru-Mohamed (1992) tested the double-edge-notched (DEN) specimen shown in Fig. 7(a), which can be subjected to a combination of shear and tension (or compression). Nooru-Mohamed performed the experiments for a number of loading paths, some of them even nonproportional. One of the most interesting loading scenarios is path 4c, which produces curved macroscopic cracks [Fig. 7(b)]. The specimen is first loaded by an increasing "shear" force, $P_s$, while keeping the "normal" force, $P$, at zero. After reaching the peak of the $P_s-\delta_s$ curve, the type of loading changes. From that moment on, $P_s$ is kept constant and the normal displacement $\delta$ is increased, which results into a nonzero reaction $P$.

The aforementioned experiment was simulated using the nonlocal rotating crack model with transition to scalar damage. The meshes were constructed following a pseudoadaptive technique proposed by Jirásek and Zimmermann (1997), starting from a mesh of uniform density. A sequence of three progressively refined meshes along with the corresponding process zones is shown in Fig. 8, in which the light regions indicate high levels of damage. The simulated process zone matches the experimentally observed cracks very well.

### Alleviation of Mesh Orientation Bias for Crack Propagation Direction

In connection with smeared cracking, it is appropriate to point out one advantageous side benefit brought about by the nonlocal averaging concept—the alleviation of mesh orientation bias. Fig. 9 shows a beam analyzed with an aligned mesh and with a deliberately rotated mesh. The local cracking models or cohesive models

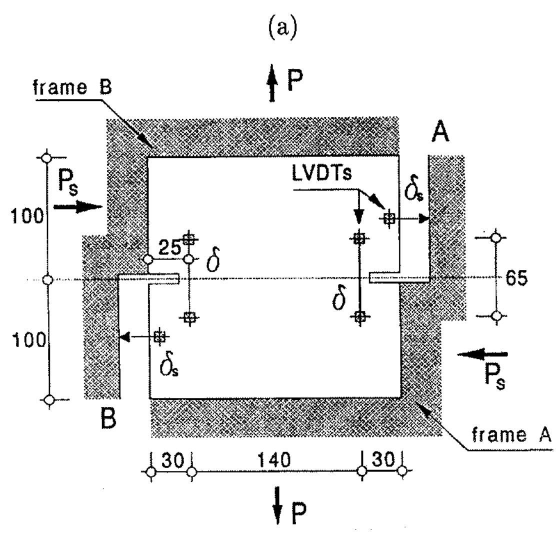


**Fig. 7.** Double-edge-notched specimen and observed cracks (after Nooru-Mohamed 1992)

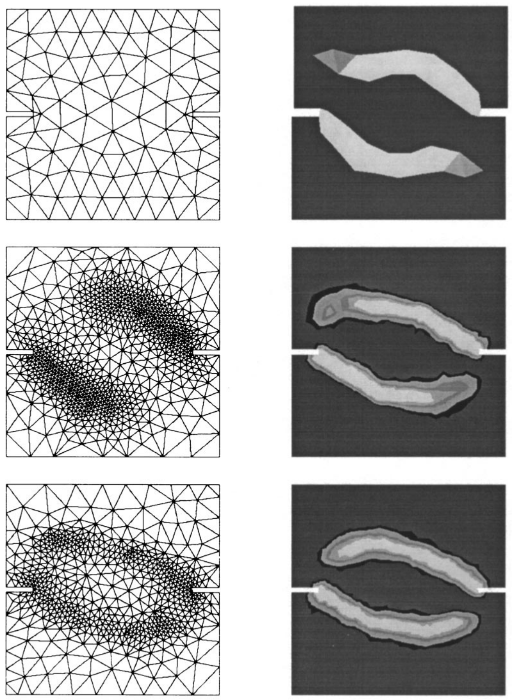

**Fig. 8.** Simulation of curved crack propagation in a double-edge-notched specimen using a pseudoadaptive technique: Sequence of meshes and fracture process zones (after Jirásek and Zimmermann 1997).

on a fixed mesh would give an inclined direction of propagation for the rotated mesh. Simply, the cracks prefer to run in the direction of mesh lines. The nonlocal model, on the other hand, gives a nearly vertical propagation of the damage band even for the rotated mesh, provided that the elements are no larger than about $\frac{1}{3}$ of the process zone width (Bažant and Lin 1988a). The $\frac{1}{3}$ rule for avoiding the mesh orientation bias seems to be generally applicable to all the models with nonlocal averaging.

The nonlocal averaging concept seems to be the most effective way to avoid mesh-induced directional bias, provided, of course, that the use of small enough finite elements is feasible. Fig. 10 shows the fracture process zone in a notched three-point bend

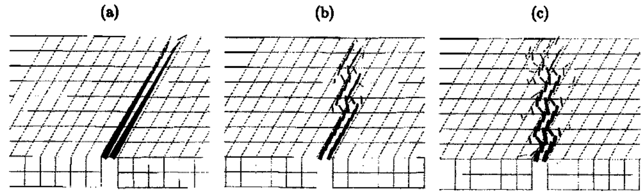


**Fig. 10.** Simulated crack pattern in the central part of a notched three-point bend specimen: (a) standard rotating crack model (local), (b) rotating crack model with transition to scalar damage (local), and (c) nonlocal rotating crack model with transition to scalar damage (after Jirásek and Zimmermann 1998b).

specimen simulated with smeared crack models on a skewed mesh. Due to symmetry, the actual crack trajectory should be straight and vertical. Of course, the real crack path is tortuous and, in one single experiment, may deviate from this ideal trajectory. However, the computational simulation is supposed to reproduce the mean trajectory, averaged over a large number of experiments, which is no doubt expected to lie on the axis of symmetry. The local version of the standard rotating crack model exhibits strong directional bias—the crack band propagates along the mesh lines over the entire depth of the specimen [Fig. 10(a)]. It is interesting to note that a partial improvement is achieved already by means of the local version of the modified rotating crack model with transition to scalar damage [Fig. 10(b)]. This can be explained by the fact that, for the standard rotating crack model, the crack trajectory shown in Fig. 10(b) would lead to stress locking (Jirásek and Zimmermann 1998a), which means that it is easier for the macroscopic crack to keep propagating along the mesh lines.

When the modified rotating crack model is reformulated as nonlocal, the results improve further, even for a rather coarse mesh for which the nonlocal interaction radius used in the present example is only 1.25 times the element size. The resulting macroscopic crack trajectory only slightly deviates from the axis of symmetry, despite the skewed character of the mesh [Fig. 10(c)]. The directions of ‘‘local cracks’’ (marked by dark rectangles at individual Gauss points) oscillate and are not aligned with the overall crack trajectory. However, this is quite natural because these local cracks, defined just for the purpose of visualization of

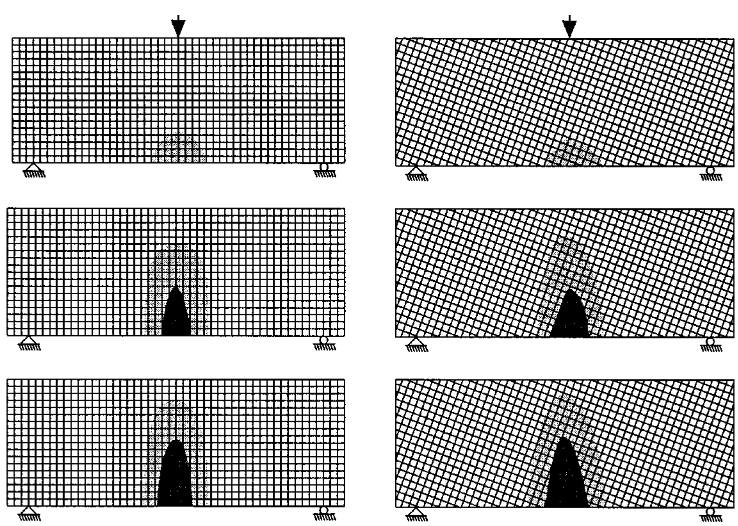

**Fig. 9.** Study of mesh-orientation bias with nonlocal smeared cracking model of Bažant and Lin (1988a)

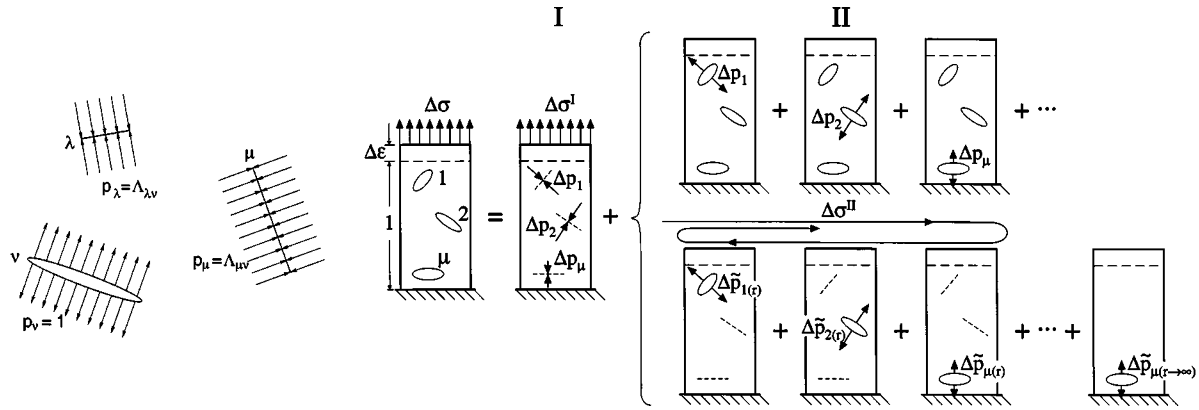


**Fig. 11.** Microcrack interaction approach: stress on cracks $\mu$ and $\lambda$ caused by unit pressure on the faces of crack $\nu$, and superposition and iteration approach to a system of interacting cracks

the results, are computed from the local inelastic strains. At very late stages of the stiffness degradation process, the local strains localize in a single layer of elements, even though the model is nonlocal. This cannot be judged as a deficiency of the model. What matters is that the load-displacement diagram and the overall direction of the cracking band are reproduced correctly, regardless of the mesh size and orientation. Still better results can be expected for a refined mesh with elements several times smaller than the interaction radius.

### Oriented Long-Range Nonlocality Due to Microcrack Interactions

The models discussed so far introduce ad hoc nonlocal averaging, without any physical justification. Bažant (1994) proposed a physical motivation of nonlocal damage, based on a micromechanical analysis of microcrack interactions. He exploited the solution procedure for a statistically homogeneous system of random cracks in an elastic matrix, developed in detail by Kachanov (Kachanov 1985; Kachanov 1987). The solution is a sum of a trivial solution of an elastic solid with all the cracks imagined closed (as if "glued"), and the solution for the case in which the cracks are "unglued" and their faces are loaded by releasing the stresses transmitted across the glued cracks (Fig. 11). Only the second case matters for the stress intensity factors of the cracks. The solution for many cracks can be constructed by an iterative relaxation scheme on the basis of the known solution of the stress transmitted across the plane of an arbitrary closed (glued) crack in an infinite space containing only one pressurized open crack. This solution is well known, not only for two but also for three dimensions (Fabrikant 1990).

Based on this kind of solution, it was shown (Bažant 1994; see also Bažant and Planas 1998, Sec. 13.3) that the nonlocal incremental constitutive law should have the general form

$$\Delta\boldsymbol{\sigma}=\mathbf{D}_e\Delta\boldsymbol{\epsilon}-\Delta\tilde{\mathbf{s}} \tag{29}$$

in which the nonlocal inelastic stress increment $\Delta\tilde{\mathbf{s}}$ is the solution of the Fredholm integral equation

$$\Delta\tilde{\mathbf{s}}(\mathbf{x})=\Delta\overline{\mathbf{s}}(\mathbf{x})+\int_V\Lambda(\mathbf{x},\boldsymbol{\xi})\Delta\tilde{\mathbf{s}}(\boldsymbol{\xi})d\boldsymbol{\xi} \tag{30}$$

averaging + crack interactions

The inelastic stress is understood as the stress drop due to the presence of cracks, as compared to the elastic stress that would be induced in an undamaged material by the same strain; see Fig. 12.

Equation (30) is justified by the fact that its approximation by a discrete sum over $\Delta\overline{\mathbf{s}}$ values at the random crack centers reduces this integral equation to a matrix equation for the solution of the many-crack system in Fig. 11. The first term on the right, which describes the standard nonlocal averaging according to Eq. (19), is justified by material heterogeneity and by the finiteness of the energy release zone of a crack. The second term, which describes microcrack interactions, contains the microcrack interaction function $\Lambda(\mathbf{x},\boldsymbol{\xi})$, which represents statistical continuum smearing of

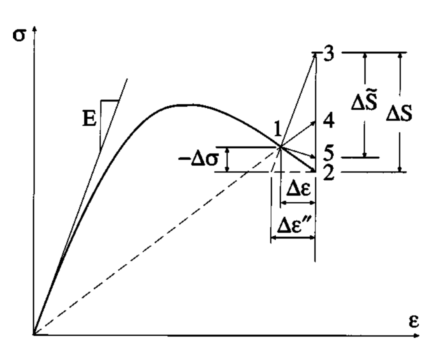


**Fig. 12.** Stress decomposition into elastic and inelastic parts

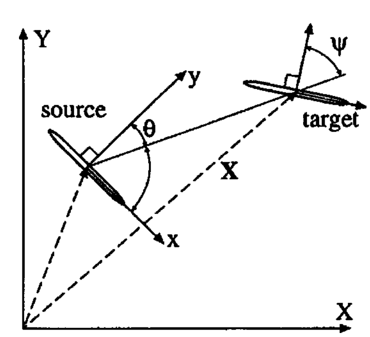


**Fig. 13.** Parameters describing relative position of two interacting cracks

discrete coefficients $\Lambda_{\mu\nu}$ of the aforementioned matrix equation, each of which is defined as the stress across a glued crack $\mu$ produced by a unit pressure applied on the faces of crack $\nu$ (Fig. 11). The domain of the integral must be taken as the entire body volume $V$, because the decay of function $\Lambda$ with distance is rather slow.

The microcrack interaction function $\Lambda$ depends not only on the distance $r=\|\mathbf{x}-\boldsymbol{\xi}\|$ between points $\mathbf{x}$ and $\boldsymbol{\xi}$, but also on the orientation of dominant cracks at those points. The dominant crack is assumed to be perpendicular to the maximum principal stress direction. In two dimensions, the interaction factor

$$\Lambda(r,\theta,\psi)\approx-\frac{k(r)}{2\ell^2}[\cos 2\theta+\cos 2\psi+\cos 2(\theta+\psi)] \quad (31)$$

is a function of the distance $r$, of the angle $\theta$ between the radial coordinate axis $r$ and the maximum principal stress direction at the source point $\boldsymbol{\xi}$, and of the angle $\psi$ between the radial coordinate axis $r$ and the maximum principal stress direction at the receiver point $\mathbf{x}$ (Fig. 13). Parameter $\ell$ is a length parameter reflecting the dominant spacing of interacting microcracks. A similar but more complicated approximate expression was proposed for function $\Lambda$ in three dimensions (Bažant 1994).

The far-away asymptotic form of radial function $k(r)$ for $r \gg \ell$ was derived from the remote asymptotic field of a single pressurized crack in an infinite space. It turns out that, asymptotically for $r\to\infty$, $k(r)$ is proportional to $r^{-d}$ where $d$ is the number of spatial dimensions (2 or 3). As the simplest match between this asymptotic field and the value 0 required for $r\to 0$, Bažant (1994) proposed to use in two dimensions the expression

$$k(r)=\left(\frac{\kappa\ell r}{r^2+\ell^2}\right)^2 \quad (32)$$

An interesting point to note is that the asymptotic decay of $k(r)$ with $r$ is very slow, in fact, so slow that the integral in Eq. (30) is at the limit of integrability for the case of an infinite solid. Another point to note is that the integral in Eq. (30) does not represent nonlocal averaging because the weight function $\Lambda$ does not satisfy the normalizing condition (20). In fact, for a body symmetric with respect to the origin we would get

$$\int_V \Lambda(\mathbf{0},\boldsymbol{\xi})\,d\boldsymbol{\xi}=0 \quad (33)$$

For angles $|\theta|>56°$ (in two dimensions), the normal stress component in the direction of the normal to a crack at $\boldsymbol{\xi}$ is tensile and tends to amplify the growth of a crack at $\mathbf{x}$. For $|\theta|\leqslant 56°$, it tends to shield the other crack from stress and thus inhibit its growth. Thus, a physically important feature of the crack interaction function is that it can distinguish between remote amplification and shielding sectors of a crack, which is ignored in the standard approach.

Eq. (31) was introduced by Ožbolt and Bažant (1996) into a nonlocal finite-element code with the microplane constitutive model for concrete. Eq. (30) was solved at each loading step iteratively, simultaneously with the equilibrium iterations. Ožbolt and Bažant (1996) found that the modified nonlocal averaging based on microcrack interactions makes it easier to reproduce, with the same set of material parameters (including the characteristic length), the results of different types of experiments, especially the failures dominated by tension and those dominated by shear and compression.

### *Measurement of Material Characteristic Length for Nonlocal Damage*

The characteristic length $\ell$ is a material parameter controlling the spread of the nonlocal weight function. It cannot be directly measured but can be indirectly inferred by inverse analysis of test results. However, a reliable identification procedure can be based only on experiments that are sufficiently sensitive to the nonlocal aspects of the material behavior.

At this point, it is important to note that there is a subtle difference between the notions of an intrinsic material length and a nonlocal characteristic length. The latter is a special case of the former, because nonlocal averaging is only one of the possible enrichments that introduce a length parameter into the continuum theory. Certain other enrichments, e.g., by higher-order gradients or independent microrotation fields, can be classified as weakly nonlocal theories and have similar regularizing effects on the solution. When properly formulated, all the nonlocal theories in the broad sense lead to continuous strain profiles and to process zones (regions of localized strain) of nonzero *width and length*.

On the other hand, there exist special techniques such as the crack band model (Bažant 1982; Bažant and Oh 1983), to be discussed in detail later on, for which the global solution characteristics (such as the load-displacement diagram and the energy dissipated during failure) remain objective with respect to the mesh refinement but the process zone eventually collapses into a surface or curve of vanishing thickness. Since the process zone keeps a finite *length*, such models still carry information about the intrinsic material length and are capable of describing size effects of the transitional type. This material length is, however, not really a *nonlocal* characteristic length. It is present, for instance, in models that work simultaneously with a fracture energy and a material strength (such as the crack band model). A classical example is provided by Irwin's parameter (Irwin 1958),

$$r_p=\frac{K_c^2}{\pi\sigma_0^2} \quad (34)$$

which was introduced to characterize the length of the plastic zone around a crack tip in elastoplastic fracture mechanics [for concrete, it was adopted by Hillerborg et al. (1976)]. Here, $K_c$ is the fracture toughness (critical value of the stress intensity factor) and $\sigma_0$ is the yield stress.

The foregoing analysis suggests that the nonlocal models have, in fact, two parameters with the dimension of length, one of them characterizing the length and the other the width of the process zone. They are, in general, independent. Any identification procedure that extracts from the given experimental data only a single length parameter and relates it to the nonlocal characteristic length is in reality based on the tacit assumption that the two length parameters are somehow linked, e.g., that their ratio is fixed. For a limited class of materials, such an assumption can be reasonable, but it can hardly be considered as a general rule. A reliable and fully general identification procedure should be capable of extracting two independent length parameters, but for this it is necessary to have a sufficiently rich set of experimental data (rich in terms of the variety of information, not just in terms of the number of specimens tested).

A typical example is provided by the size effect method. Sometimes it is assumed that the nonlocal characteristic length can be determined by fitting the size effect on the nominal strength of specimens, in other words, by trying to reproduce the dependence of the peak load on the structure size for a set of geometrically similar fracture specimens covering a sufficiently

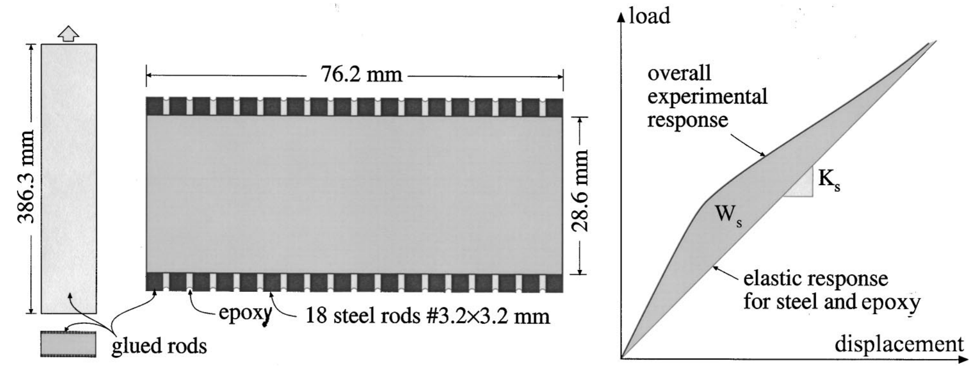


**Fig. 14.** Left: cross section of concrete specimen restrained by elastic rods; right: load-displacement curves of restrained specimen and area $W_s$

wide range of sizes (or, more generally, brittleness numbers, in the case of dissimilar specimens). In reality, this procedure can give only one material length parameter, which is related (through a geometry factor determined from fracture mechanics) to the length of the process zone rather than to its width.

A rigorous determination of the nonlocal characteristic length merely from the size effect on the peak loads for one test geometry is, therefore, impossible. Only if one takes into account the fact that both dimensions of the process zone are related to the same microstructural property, e.g., to the maximum aggregate size and spacing in concrete, it is possible to assume that the ratio between the two length parameters is equal to a certain constant and relate the length provided by the size effect method to the nonlocal characteristic length. In practice, this assumption can be hidden in fixing certain other parameters, e.g., those that define the local softening law.

For a gradient damage model, the optimal estimation of the characteristic length of cement mortar reinforced by cellulose fibers was studied in detail by Carmeliet (1999). His conclusions remain valid for nonlocal models of the integral type. Carmeliet used the Markov estimation method, which gives not only the mean values of the material parameters, but also their standard deviations and correlation coefficients, provided that a sufficient number of test results are available. The identification procedure exploited the results of uniaxial tensile tests on a double-notched specimen, and of four-point bending tests on single-notched specimens of different sizes in the range 1:8 and with two different relative notch depths. Young's modulus was determined directly from the tensile tests, and the Markov method aimed at determining the tensile strength, the nonlocal characteristic length, and a parameter that appears in the damage law and controls the area under the local stress–strain curve, i.e., the dissipation density per unit volume in the absence of localization. When the identification was based on the peak loads from all the tests, Carmeliet found quite a high correlation between the strength and the characteristic length (correlation coefficient 0.88). This indicates a strong dependence between these parameters, meaning that the procedure just outlined cannot determine them independently—they are somehow constrained by an implicit hypothesis. Even if the identification procedure is refined by taking into account the entire load-displacement curves (not only the peak loads), the correlation coefficient remains between 0.84 and 0.95, depending on which experiments are taken into account. Correlation coefficient 0.95 is obtained when the parameters are evaluated only from the geometrically similar specimens with the same relative notch size, which is the typical case considered by the size effect method. This again confirms that the nonlocal characteristic length cannot be uniquely determined from the size effect alone.

For a reliable identification of the nonlocal length parameter, one needs the results of tests that are sensitive to the nonlocal aspects of the material behavior. One possible solution is to combine two kinds of tests: (1) those leading to localized damage patterns with (2) those where the damage remains distributed. The former kind is sufficiently represented by standard fracture tests on notched specimens. The latter kind is represented, for instance, by the elastically restrained tensile test, proposed by Bažant and Pijaudier-Cabot (1989) and later modified by Mazars and Berthaud (1989), and Mazars et al. (1990). In this kind of experiment, tension is applied to a long prismatic specimen of concrete, restrained on the surface by gluing to it many thin longitudinal aluminum rods (Fig. 14). The cross section of the rods must be sufficient to prevent the tangential stiffness of the specimen from ever being negative, despite softening damage in concrete. Furthermore, to permit simple analysis, the concrete thickness must be small enough (only a few aggregate sizes) to force the deformation to be essentially uniform across the thickness. The area of the load-displacement curve lying above the line that corresponds to the stiffness of the elastic rods alone (without the concrete) represents the energy $W_s$ dissipated by strain-softening damage within the specimen. Dividing $W_s$ by the volume $V$ of tensioned concrete in the specimen, one gets an estimate of the dissipation density per unit volume, $g_F = W_s/V$, which is valid in the absence of localization and should correspond to the area under the local stress–strain diagram. Once the parameters of the local stress–strain law are fixed, the nonlocal characteristic length remains the only undefined parameter. It can be determined from the ratio $G_F/g_F$ where $G_F$ is the fracture energy (dissipation per unit area), which can be determined from fracture tests. In fact, it is not even necessary to use the size effect method; the average peak load for a single specimen size would suffice, provided that the specimen is large enough to make the process zone substantially smaller than the whole specimen.

It must, nevertheless, be noted that the aforementioned experimental procedure would be perfect only if the elastically restrained tensile test provided reliable information on the complete stress–strain law, including the value of the tensile strength. However, a refined analysis by Boudon-Cussac et al. (1999) revealed that this technique can have a large error due to partial debonding at the interface between the concrete and the glued rods. If any debonding occurs, the true behavior of concrete is not easy to deduce, because the stress state in concrete ceases to be uniform, even in the central part of the specimen. Instead of a straightforward interpretation of the test results, one would have to resort to inverse analysis. Another spoiling effect, albeit no doubt less serious, is that a surface restraint by homogeneously deforming rods does not exactly replicate the ideal microstructural deformation at a cross section of a large concrete specimen with statistically homogeneous deformation.

An alternative identification procedure can be based on local measurements in or around the process zone. Knowing the details of the local strain distribution, one can evaluate the nonlocal characteristic length by looking for the best match between the experimental and numerical strain profiles. Geers et al. (1996a) applied this idea with success to short glass-fiber-reinforced polypropylene, based on the experimental results reported by Geers et al. (1996b), who measured the displacements at a number of points arranged in a grid on the surface of the compact tension (wedge splitting) specimen. For concrete, Denarié et al. (2001) tried to measure strains inside the process zone using optical fibers with Bragg gratings, but the accuracy of this technique in the presence of cracking and the correct interpretation of the results are not completely clear. Nevertheless, these experiments confirmed that the process zone has a finite width, because the measured strains were way above the elastic limit, even though the fibers were not crossed by the final macroscopic crack. Another interesting point is that the residual strains measured at complete failure were rather high. This confirms the fact, well known from fracture testing of concrete, that the assumption of unloading to the origin, characteristic of pure damage models, is simplistic. Realistic models should incorporate permanent strains, even under tension.

Although detailed information on the precise distribution of strains in the process zone is highly valuable, it is difficult to obtain. For identification of the nonlocal characteristic length, it is nevertheless sufficient to know at least the width of the process zone. An estimate can be obtained by monitoring and evaluating the acoustic emissions produced by cracking, or by x-ray inspection. Such experimental techniques have been developed among others by Mihashi and Nomura (1996), Landis (1999), Otsuka and Date (2000), or Labuz et al. (2001).

### Nonlocal Adaptation of General Constitutive Models

The section on nonlocal damage models dealt with various nonlocal formulations of the simple isotropic damage model with one scalar damage variable. It would be very useful to develop a unified nonlocal formulation applicable to any inelastic constitutive model with softening and acting as a reliable localization limiter, insensitive to mesh bias and free of locking effects. However, such a formulation is not available.

The first four formulations from Table 1 rely on the concept of damage and, consequently, they can be generalized only to anisotropic damage models or to the closely related smeared crack models. For other constitutive frameworks, such as plasticity or microplane theory, a straightforward generalization is not available.

The formulations based on the notion of inelastic stress may seem to be ideal candidates for a unifying averaging scheme, because the inelastic stress **s** can be defined for any type of constitutive model, simply as the difference between the elastically evaluated stress, $\mathbf{D}_e\boldsymbol{\epsilon}$, and the actual stress, $\boldsymbol{\sigma}$; see Fig. 12. In the elastic range, the inelastic stress by definition vanishes, and so the model response remains local. In the present context, this is a desirable feature, because we are interested mainly in strain localization due to softening and the deviations from locality in the elastic range are not expected to play an important role.

In the inelastic range, the nonlocal constitutive law could be defined as (Bažant et al. 1996)

$$\boldsymbol{\sigma}=\mathbf{D}_e\boldsymbol{\epsilon}-\overline{\mathbf{s}(\boldsymbol{\epsilon})} \tag{35}$$

where $\overline{\mathbf{s}(\boldsymbol{\epsilon})}$ = nonlocal average of the inelastic stress **s** evaluated from the local strain $\boldsymbol{\epsilon}$. Unfortunately, this averaging scheme is suitable only at the early stages of softening but later inevitably leads to spurious locking effects. This has already been documented in Fig. 5(b) for the case when the local constitutive law is based on damage mechanics. In a general case, the reason for locking can be explained as follows.

Under uniaxial tension with uniform strain, the actual stress $\sigma=E\epsilon-s$ is expected to be non-negative. Consequently, in the general case with nonhomogeneous strain, the inelastic stress evaluated from the local stress–strain law cannot exceed the elastic stress, which can be written as $s(x)\leqslant E\epsilon(x)$ for all $x$. Now, let $x_{\max}$ be the spatial coordinate of the point at which the strain attains its maximum value $\epsilon_{\max}$. It is easy to show that the nonlocal inelastic stress at $x_{\max}$ must satisfy the inequality

$$\overline{s}(x_{\max})=\int_L\alpha(x_{\max},\xi)s(\xi)\mathrm{d}\xi\leqslant\int_L\alpha(x_{\max},\xi)E\epsilon(\xi)\mathrm{d}\xi$$

$$\leqslant E\epsilon_{\max}\int_L\alpha(x_{\max},\xi)\mathrm{d}\xi=E\epsilon_{\max} \tag{36}$$

The derivation exploits the basic properties of the nonlocal weight function, namely, the fact that $\alpha(x,\xi)$ is non-negative and normalized according to Eq. (20). From the one-dimensional version of Eq. (35) it is clear that the stress at $x_{\max}$ can vanish only if $E\epsilon(x_{\max})=\overline{s}(x_{\max})$, i.e., if Eq. (36) holds with equality signs. But, this is possible only if $\epsilon(\xi)=\epsilon_{\max}$ for all the points within the domain of influence of point $x_{\max}$, i.e., for all $\xi$ such that $\alpha(x_{\max},\xi)>0$. Note that under uniaxial tension with negligible body forces the stress is uniform along the entire bar. The foregoing analysis means that the stress in the bar can vanish only if the strain distribution is uniform along the entire bar. In other words, the strain distribution cannot remain localized until complete failure. This is true independently of the type of constitutive law, as long as this law does not generate compressive stress under monotonic uniaxial tension.

Consequently, the nonlocal formulation with averaging of the inelastic stress is inherently incapable of describing complete failure, and so it cannot be used as a unifying concept. The same holds for the modifications that define the inelastic stress incrementally or evaluate the inelastic stress from the nonlocal strain.

## Nonlocal Plasticity

### Associated Plasticity with Isotropic Hardening or Softening

In the standard (local) version of the flow theory of plasticity with isotropic hardening or softening, the yield function, typically, has the form

$$f(\boldsymbol{\sigma},\kappa)=F(\boldsymbol{\sigma})-\sigma_Y(\kappa) \tag{37}$$

where $\boldsymbol{\sigma}$ = stress, $\kappa$ = scalar hardening-softening variable, $F(\boldsymbol{\sigma})$ = equivalent stress (e.g., the von Mises equivalent stress for $J_2$ plasticity, or the maximum principal stress for Rankine plasticity), and $\sigma_Y$ = current yield stress. The evolution of the yield stress as a function of the hardening variable is described by the hardening law. This law is, for future use, conveniently written as

$$\sigma_Y(\kappa)=\sigma_0+h(\kappa) \tag{38}$$

where $\sigma_0$ = initial yield stress and $h$ = hardening function. The derivative $H=\mathrm{d}h/\mathrm{d}\kappa$ is called the plastic modulus. The fundamental equations of associated elastoplasticity include also the elastic stress–strain law,

$$\boldsymbol{\sigma}=\mathbf{D}_e(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_p) \tag{39}$$

the flow rule,

$$\dot{\boldsymbol{\epsilon}}_p=\dot{\lambda}\mathbf{f}(\boldsymbol{\sigma}) \tag{40}$$

and the loading–unloading conditions

$$\dot{\lambda}\geqslant 0, \quad f(\boldsymbol{\sigma},\kappa)\leqslant 0, \quad \dot{\lambda}f(\boldsymbol{\sigma},\kappa)=0 \tag{41}$$

Here, $\boldsymbol{\epsilon}$ = (total) strain, $\boldsymbol{\epsilon}_p$ = plastic strain, $\mathbf{D}_e$ = elastic material stiffness, $\dot{\lambda}$ = rate of the plastic multiplier, and $\mathbf{f}\equiv\partial f/\partial\boldsymbol{\sigma}$ = gradient of the yield function, defining the direction of plastic flow.

The hardening variable $\kappa$ reflects the changes in the microstructure induced by plastic flow. Its rate is usually related to the plastic strain rate by the strain-hardening hypothesis

$$\dot{\kappa}=\|\dot{\boldsymbol{\epsilon}}_p\| \tag{42}$$

or by the work-hardening hypothesis

$$\dot{\kappa}=\frac{\boldsymbol{\sigma}^T\dot{\boldsymbol{\epsilon}}_p}{F(\boldsymbol{\sigma})}. \tag{43}$$

Substituting the plastic strain rate (40) into Eq. (42) or (43), we can write

$$\dot{\kappa}=k\dot{\lambda} \tag{44}$$

where $k=\|\mathbf{f}(\boldsymbol{\sigma})\|$ for strain hardening and $k=\boldsymbol{\sigma}^T\mathbf{f}(\boldsymbol{\sigma})/F(\boldsymbol{\sigma})$ for work hardening.

During plastic yielding, the yield function must remain equal to zero, and so the rates of the basic variables must satisfy the consistency condition $\dot{f}=0$. This makes it possible to derive the well-known expression for the rate of the plastic multiplier,

$$\dot{\lambda}=\frac{\langle\mathbf{f}^T\mathbf{D}_e\dot{\boldsymbol{\epsilon}}\rangle}{\mathbf{f}^T\mathbf{D}_e\mathbf{f}+kH} \tag{45}$$

where the Macauley brackets $\langle\cdots\rangle$ denote the positive part; for more details, see, e.g., Jirásek and Bažant (2001), Sec. 20.1.

In the presence of softening, characterized by a negative value of the plastic modulus $H$, the boundary value problem becomes ill posed and must be regularized. This can be achieved by a suitable nonlocal formulation.

### *Eringen's Nonlocal Plasticity*

Historically, the first nonlocal formulations of plasticity were proposed by Eringen in the early 1980s. He set up the framework for three classes of nonlocal plasticity models, based on the deformation theory, flow theory, and strain–space plasticity. Eringen did not consider the case of softening and did not intend his models to serve as localization limiters. Rather, he was interested in the continuum-based description of interacting dislocations and in the nonlocal effects on the distribution of stress around the crack tip in elastoplastic fracture mechanics. Nevertheless, for the sake of comparison it is useful to briefly describe his approach.

Eringen (1981) started from the plasticity theory formulated in the strain space by Green and Naghdi (1965) and Naghdi and Trapp (1975). The yield function is here written in terms of the strain, plastic strain, and hardening variable(s). The original theory was developed for large-strain applications, but for the present purpose we could write the yield condition as

$$g(\boldsymbol{\epsilon},\boldsymbol{\epsilon}_p,\kappa)=0 \tag{46}$$

This condition is equivalent to the classical yield condition in the stress space, $f(\boldsymbol{\sigma})=\sigma_Y(\kappa)$, if the function $g$ is defined as

$$g(\boldsymbol{\epsilon},\boldsymbol{\epsilon}_p,\kappa)=F[\mathbf{D}_e(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_p)]-\sigma_Y(\kappa) \tag{47}$$

The evolution laws for the internal variables and the loading–unloading conditions are postulated in the general form

$$\dot{\boldsymbol{\epsilon}}_p=\dot{\lambda}\boldsymbol{\rho}(\boldsymbol{\epsilon},\boldsymbol{\epsilon}_p,\kappa) \tag{48}$$

$$\dot{\kappa}=\dot{\lambda}k(\boldsymbol{\epsilon},\boldsymbol{\epsilon}_p,\kappa) \tag{49}$$

$$g(\boldsymbol{\epsilon},\boldsymbol{\epsilon}_p,\kappa)\leqslant 0, \quad \dot{\lambda}\geqslant 0, \quad \dot{\lambda}g(\boldsymbol{\epsilon},\boldsymbol{\epsilon}_p,\kappa)=0 \tag{50}$$

where $\boldsymbol{\rho}$ and $k$ are given functions. In analogy to the stress–space formulation, the rate of the plastic multiplier

$$\dot{\lambda}=-\frac{\left\langle\left(\frac{\partial g}{\partial\boldsymbol{\epsilon}}\right)^T\dot{\boldsymbol{\epsilon}}\right\rangle}{\left(\frac{\partial g}{\partial\boldsymbol{\epsilon}_p}\right)^T\boldsymbol{\rho}+\frac{\partial g}{\partial\kappa}k} \tag{51}$$

can be evaluated from the consistency condition $\dot{g}=0$, valid during the plastic flow.

In the nonlocal plasticity theory of Eringen (1981), the stress is computed by averaging the local stress that would be obtained from the local model. This is equivalent to rewriting the stress–strain law (39) in the form

$$\boldsymbol{\sigma}=\overline{\mathbf{D}_e(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_p)} \tag{52}$$

If the elastic stiffness is constant in space, Eq. (52) is equivalent to

$$\boldsymbol{\sigma}=\mathbf{D}_e\overline{(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_p)} \tag{53}$$

which is a straightforward extension of the nonlocal elastic law (12). Thus, the nonlocal approach of Eringen can be characterized as averaging of the stress or of the elastic strain. The response of the model is nonlocal already in the elastic range, which is not a desirable feature in applications to problems with strain localization due to softening. Stress averaging does not act as a localization limiter anyway. For instance, in the one-dimensional tensile test, all the solutions obtained with the corresponding local model would remain admissible, since the stress is constant due to the equilibrium condition and the constant field is not modified by the averaging operator.

As a further step, Eringen (1983) formulated nonlocal theories of plasticity in the stress space. In the flow theory he considered only perfect von Mises plasticity with an associated flow rule, but the logic he followed would, in a general case, lead to the stress–strain law

$$\boldsymbol{\sigma}=\overline{\mathbf{D}_e\boldsymbol{\epsilon}}-\mathbf{D}_e\boldsymbol{\epsilon}_p \tag{54}$$

which is in the case of a spatially constant elastic stiffness equivalent to

$$\boldsymbol{\sigma}=\mathbf{D}_e(\overline{\boldsymbol{\epsilon}}-\boldsymbol{\epsilon}_p) \tag{55}$$

Thus, one could say that the averaged quantity is the elastic stress or the total strain. Again, the response is nonlocal already in the elastic range, and the formulation cannot be used as a localization limiter, but this time the reason is different. While the previously mentioned model with nonlocal stress would not prevent localization into a set of zero measure, the present model with nonlocal strain would not allow any localization at all (in the one-dimensional test problem). This becomes clear if one realizes that, at the bifurcation from a uniform strain state, the nonlocal strain rate in the elastically unloading region would have to be equal to a negative constant while in the plastically softening region it would have to be equal to a positive constant. However, the nonlocal strain obtained by applying an integral operator with a continuous weight function is always continuous, even in situations when the local strain has the character of a Dirac distribution. Consequently, the nonlocal strain cannot have a jump at the elastoplastic boundary, and the plastic strain cannot localize into any region smaller than the entire bar.

### Nonlocal Plasticity Model of Bažant and Lin

The first nonlocal formulation of *softening* plasticity was proposed by Bažant and Lin (1988b); it was applied in finite-element analysis of the stability of unlined excavation of a subway tunnel in a grouted soil. The underlying cohesive–frictional plasticity model was based on the Mohr–Coulomb yield condition and a linear softening law with the work-softening hypothesis. Bažant and Lin (1988b) proposed to replace the plastic strain in the stress–strain law (39) by its nonlocal average,

$$\overline{\boldsymbol{\epsilon}_p}(\mathbf{x}) = \int_V \alpha(\mathbf{x},\boldsymbol{\xi})\boldsymbol{\epsilon}_p(\boldsymbol{\xi})\,d\boldsymbol{\xi} \tag{56}$$

where the local plastic strain $\boldsymbol{\epsilon}_p$ is obtained by integrating in time the rate $\dot{\boldsymbol{\epsilon}}_p$ evaluated from the standard expressions (45) and (40). As an alternative, Bažant and Lin (1988b) also proposed to average the rate of the plastic multiplier (45) and then substitute the nonlocal average

$$\overline{\dot{\lambda}}(\mathbf{x}) = \int_V \alpha(\mathbf{x},\boldsymbol{\xi})\dot{\lambda}(\boldsymbol{\xi})\,d\boldsymbol{\xi} \tag{57}$$

into Eq. (40) instead of $\dot{\lambda}$. They noted that this modification is computationally more efficient (since the averaged quantity is a scalar and not a tensor) and that it leads to very similar numerical results as the formulation with averaging of the plastic strain.

Fig. 15(a) shows four successive mesh refinements used in the tunnel excavation analysis, and the (exaggerated) deformation of the mesh caused by the excavation (Bažant and Lin 1988b) is plotted in Fig. 15(b). The contours of the strain-softening zone given in Fig. 15(c) demonstrate negligible differences, which confirm that averaging of the plastic strain or of the plastic multiplier indeed acts as a localization limiter. However, this particular model with

$$\boldsymbol{\sigma} = \mathbf{D}_e(\boldsymbol{\epsilon} - \overline{\boldsymbol{\epsilon}_p}) \tag{58}$$

turns out to be a special case of the nonlocal formulation with averaging of the inelastic stress, already discussed in the section on nonlocal adaptation of general constitutive models. In local plasticity, the expression for the inelastic stress reads

$$\mathbf{s} = \mathbf{D}_e\boldsymbol{\epsilon} - \boldsymbol{\sigma} = \mathbf{D}_e\boldsymbol{\epsilon} - [\mathbf{D}_e(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_p)] = \mathbf{D}_e\boldsymbol{\epsilon}_p \tag{59}$$

and the corresponding nonlocal inelastic stress is

$$\overline{\mathbf{s}} = \overline{\mathbf{D}_e\boldsymbol{\epsilon}_p} = \mathbf{D}_e\overline{\boldsymbol{\epsilon}_p} \tag{60}$$

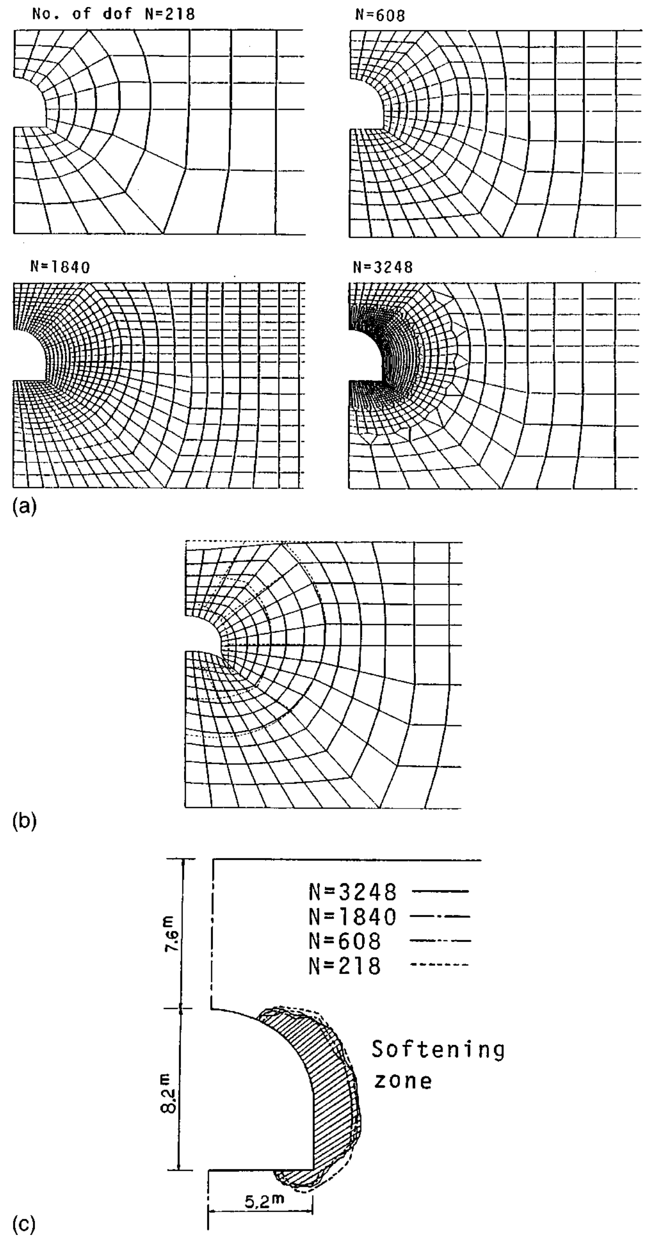


**Fig. 15.** Nonlocal plastic analysis of subway tunnel excavation in a grouted soil (after Bažant and Lin 1988b): (a) finite-element meshes, (b) deformed mesh with $N = 608$ degrees of freedom, and (c) contours of the strain-softening zone.

provided that the elastic stiffness $\mathbf{D}_e$ is constant in space. Consequently, this nonlocal plasticity model is expected to exhibit locking at later stages of the process of strain softening.

### Plasticity with Nonlocal Softening Variable

Perhaps the simplest nonlocal formulation of plasticity with isotropic strain softening is obtained if the current yield stress is computed from the nonlocal average of the softening variable. The softening law (38) is replaced by

$$\sigma_Y = \sigma_0 + h(\overline{\kappa}) \tag{61}$$

where

$$\overline{\kappa}(\mathbf{x}) = \int_V \alpha(\mathbf{x},\boldsymbol{\xi})\kappa(\boldsymbol{\xi})\,d\boldsymbol{\xi} \tag{62}$$

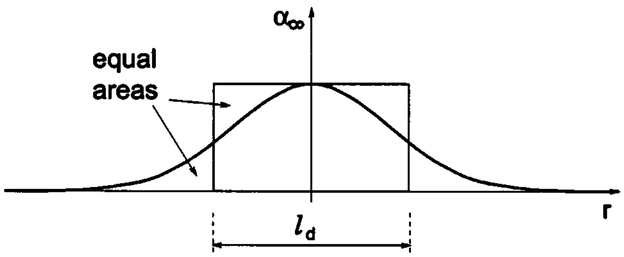


**Fig. 16.** Geometrical meaning of dissipation length $\ell_d$

is the nonlocal softening variable. The flow rule and the evolution law for the local softening variable $\kappa$ are kept unchanged.

As shown by Planas et al. (1993), this simple nonlocal plasticity model is, in fact, equivalent to a cohesive crack model (Hillerborg et al. 1976). The plastic strain localizes into a set of zero measure but, in contrast to the local model, the energy dissipation does not vanish. The total work $G_F$ spent by the complete failure (material separation) over a unit area of the cohesive crack under uniaxial tension is equal to the product $g_F \ell_d$, where $g_F$ is the area under the local stress–strain curve (work per unit volume in a local continuum) and $l_d$ is the equivalent dissipation length imposed by the nonlocal model. It can be shown that, for the present model,

$$\ell_d = \frac{1}{\alpha(\mathbf{x}_s, \mathbf{x}_s)} \tag{63}$$

where $\mathbf{x}_s$ = point at which the localized cohesive crack forms. Far from the boundary, $\ell_d = 1/\alpha_\infty(0)$ is a constant proportional to the internal length parameter of the nonlocal weight function. It has the geometrical meaning of the width of a rectangle that has the same area as the nonlocal weight function and the same height at the origin, see Fig. 16. In the proximity of a physical boundary, the dissipation length (63) decreases, which means that the local fracture energy is smaller than in an infinite body. It has been suggested (Hu and Wittmann 2000) that this could partially explain the size effect on the fracture energy observed in experiments with concrete and mortar.

### Formulation Combining Nonlocal and Local Softening Variables

The singular character of the strain distribution generated by the simple nonlocal plasticity model from the previous section is an exception. As a rule, nonlocal models lead to solutions characterized by a high degree of regularity. This is the case, for instance, for nonlocal damage and smeared crack models. For nonlocal softening plasticity, regularity can be achieved by a modification of the variable driving the yield stress degradation. Vermeer and Brinkgreve (1994) defined the variable $\overline{\kappa}$ that enters the softening law (61) as a linear combination of the local cumulative plastic strain and its nonlocal average:

$$\overline{\kappa(\mathbf{x})} = (1-m)\kappa(\mathbf{x}) + m \int_V \alpha(\mathbf{x},\boldsymbol{\xi})\kappa(\boldsymbol{\xi}) \; d\boldsymbol{\xi} \tag{64}$$

The same definition of the softening variable was also considered by Planas et al. (1996) and by Strömberg and Ristinmaa (1996).

Note that Eq. (62) is obtained as a special case with $m=1$. Another special case with $m=0$ corresponds to the local softening plasticity model. One might expect the typical values of $m$ used by the generalized model to lie between these two cases but, interestingly, the plastic region has a nonzero width only for $m > 1$ (which may be called an overnonlocal formulation).

The localized strain distributions in a one-dimensional tensile test obtained with various values of $m$ are plotted in Fig. 17. Far from the boundary, the distribution of plastic strain inside the plastic region is symmetric [Fig. 17(a)]. If the plastic region is adjacent to the physical boundary of the bar, the largest plastic strain is attained right on the boundary [Fig. 17(b)]. The width of the plastic region is an increasing function of parameter $m$ and tends to zero for $m \to 1^+$. For a given $m$, the width of the plastic region is proportional to the internal length imposed by the nonlocal averaging function. It is worth noting that the shape of the plastic strain profile does not depend on the softening modulus, and that the width of the plastic region does not change during the softening process (in the uniaxial test).

Since the yield stress in Eq. (61) depends on the nonlocal softening variable, the yield function and the loading–unloading conditions have a nonlocal character, and the stress evaluation cannot be performed at each Gauss integration point separately. This is the price to pay for the enhancement of the standard continuum description, which restores the well posedness of the boundary value problem and leads to numerical solutions that exhibit no pathological sensitivity to finite-element discretization.

### Thermodynamics-Based Symmetric Nonlocal Plasticity

One noteworthy aspect of the nonlocal models presented so far has not yet been mentioned—the fact that they yield nonsymmetric tangential stiffness matrices. This point was studied, and the cause explained for the nonlocal damage model (Bažant and

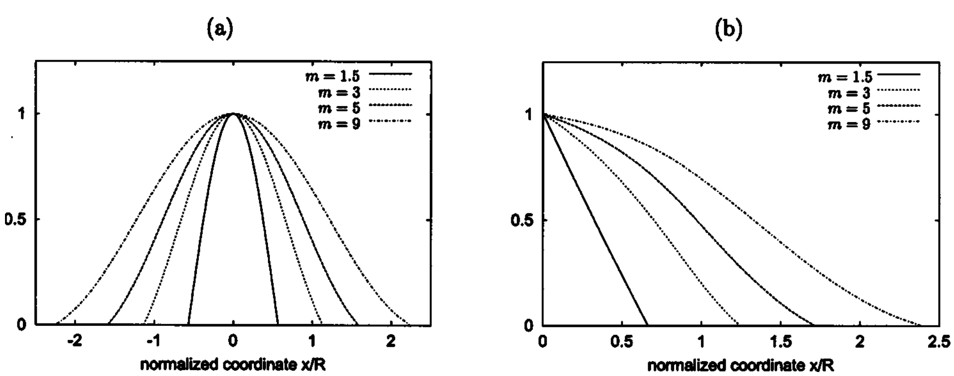


**Fig. 17.** Plasticity model combining local and nonlocal softening variables: Effect of parameter $m$ on the plastic strain profiles localized (a) inside the bar, and (b) at the boundary.

Pijaudier-Cabot 1988). The structure of the nonlocal stiffness matrix was described by Huerta and Pijaudier-Cabot (1994), and the numerical implementation and convergence rates were studied by Jirásek and Patzák (2002). No convergence problems attributed to nonsymmetry have so far been encountered in computations with the aforementioned models.

Modifications achieving symmetry would nevertheless be highly desirable, not only for theoretical reasons but also for improving computational efficiency of implicit solution schemes (through the use of more efficient equation solvers). A symmetric structure of nonlocal models can be obtained by an appropriate generalization of the postulate of maximum plastic dissipation (understood here in the global sense). The thermodynamic framework established for nonlocal elasticity by Edelen and Laws (1971) and Edelen et al. (1971) served as a basis for thermodynamically consistent formulations of nonlocal softening plasticity (Svedberg 1996; Svedberg and Runesson 1998; Polizzotto et al. 1998).

From the thermodynamic point of view, the elastic stress–strain law and the plastic hardening law are state laws that can be derived from a suitably defined free-energy potential. For models formulated within the framework of generalized standard materials (complying with the postulate of maximum plastic dissipation), the flow rule, the definition of the hardening variable and the loading–unloading conditions are complementary laws that can be derived from the dual dissipation potential defined as the indicator function of the set of plastically admissible states; for a detailed discussion, see, e.g., Jirásek and Bažant (2001, Chap. 23). The postulate of maximum plastic dissipation restricts the class of models covered by this framework to those that satisfy the conditions of convexity (of the yield function) and normality (of the plastic flow and of the evolution of hardening variables). For instance, models with nonassociated flow rules remain outside the class of generalized standard materials, even though they can still be thermodynamically admissible (Bažant and Cedolin 1991). Nevertheless, the postulate of maximum dissipation leads to a symmetric structure of the constitutive equations, and it is interesting to explore its possible extensions to nonlocal material models.

For plasticity with isotropic softening, Borino et al. (1999) considered the free-energy density in the form

$$\rho\psi(\boldsymbol{\epsilon},\boldsymbol{\epsilon}_p,\bar{\kappa})=\rho\psi_e(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_p)+\rho\psi_p(\bar{\kappa}) \tag{65}$$

where $\rho\psi_e(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_p)=\frac{1}{2}(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_p)^T\mathbf{D}_e(\boldsymbol{\epsilon}-\boldsymbol{\epsilon}_p)=$ elastically stored energy and $\rho\psi_p(\bar{\kappa})=$ plastic part of the free energy, usually interpreted as the energy stored in microstructural changes. The postulate of maximum plastic dissipation then leads to an associated nonlocal plasticity model with the softening law (38) replaced by

$$\sigma_Y=\sigma_0+\widetilde{h(\bar{\kappa})} \tag{66}$$

where $\bar{\kappa}=$ (usual) nonlocal average of the softening variable $\kappa$ (cumulative plastic strain), and the tilde over $h(\bar{\kappa})$ means that the thermodynamic force $q=\rho\partial\psi/\partial\bar{\kappa}=\rho d\psi_p/d\bar{\kappa}=h(\bar{\kappa})$ is subjected to the so-called dual averaging. The dual averaging operator is defined implicitly by the identity

$$\int_V \tilde{f}g\,dV=\int_V f\bar{g}\,dV \tag{67}$$

which must hold for all functions $f$ and $g$ for which the right-hand side makes sense. Since

$$\int_V f(\mathbf{x})\bar{g}(\mathbf{x})\;d\mathbf{x}=\int_V f(\mathbf{x})\int_V \alpha(\mathbf{x},\boldsymbol{\xi})g(\boldsymbol{\xi})d\boldsymbol{\xi}d\mathbf{x}$$

$$=\int_V\int_V \alpha(\mathbf{x},\boldsymbol{\xi})f(\mathbf{x})d\mathbf{x}\;\;g(\boldsymbol{\xi})d\boldsymbol{\xi} \tag{68}$$

we obtain $\tilde{f}(\boldsymbol{\xi})=\int_V\alpha(\mathbf{x},\boldsymbol{\xi})f(\mathbf{x})\;d\mathbf{x}$, or, equivalently,

$$\tilde{f}(\mathbf{x})=\int_V \alpha(\boldsymbol{\xi},\mathbf{x})f(\boldsymbol{\xi})d\boldsymbol{\xi} \tag{69}$$

This formula indicates that the dual averaging operator differs from the original (primal) one only in the order of arguments of the weight function $\alpha$. For an infinite domain $V_\infty$, the weight function is symmetric (because it depends only on $\|\mathbf{x}-\boldsymbol{\xi}\|$), and so there is no difference between the primal and dual averaging (thus the nonlocal operator is self-adjoint). For a finite domain $V$, the weight function is nonsymmetric in the proximity of the boundary, and $\tilde{f}$ is, in general, different from $\bar{f}$. In particular, the dual averaging operator does not satisfy the normalizing condition that a constant field $f$ is transformed into a constant field $\tilde{f}\equiv f$.

As pointed out by Rolshoven and Jirásek (2001), the nonlocal plasticity formulation with softening law (66) does not provide full regularization, similar to the simpler formulation with softening law (61). Plastic strain still localizes into a set of zero measure. This is not possible in the nonlocal plasticity theory proposed by Svedberg and Runesson (1998), who considered the free energy as a function of both the local and nonlocal internal variables. A particular model of this type defines the plastic part of the free energy in the form $\psi_p(\kappa,\bar{\kappa})=\psi_{p1}(\kappa)+\psi_{p2}(\bar{\kappa}-\kappa)$, i.e., as a sum of two terms that depend on the local softening variable and on the difference between the nonlocal and local softening variable, respectively. The consistent thermodynamic approach then leads to the hardening–softening law

$$\sigma_Y=\sigma_0+h_1(\kappa)+\widetilde{h_2(\bar{\kappa}-\kappa)}-h_2(\bar{\kappa}-\kappa) \tag{70}$$

where function $h_1\equiv\rho\psi'_{p1}$ describes softening under homogeneous conditions (uniform distribution of plastic strain) and function $h_2\equiv\rho\psi'_{p2}$ is the correction due to nonlocal effects. This model has been shown to give a localized plastic zone of a nonzero size (Borino and Failla 2000). However, its scope of application is limited; while it reasonably captures the initial localization profile, it cannot describe complete failure with zero residual resistance of the material. This can be easily explained for the uniaxial tensile test. If the material completely loses cohesion at late stages of the deformation process, the local softening law must be such that $\sigma_0+h_1(\kappa)=0$ for $\kappa$ exceeding a critical value $\kappa_c$. In the zone where the plastic strain exceeds $\kappa_c$, the actual yield stress (70) can vanish only if the nonlocal correction vanishes, i.e., if

$$\widetilde{h_2(\bar{\kappa}-\kappa)}=h_2(\bar{\kappa}-\kappa) \tag{71}$$

Sufficiently far from the boundary, the dual averaging operator is identical with the primal one. Arguments similar to those used by Jirásek and Rolshoven (2002) in their analysis of the Vermeer–Brinkgreve nonlocal plasticity model (previous section) lead to the conclusion that Eq. (71) is satisfied only if $h_2(\bar{\kappa}-\kappa)$ is constant, and this is possible only if $\bar{\kappa}-\kappa$ is constant, provided that the hardening function $h_2$ is monotonic. Applying the same reasoning once again, it can be shown that the difference $\bar{\kappa}-\kappa$ is constant across the plastic zone of a nonzero size only if $\kappa$ is constant along the entire bar. This means that when the plastic strain at the center of the localized plastic zone reaches $\kappa_c$, the plastic zone starts expanding and the strength degradation gets delayed, which is perceived as a stress locking effect. For nonlinear softening laws, this effect builds up gradually.

The foregoing thermodynamically based nonlocal plasticity has apparently not yet been applied in multidimensional finite-element codes. The same general idea was adapted to nonlocal damage by Benvenuti et al. (2000). Comi and Perego (2001) proposed a simpler formulation that permits an explicit evaluation of nonlocal averages, without the need for solving a nonlocal consistency condition, which has the character of an integral equation. However, since they used a primal averaging operator based on the scaled weight function (21), the dual operator did not preserve a uniform field, which caused certain problems in the vicinity of physical boundaries. A remedy was found by Borino et al. (2002), who constructed a self-adjoint operator that preserves a uniform field. The resulting thermodynamically based nonlocal damage model seems to perform well, but it still requires a two-fold nonlocal averaging, and so it is more computationally expensive than the simple nonlocal damage theory of Pijaudier-Cabot and Bažant (1987).

The thermodynamically based formulation is very attractive from the theoretical viewpoint, and the fact that it leads to symmetric stiffness matrices is an asset from the computational viewpoint. The dissipation inequality is automatically satisfied. However, there are two penalties to pay: (1) a greater complexity of the constitutive model; and (2) a broadened bandwidth in the stiffness matrix, caused by the double application of the nonlocal operator. Furthermore, while a very appealing theoretical framework has been developed, much of its physical foundation remains clouded.

The way to symmetrize the nonlocal model based on crack interactions is not known at present. It is even unclear whether an effort in that direction would be physically justified.

### Nilsson's Model

In his thesis (Nilsson 1994) and journal article (Nilsson 1997), Nilsson proposed a thermodynamically motivated nonlocal plasticity model that shares many similar features with the model of Borino et al. (1999). However, there are also some important differences. In the most general version of his model, Nilsson considered all the arguments of the free-energy potential as nonlocal. Since this would lead to a stress–strain law that is not easily invertible, Nilsson focused his attention on what he called the model with restricted nonlocality, which deals with total strain only in its local form. For isotropic softening, the free-energy density is written as

$$\psi(\boldsymbol{\epsilon},\overline{\boldsymbol{\epsilon}}_p,\overline{\kappa})=\psi_e(\boldsymbol{\epsilon}-\overline{\boldsymbol{\epsilon}}_p)+\psi_p(\overline{\kappa}) \tag{72}$$

In contrast to Eq. (65), the plastic strain that appears in the elastic part of the free energy is nonlocal. Consequently, the elastic part of the stress–strain law (39) is now replaced by

$$\boldsymbol{\sigma}=\mathbf{D}_e(\boldsymbol{\epsilon}-\overline{\boldsymbol{\epsilon}}_p) \tag{73}$$

The softening law (38) is the same as for the simple formulation of nonlocal plasticity, and the flow rule (40) and loading–unloading conditions (41) remain standard. However, this is not consistent with the postulate of maximum plastic dissipation, as pointed out by Borino and Polizzotto (1999) and admitted by Nilsson (1999). It even turns out that, under certain circumstances, the global dissipation could become negative (Jirásek and Rolshoven 2002), which means that the model is not really thermodynamically consistent. Moreover, the plastic strain again localizes into a set of zero measure, similar to the model with a nonlocal softening variable from the section on plasticity with nonlocal softening variable.

### Nonlocal Version of Dislocation-Based Gradient Plasticity for Micrometer Scale

While some numerical experts assert that the gradient models are more friendly to a computer programer than the nonlocal averaging models, Gao and Huang (2001) recently took the opposite view. For the purpose of finite-element analysis, they converted the dislocation-based gradient plasticity model for metals on the micrometer scale (Gao et al. 1999; Huang et al. 2000) to a form in which the strain gradient is approximated by a nonlocal integral over a representative volume. A generalized form of their idea is as follows.

Within a certain representative volume $V_c$ surrounding a given point $\mathbf{x}$, the strain tensor may be approximated as $\boldsymbol{\epsilon}(\mathbf{x}+\mathbf{s})\approx\boldsymbol{\epsilon}(\mathbf{x})+\boldsymbol{\epsilon}_{,m}(\mathbf{x})s_m$, where the subscript preceded by a comma denotes a partial derivative and repeated indices imply summation. Multiplying this equation by $s_k$ and integrating over $V_c$, one gets

$$\int_{V_c}\boldsymbol{\epsilon}(\mathbf{x}+\mathbf{s})s_k d\mathbf{s}\approx\boldsymbol{\epsilon}(\mathbf{x})\int_{V_c}s_k d\mathbf{s}+\boldsymbol{\epsilon}_{,m}(\mathbf{x})\int_{V_c}s_m s_k d\mathbf{s} \tag{74}$$

The integrals on the right-hand side characterize the geometry of the representative cell. If point $\mathbf{x}$ is located at the center of gravity of the cell, the first-order moments $\int_{V_c} s_k\, d\mathbf{s}$ vanish. The second-order moments

$$I_{mk}=\int_{V_c}s_m s_k d\mathbf{s} \tag{75}$$

are components of the inertia tensor, which is always regular, and thus invertible. In the simplest case of a cubic or spherical cell, $I_{mk}$ is a multiple of the unit tensor. Multiplying Eq. (74) from the right by the inverse tensor $I_{kl}^{-1}$, we obtain for the strain gradient the approximation

$$\boldsymbol{\epsilon}_{,l}(\mathbf{x})\approx\mathbf{I}_{kl}^{-1}\int_{V_c}\boldsymbol{\epsilon}(\mathbf{x}+\mathbf{s})s_k d\mathbf{s} \tag{76}$$

If the strain gradient terms in a gradient plasticity model are replaced by this approximation, the numerical solution can be programed in the same manner as the nonlocal averaging models.

In strain-gradient plasticity, the size of the representative cell is dictated by the characteristic length

$$\ell_g=\alpha_g(G/\sigma_0)^2 b \tag{77}$$

where $G$ = elastic shear modulus, $\sigma_0$ = yield stress, $b$ = Burgers vector of edge dislocation, and $\alpha_g$ = dimensionless semiempirical constant of the order of 1 (Gao et al. 1999). Alternatively, one could consider the approximation of the gradient by an integral as a purely numerical procedure and select $V_c$ as the smallest neighborhood of the given point $\mathbf{x}$ that contains a sufficient number of material points traced by the finite-element program. Eq. (76) can then be interpreted as a generalized finite-difference scheme. In one dimension, the minimum required number of integration points would be 2, and if these points are located symmetrically with respect to $x$, Eq. (76) yields the standard finite-difference formula

$$\epsilon'(x)\approx\frac{\epsilon(x+h)-\epsilon(x-h)}{2h} \tag{78}$$

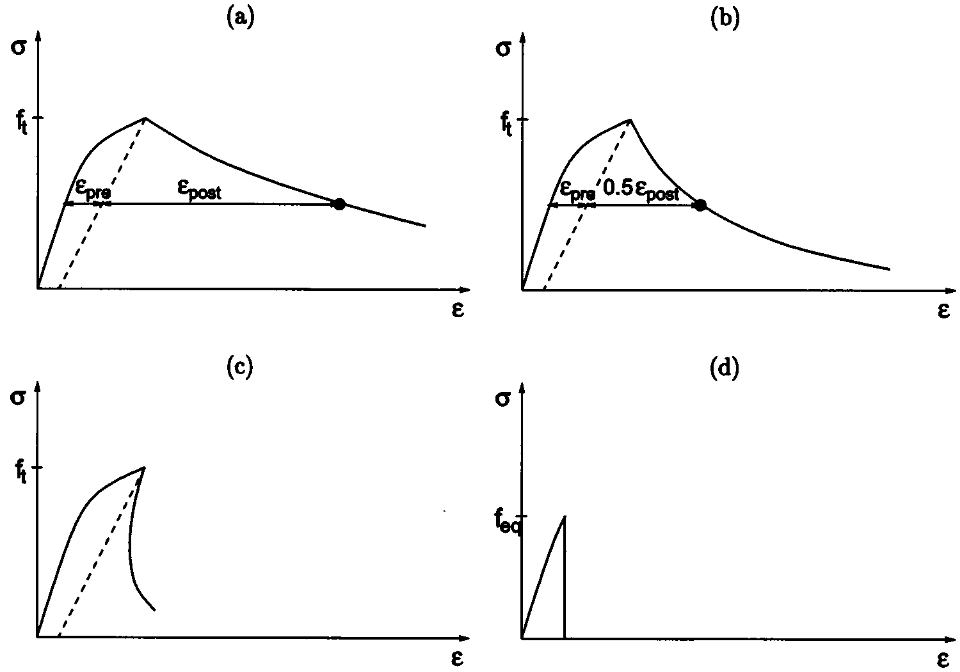


**Fig. 18.** Adjustment of the stress–strain diagram according to the crack band approach: (a) master curve and decomposition of strain into the prelocalization and postlocalization parts, (b) horizontal scaling of the postlocalization part for an element twice as large as the physical process zone, (c) snap back occurring for a very large element, and (d) conservation of fracture energy by strength reduction.

However, in a general case (with more integration points or irregular point arrangement), this approach does not provide the optimal accuracy, and it would be preferable to replace $s_k$ in Eq. (74) by a suitably modified weight function.

## Partial Regularization and Substitutes for Nonlocality

### *Crack Band Model*

The crack band model (Bažant 1982; Bažant and Oh 1983) is (aside from viscosity) practically the simplest, and computationally most effective, way to avoid the pathological sensitivity to mesh refinement. Since the beginning, it has been the most widely used model for distributed cracking of concrete and geomaterials and has been incorporated in a number of commercial finite-element codes. The basic idea of the crack band model is very similar to the mesh-adjusted softening modulus technique, proposed in the context of softening plastic shear bands by Pietruszczak and Mróz (1981). Essentially, the same approach was developed by Willam et al. (1986) under the name of the composite fracture model.

This simple model is endowed with some, but not all, of the characteristics of nonlocal models. It ensures the correct energy dissipation in a localized damage band (or equivalent fracture), and it gives the correct transitional size effect. The main difference from the nonlocal models is that the nonlocal spatial averaging is replaced by an energy-based rescaling of the postlocalization part of the stress–strain relation, which takes into account the density of the computational mesh (size of the finite elements). The global load–displacement diagram can be captured correctly, but the width of the numerically resolved fracture process zone depends on the element size and tends to zero as the mesh is refined. This is why the crack band model (or fracture energy approach) cannot be considered as a true localization limiter. It provides only a partial regularization of the problem, in the sense that the global response characteristics do not exhibit spurious mesh sensitivity. However, the mesh-induced directional bias is still present.

The crack band approach is based on the observation that, if a local softening model is used, the numerically resolved process zone typically localizes into one layer of finite elements, called the crack band. The effective width of the crack band, $h_{ef}$, can be estimated using the rules proposed by Bažant (1985b) and refined by Rots (1988). A reasonable estimate is obtained by projecting one element onto the direction normal to the band, which is (for quasibrittle failure) usually close to the direction of maximum principal strain. The user specifies the stress–strain relation for the basic case when $h_{ef}$ is equal to the actual effective width of the physical fracture process zone, $L_s$, which is proportional to the intrinsic material length $\ell$. As far as possible, the size of the finite elements in the strain-softening distributed damage zone is kept equal to $L_s$. But if, for reasons of computational efficiency, the finite elements need to be larger, then the postlocalization portion of the master constitutive diagram (on which the tensorial constitutive law is based) is scaled horizontally (i.e., in the direction of the strain axis) by the factor $L_s/h_{ef}$; see Fig. 18(b). This makes the strain softening steeper and a proper scaling ratio achieves that the energy dissipated by the cracking band per its unit advance remains the same. Note that the postlocalization portion is the difference, for the same stress, of the strain at the softening curve from the strain at the unloading diagram emanating from the point at which localization occurs (typically the peak), not from a vertical line dropping down from that point [Fig. 18(a)]. In the simple case of a triangular stress–strain diagram with linear strain softening in uniaxial tension characterized by the tangent modulus $E_t<0$, the adjusted postpeak tangent modulus $\tilde{E}_t$ must satisfy the condition (Bažant 1982; Bažant and Oh 1983)

$$-\frac{1}{\tilde{E}_t}=\frac{2G_f}{h_{ef}f_t^2}-\frac{1}{E} \tag{79}$$

where $E$ = initial Young's modulus and $G_f=L_s(E^{-1}-E_t^{-1})f_t^2/2$.

The increase of finite-element size, however, has a limit. For a certain large enough critical element size, $h_{\text{crit}}$, the adjusted postpeak diagram develops a snapback [Fig. 18(c)], which is inadmissible (the stress would cease to be a unique functional of the strain history). To avoid the spurious snapback, one can use a vertical stress drop and scale down the strength limit $f_t$ to some equivalent value $f_{eq}$ such that $h_{ef}$ times the area under the stress–strain diagram with the vertical drop would remain equal to $G_f$ [Fig. 18(d)]. It turns out that, generally, $f_{eq}\propto 1/\sqrt{h_{ef}}$, and for the special case of a triangular stress–strain diagram

$$f_{eq}=f_t\sqrt{2l_0/h_{ef}} \tag{80}$$

where $l_0=EG_f/f_t^2$ = Irwin-type characteristic length of the material. This expedient was proposed by Bažant and Cedolin (1979) and Cedolin and Bažant (1980), and good practical results with finite elements much larger than $L_s$ were obtained in situations where failure is driven by fracture.

To model a very narrow cracking band, or to approximate a line fracture, finite elements smaller than $L_s$ may be used in the crack band model. The same scaling of the postpeak diagram can again be applied, making the strain softening less steep. As the mesh is refined, the band of cracking elements converges to a layer of zero thickness, i.e., to a line (in two dimensions) or surface (in three dimensions). Thus, the limit corresponds to the solution of a cohesive crack model. The mathematical aspects of this correspondence were studied by Simo et al. (1993).

Scaling of the stress–strain diagram in the above sense is straightforward only for models that explicitly control the evolution of inelastic strain, e.g., for softening plasticity or smeared crack models. In that case, the desired scaling effect is achieved by a modification of the hardening modulus (derivative of stress with respect to inelastic strain). In continuum damage mechanics, nonlinearity and softening are controled by the damage evolution law, and the reduction factor $1-\omega$ multiplies the total strain. It is, therefore, not easy to scale only the postlocalization part of strain while keeping the unloading part unaffected. An exact scaling procedure leads to a nonlinear equation (except for the special case of linear softening), which must be solved iteratively during each stress evaluation. It is also difficult to extend the crack band approach to mixed-mode failure or failure under complex three-dimensional stress states. In such general cases, the standard fracture energy is not sufficient to characterize the dissipation in localized failure modes. Generalized forms of the crack band approach that cover a variety of failure modes have been proposed, e.g., for softening plasticity models (Etse and Willam 1994; Kang 1997) and for the microplane constitutive model M4 for concrete (paper in preparation by Bažant, Zi, Jendele, and Novák).

Whereas the nonlocal models with at least three finite elements across the width of the localization band are the best known way to avoid mesh orientation bias for the propagation direction (in fact, better than the known mesh-adaptive schemes), the crack band model is poor in this regard. It performs best if the path of the cracking band (or the fracture to be approximated by it) is known in advance and if the mesh is laid out so that a mesh line would coincide with this path. If the crack path is not known, one should solve the problem with meshes of different inclinations. The most reasonable solution is usually that providing the smallest peak load or the steepest postpeak load deflection diagram.

A caveat needs to be mentioned with respect to those infrequent situations where the strain-softening damage does not localize, being stabilized, for example, by a heavy enough reinforcement net or by an adjacent layer of compressed material (see the examples mentioned later in the section on limitations of cohesive models). In such situations, the postpeak strain-softening behavior must not be rescaled. So, in using the crack band approach, the user must separately assess whether it is reasonable to expect localization. However, in some cases, diffuse softening damage patterns in some parts of the structure can coexist with localized cracks in other parts, and these parts may even change during the loading process. In such cases it is then next to impossible to define a reasonable rule for the adjustment of the stress–strain diagram according to the element size. A consistent solution can be obtained only when finite-element sizes $h=L_s$ can be used (in which case no rescaling is needed for the crack band approach), or with a fully regularized nonlocal model.

As explained before, the crack band model uses a modification of the postpeak part of the constitutive law enforcing the correct energy dissipation by a localized crack band. When large finite elements need to be used (larger than the actual width of the fracture process zone), the resolution of the localization process can be improved by special enrichments of the finite-element approximation that allow capturing narrow bands of highly localized strain inside the elements. The pioneering paper by Ortiz et al. (1987) paved the way to special elements with embedded localization bands (Belytschko et al. 1988; Sluys 1997; Oliver et al. 1998). These elements can describe a layer of softening material separated from the surrounding, elastically unloading material by two parallel weak discontinuity planes across which the displacement remains continuous but the out-of-plane strain components can have a jump. The width of the band can be specified independently of the mesh (provided that the elements are not too small), and it can be set equal to the actual width of the process zone, $L_s$, considered as a material parameter. A one-dimensional element of this kind would give exactly the same response (in terms of the relationship between the nodal displacements and nodal forces) as a standard element with the softening modulus adjusted according to the element size, as described for the crack band model. For two- and three-dimensional elements, the formulation with embedded discontinuities can better reproduce the kinematics of highly localized deformation modes and thus avoid certain locking effects (Jirásek 2000a).

In a recent study (Bažant et al. 2001; Červenka et al. 2002), a new model for a damage localization layer within a finite element has been developed. For the special case of an orthogonal brick element (whose nodal displacements are compatible with uniform strain), the model reduces to a combination of one brick element for the strain-softening localization band and another brick element for the parallel layer of elastically unloading material, such that the interface conditions of stress equilibrium and of kinematic compatibility are exactly satisfied. In the general case of an arbitrary finite element containing a localization layer of arbitrary orientation, the incremental stress–strain relation is assumed to be the same as in an "equivalent" orthogonal brick element that has the same total volume, and the same volume and width of the localization layer. The interface conditions (numbering 6) are solved within each loading step or iteration, together with iterations for the nonlinear constitutive law. Implementation of this localization element together with microplane model M4 for concrete (Bažant et al. 2000) led to good results for various localization problems.

### Cohesive Crack and Cohesive Zone Models

#### Traction-Separation Law

The cohesive crack model for concrete, formulated by Hillerborg et al. (1976) under the name of fictitious crack model, represents a refinement of the classical cohesive models, with the main difference that a cohesive crack is not considered to initiate and propagate along a predetermined path but is assumed to initiate anywhere in the structure if the tensile stress reaches the strength limit. Although this approach is not really a nonlocal model, it provides an alternative to achieve one objective of the nonlocal models—a mesh-size independent response in the presence of strain-softening damage or fracture. The early applications in the field of concrete fracture were limited to mode-I situations. An extension to cracks and interfaces opening under mixed-mode conditions was developed, e.g., by Červenka (1994), who exploited a loading function in the space of normal and shear tractions, originally proposed by Carol and Prat (1990) in the context of statically constrained microplane models. Scaling properties and asymptotic solutions of cohesive crack models were studied by Planas and Elices (1992, 1993) and Bažant and Li (1995).

Phenomena such as debonding, delamination, or intergranular damage in composites and metals are often described by cohesive zone models, going back to the ideas of Dugdale (1960) and Barenblatt (1962) and recently developed in a modern computational framework by Needleman (1987), Tvergaard and Hutchinson (1992), Tvergaard and Hutchinson (1993), and Ortiz and coworkers (Camacho and Ortiz 1996; Pandolfi et al. 1999; Ortiz and Pandolfi 1999).

Cohesive models describe highly localized inelastic processes by traction-separation laws that link the cohesive traction transmitted by a discontinuity line or surface to the displacement jump, characterized by the separation vector. The physical interpretation and the numerical implementation of such models can be manifold. The discontinuity surface can be an internal boundary between two different materials (e.g., a matrix-inclusion interface), a plane of weakness (e.g., a rock joint), or a simplified representation of a narrow localization zone (e.g., of a fracture process zone or a shear band). Some formulations assume a nonzero initial elastic compliance of the cohesive zone, which may be either physically motivated by the reduced stiffness of the interface layer as compared to a perfect bond between two materials, or considered as a purely numerical artifact corresponding to a penalty-type enforcement of displacement continuity in the elastic range. Other formulations enforce displacement continuity directly, and allow a nonzero separation only after a certain initiation condition has been met.

Cohesive models for which the traction-separation law begins with a positive slope (i.e., models with a nonzero elastic compliance, or models with softening preceded by hardening) must be used with care. If the stress increases above the initiation limit $f_0$ (which is equal to zero for models with a nonzero elastic compliance), it must also increase above $f_0$ at some material points sufficiently close to the crack face, yet a new cohesive crack cannot initiate at that point. Because of this inconsistency, cracks described by such cohesive models cannot be assumed to initiate at an arbitrary location. Their path must be specified in advance or restricted to a predefined finite number of segments.

Needleman's original model is based on an elastic potential without any internal variables, i.e., it has the character of a nonlinear elastic model with a path-independent work of separation. Such a simple approach is suitable only if the crack opening grows monotonically, but it fails to give reasonable results for (even partially) closing cracks. Moreover, the model has a nonzero initial compliance, which means that the displacement jump starts growing whenever nonzero tractions are applied, no matter how small they are. This is appropriate for simulations of preexisting material interfaces (grain boundaries, matrix-inclusion interfaces, etc.) with a well-defined geometrical structure that can be taken into account by the finite-element mesh. When the aim is to simulate a crack propagating along an arbitrary path that is not known in advance, potential surfaces (or lines, in two dimensions) of decohesion must be interspersed throughout the material (Xu and Needleman 1994). Their high initial stiffness has an adverse effect on the conditioning of the global stiffness for implicit methods, or on the critical time step for explicit methods. Moreover, due to the finite number of potential discontinuity segments, the crack propagation path is locally always constrained to a discrete set of directions, typically spaced by 45 or 60°.

A cohesive zone model with a nonzero initial compliance was also used by Tvergaard and Hutchinson (1993) and Wei and Hutchinson (1999), who studied the interplay between plastic yielding in a small process zone and separation processes at an interface between two materials.

Ortiz (1988) suggested a derivation of the traction-separation law for mode I from a micromechanical model based on an array of collinear microcracks. Camacho and Ortiz (1996) studied fracture and fragmentation in brittle materials using a cohesive model with possible crack initiation under mixed mode. The model was then reformulated within a thermodynamic framework (Pandolfi et al. 1999) and extended to the area of finite opening displacements (Ortiz and Pandolfi 1999). Due to insistence on a simple symmetric formulation with a potential, the resultant of the shear and normal stresses in the cohesive crack has in these models the same direction as the relative (normal and shear) displacement across the crack. For shear with a nearly vanishing normal stress, the relative displacement is predicted to be nearly parallel to the crack plane. This means that the dilatancy due to shear, observed in most nonmetallic materials, is not taken into account (at least not on the element level, although some dilatancy gets produced globally if the sliding interelement crack has a zig-zag path).

#### Discretization Techniques for Cohesive Models

Discretization techniques used in conjunction with cohesive models can be divided into two broad categories, depending on whether the displacement discontinuity can occur only between adjacent elements, or can run across the finite-element mesh along an arbitrary trajectory.

In the first category, one can distinguish methods that assume an a priori given trajectory of the discontinuity, methods that trace the trajectory by continuous remeshing, and methods that admit a potential discontinuity between any two adjacent finite elements.

The second category consists of finite-element formulations with embedded discontinuities, based on the enhanced assumed strain method, and extended finite elements based on the partition-of-unity method.

In analogy to elements with embedded localization bands, which use discontinuous strain approximations, it is possible to construct elements with embedded localization lines (in 2D) or planes (in 3D), across which the displacement field is discontinuous (Dvorkin et al. 1990; Klisinski et al. 1991; Olofsson et al. 1994; Simo and Oliver 1994; Oliver 1996). For this numerical scheme, the trajectory of the cohesive crack or cohesive zone is independent of the finite-element mesh. This technique is much more flexible than the standard scheme with discontinuities allowed only at element interfaces, and it eliminates the need for continuous remeshing.

A vast majority of embedded crack formulations developed in the 1990s use a nonconforming approximation of the displacement jump. On one hand, this is convenient, because the approximation of the displacement jump in one element can be completely decoupled from the other elements and eliminated on the local level. However, the elements then become sensitive to the orientation of the discontinuity line with respect to the nodes, and in some unfavorable situations the response of the element ceases to be unique (Jirásek 2000b).

A much more robust implementation can be achieved if the discontinuous enrichment is based on the partition-of-unity concept (Melenk and Babuška 1996; Duarte and Oden 1996), with a conforming approximation of the displacement jump and completely independent strain components in the two parts of the element separated by the crack (Wells 2001). This formulation is intimately related to the manifold method, developed by Shi (1991) for preexisting material discontinuities such as rock joints, and to the extended finite-element method (XFEM), developed by Belytschko and co-workers for stress-free cracks and material interfaces (Moës et al. 1999; Sukumar et al. 2000; Daux et al. 2000) and adapted by Moës and Belytschko (2002) for cohesive cracks.

The traction-separation laws for embedded discontinuities are usually postulated in a plasticity format (Simo and Oliver 1994; Oliver 1996; Olofsson et al. 1994; Armero and Garikipati 1995; Larsson et al. 1996; Ohlsson and Olofsson 1997) or constructed as extensions of the cohesive crack model to mixed-mode situations (Dvorkin et al. 1990; Klisinski et al. 1991; Lotfi and Shing 1995) using concepts similar to fixed crack versions of smeared crack models. However, the plasticity-based models do not properly describe unloading of a brittle material. This problem becomes especially severe at late stages of the degradation process when the crack is stress free and, according to the plasticity theory, a reversal of the opening rate immediately generates a compressive traction, which is not physically realistic. The theories inspired by the cohesive crack model for concrete are close to interface damage mechanics (Simo and Oliver 1994; Oliver 1996; Armero 1997), which provides a natural description of the gradual loss of integrity. In this framework, a difficult issue is the proper treatment of stiffness recovery upon complete crack closure, with possible frictional sliding. Such effects were consistently taken into account by Cangemi et al. (1996) and by Chaboche and coworkers (Chaboche et al. 1997a,b) in the context of interface models for delamination and debonding of fiber–matrix composites. For cracks, a similar approach was proposed by Jirásek (1998a) and Jirásek and Zimmermann (2001).

## Limitations of Cohesive Models

The cohesive crack or cohesive zone models provide an objective description of fully localized failure. In this case, the cohesive traction-separation law with softening does not need any adjustment for the element size, because mesh refinement does not change the resolved crack pattern. However, the mesh independence is questionable if the cracking pattern is diffuse. This can happen, e.g.,

1. in concrete with a sufficiently dense and strong reinforcing net;
2. on the tensile side of a reinforced concrete beam, where distributed cracking is stabilized by the tensile reinforcement;
3. in a system of cooling (or drying) cracks propagating into a halfspace, especially if the temperature profile has a steep front (Bažant and Ohtsubo 1977; Bažant and Cedolin 1991, Sec. 12.5);
4. in dynamic problems when stress waves travel across the structure; and
5. if heterogeneity of the material is explicitly taken into account as the mesh is refined.

Mesh refinement can cause parallel potential cohesive cracks to become arbitrarily close. Thus, if they are under the same stress, their cumulative opening displacements per unit volume of material (if uniformly stressed) can become arbitrarily large in such situations. Such behavior was illustrated by Bažant's example (Bažant 1985a,b, 1986) of a concrete bar under uniaxial tension, with strong enough axial steel reinforcement such that the overall tangential stiffness of the bar never becomes negative, despite softening in the cracks (same as Fig. 14).

Consequently, cohesive models with an assumed mesh-independent softening law are applicable only when the strain-softening damage is a priori known to localize. In a certain sense, they are complementary to continuum-based models, for which a mesh-independent stress–strain law with softening can cover only the cases when cracking remains perfectly distributed. In a general case with regions of diffuse and localized cracking (and perhaps with gradual transition from diffuse damage to localized fracture), both classes of models would need an adjustment taking into account the element size, shape, and orientation.

For example, the model developed at LCPC Paris (Rossi and Wu 1992) works with elastic elements separated by cohesive interfaces, and the material parameters describing those interfaces are considered as random variables. To obtain mesh-independent results, at least in the global sense, it is necessary to adjust the statistical characteristics of the random strength distribution to the size of the elastic elements. For small elements, the strength characterizes the properties on a small scale, and it has a large variation. For large elements, the strength should characterize the overall behavior on a larger scale, resulting from the combined effect of small-scale processes that cannot be captured explicitly on the given resolution level. Experimental size effect investigations indicate that both the mean value and the variation of strength decrease as the size of the sample or specimen increases. The empirically derived rule for the size dependence of the statistical characteristics of random strength distribution developed at LCPC seems to give globally objective results when used in simulations on different meshes.

Another limitation of the cohesive cracks is that they cannot capture stress multiaxiality in the fracture process zone. The usual cohesive crack or cohesive zone models postulate a traction-separation relation that takes into account only the out-of-plane components of the stress tensor at the discontinuity surface. This approach suits just fine all the notched fracture specimens in common use but is not representative of many applications. The effect of the compressive normal stresses acting parallel to the crack plane (called sometimes the "$T$ stresses") is ignored. The fact that this cannot be universally correct is clear from the observation that if one of the $T$ stresses approaches the uniaxial compressive strength $f_c$ of the material, the strength limit of the cohesive crack model should approach zero. Thus, for general situations, a multiaxial cohesive crack model would have to be developed. Even though this would, in principle, be possible, the multiaxial behavior in the fracture process zone can be captured more easily by using a nonlocal model or crack band model with a good triaxial constitutive law.

Finally, since the softening damage in the early stage of opening of a cohesive crack occurs within a band of a certain width, the cohesive stress at location *x* should depend not merely on the crack opening at *x* but on the average of crack opening taken over a certain neighborhood of *x*. In other words, the cohesive law itself may need to be considered as nonlocal (Bažant 2001, 2002b), which was taken into account in the gradient-enhanced formulation of the cohesive crack model proposed by van Gils (1997).

### Regularization by Real or Artificial Viscosity

In dynamics, nonlocal averaging can sometimes be avoided, as a convenient approximation, by considering material viscosity, rate effect, or damping. Introducing artificial viscosity may serve as an expedient substitute for nonlocality.

Indeed, since the dimension of viscosity $\eta$ is kg/(m s), the dimension of Young's modulus $E$ (or strength) is kg/(m s$^2$), and the dimension of mass density $\rho$ is kg/m$^3$, there exists a material length associated with viscosity, given by

$$\ell_v = \frac{\eta}{v\rho} = \frac{\eta}{\sqrt{E\rho}} \tag{81}$$

where $v = \sqrt{E/\rho}$ = longitudinal wave propagation velocity. Consequently, any rate dependence in the constitutive law combined with inertial effects introduces a length scale. This effect was exploited as a localization limiter regularizing the boundary value problem, for example, by Needleman (1988). In a similar fashion, a length scale is introduced by the delay effects in the damage evolution law (Ladevèze 1992, 1995; Ladevèze et al. 2000).

There is, however, an important difference from the nonlocal models—the viscosity-induced nonlocality gradually disappears with the passage of time. For load durations much larger than the relaxation (or retardation) time associated with the type of viscosity used, the modeling is not completely objective. Thus, the viscosity or rate effect can be used as a substitute for a nonlocal model only within a narrow range of time delays and rates, generally not spreading more than one order of magnitude. Therefore, if an artificial viscosity is used, cautious insight is needed.

If a characteristic length, $\ell_v$, is considered in dynamic problems, then a characteristic time $\tau_0$ is automatically implied as the time of passage of a wave front over the distance $\ell_v$:

$$\tau_0 = \frac{\ell_v}{v} = \frac{\eta}{v^2\rho} = \frac{\eta}{E} \tag{82}$$

In problems of impact of missiles on concrete walls, for example, the duration of the dynamic event is not too long compared to $\tau_0$. Therefore, nonlocal averaging is generally not needed. Due to inertia effects, there is not enough time for spurious localization of strain-softening damage to develop, whether or not any viscosity or damping is taken into account.

If the viscosity is introduced in the constitutive law in such a manner that there is no softening for fast enough loading [e.g., Bažant and Li's (1997) model for the rate dependence or cohesive cracks in concrete], then the boundary value problem is regular (well posed) for fast enough loading even in the absence of inertia forces (i.e., in quasistatics), simply because there is no strain softening in the structure.

## Nonlocal Probabilistic Models of Failure

It is impossible to do justice to the subject without pointing out the importance of nonlocality in probabilistic modeling of failure. The evolution of the classical probabilistic theory of strength, begun in qualitative terms by Mariotte (1686), culminated with the studies on extreme value statistics and discovery of Weibull distribution by Fischer and Tippett (1928), and the application of this distribution to fracture by Weibull (1939). This theory rests on the hypothesis that the structure fails if a material element that is infinitesimal compared to structure size $D$ fails. The theory works just fine for fatigue-embrittled metals and fine-grained ceramics. It is noteworthy that since the size effect of this statistical theory is a power law, self-similarity is implied, and no characteristic structure size $D_0$ exists. The model possesses no characteristic length. This is not realistic for quasibrittle materials.

This deficiency gets manifested in structures that do not fail immediately after the failure of one infinitesimal element. It requires modification for quasibrittle structures, consisting of materials such as concrete, composites, or ice, which are characterized by a sizable fracture process zone (FPZ), not negligible compared to the structure size $D$. Two types of statistical behavior must be distinguished:

1. The structure fails only after a long fracture or a long damage band develops, which is typical of reinforced concrete, fiber composites, and compression dominated problems. The path of the fracture or band and the location of the FPZ are dictated mainly by mechanics on the macroscale and are little sensitive to material randomness when the structure is large. Within the FPZ the material behaves randomly, but if the size of the FPZ, which is of the same order as the intrinsic material length $\ell$, is negligible compared to structure size $D$ (i.e., if $D \gg \ell$), the statistical influence disappears and the size effect is purely deterministic (Bažant and Xi 1991). The classical Weibull theory cannot treat the large size limit for which the fracture or band front, in relative coordinates, becomes perfectly sharp. The reason is that the Weibull integral, see Eq. (83) below, diverges for a singular stress distribution, which is approached as $D/\ell \to \infty$.

2. The structure fails at the initiation of fracture from the surface, which is exemplified by flexural failure of an unreinforced beam. If the structure is not large compared to $\ell$, the stress redistribution in the cross section engendered by the FPZ is important and dominates the failure load. The statistical effects are then negligible, and the classical Weibull statistical theory does not apply. However, for $D \gg \ell$, the FPZ relative to $D$ becomes a point, and the structure fails as soon as the first point fails. Since this point can have many random locations, the statistical effect on failure is important, and in this limit case the Weibull theory does apply. The larger the structure, the smaller is the strength that can be encountered by this randomly located point.

The nonlocal concept has been shown to provide a remedy to the abovementioned problems. In the simplest theory, which deals only with the failure load and ignores the randomness of the deformations and stresses prior to failure, the probability of structural failure, $p_f$, is given by the Weibull integral whose integrand involves the (local) maximum principal stress $\sigma_I(\mathbf{x})$. The nonlocal remedy simply consists in replacing $\sigma_I(\mathbf{x})$ with the nonlocal maximum principal stress $\tilde{\sigma}(\mathbf{x})$. Thus, the Weibull integral over structure volume (in the case of tensile failures) takes the form (Bažant and Xi 1991; Bažant and Novák 2000b):

$$p_f = 1 - \exp\left\{ -\int_V \left\langle \frac{\tilde{\sigma}_I(\mathbf{x})}{\sigma_0} \right\rangle^m \frac{d\mathbf{x}}{V_r} \right\} \tag{83}$$

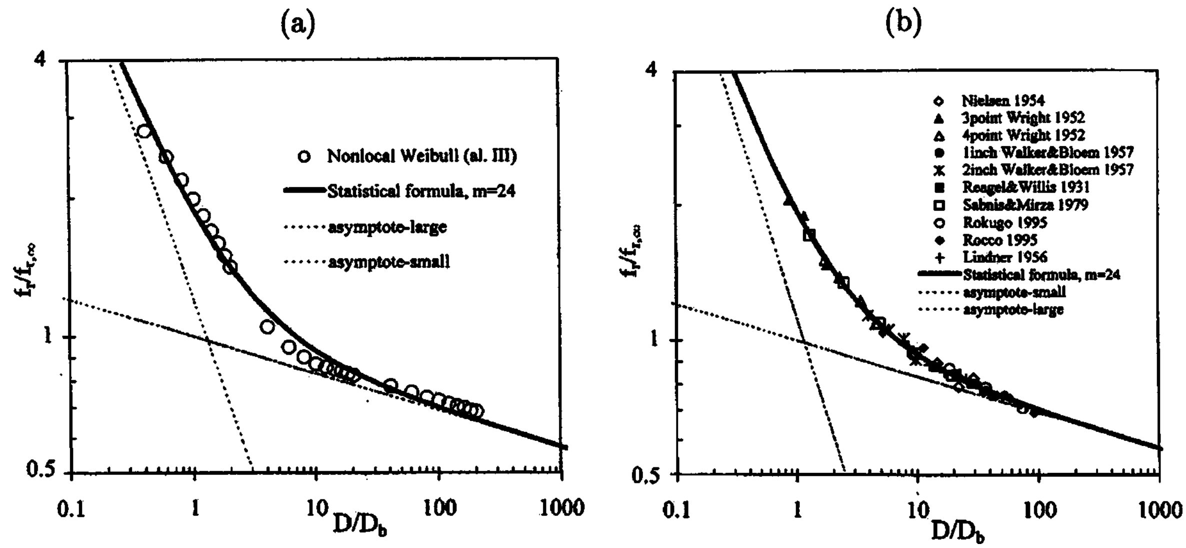


**Fig. 19.** Logarithmic size effect plots of (a) the numerical results obtained with the nonlocal Weibull-type probabilistic theory, and (b) previously published test data on the modulus of rupture of concrete, $f_r$ (i.e., flexural strength of unreinforced beams), and their fit by an asymptotic formula [in (b), the optimum fit is obtained first for each data set separately and then the data are plotted in relative coordinates]; after Bažant and Novák (2000a).

where $m$ and $\sigma_0$ are material constants; $m$ is the Weibull modulus ($m \approx 24$ for concrete), and $\sigma_0$ is the Weibull scale parameter, with the dimension of stress. The nonlocal stress cannot be defined simply as the average stress. Rather, it must be defined in terms of the nonlocal inelastic stress or nonlocal inelastic strain. Choosing the latter, one has

$$\tilde{\boldsymbol{\sigma}} = \mathbf{D}_e(\boldsymbol{\epsilon} - \overline{\boldsymbol{\epsilon}_i}) \tag{84}$$

where $\boldsymbol{\epsilon}_i$ is the inelastic strain and $\overline{\boldsymbol{\epsilon}_i}$ is its nonlocal average. For details, consult Bažant and Novák (2000b,c), who showed that the theory gives good agreement with extensive test data regarding the combined probabilistic-energetic size effect on the modulus of rupture of unreinforced concrete beams (Bažant and Novák 2000a). The logarithmic size effect plots of the numerical results obtained with the nonlocal Weibull-type probabilistic theory are shown in Fig. 19(a). Note that a deterministic nonlocal theory gives almost the same results for the left half of the plot but terminates with a horizontal asymptote. The inclined large-size asymptote corresponds to the classical Weibull statistical size effect without nonlocality. The combination of nonlocality and statistics is needed for the transition, through the intermediate sizes. Evidently, amalgamation of the statistical and nonlocal theories is crucial for correct extrapolation from small-scale laboratory tests to large structures (such as the vertical bending fracture of an arch dam).

A much more difficult proposition is the development of a nonlocal probabilistic theory that would give not only the failure probabilities, independent of the loss of positive definiteness of the structural stiffness matrix, but would also describe the probability distributions of deflections and stresses prior to failure. The theory must yield the probability distribution of the first eigenvalue $\lambda_1$ of the structural stiffness matrix as the failure is approached. The main difficulty is that what matters is the distribution tail, with probabilities of the order of $10^{-7}$. For $D \to \infty$, the tail of the distribution of $\lambda_1$ as $\lambda_1 \to 0$ must have the form of Weibull distribution when the relative structure size $D/\ell$ is approaching infinity, because in that limit the classical Weibull theory must be recovered. The existing stochastic finite-element methods do not meet this requirement, and thus their use for predicting failure loads of extremely small probability is doubtful. Formulation of a realistic general statistical nonlocal theory is a challenge for the future.

## Concluding Thoughts and Future Path

While two decades ago strain-softening damage models were regarded as controversial, the nonlocal concept has by now rendered them respectable and their use realistic. Nonlocality is now generally accepted as the proper approach for regularizing the boundary problems of continuum damage mechanics, for capturing the size effect, and for avoiding spurious localization, giving rise to pathological mesh sensitivity.

The last two decades of research gave birth to a wide variety of nonlocal models, with differences that are not quite justified by diversity in the types of materials and practical application. One may now expect a period of crystalization in which many artificially complex or oversimplified models will fade, being recognized as superfluous, and only a few will gain a permanent pedestal in the pantheon of knowledge.

Speculations though could be made in this regard, they are better left to future scrutiny. The strength of each model's pedestal will be judged by the physics of the microstructure, and the permanence of that pedestal will be decided by passage of each model through the sieve of practical applications. Doubtless much further research lies ahead, and polemics will enliven the path into the future.

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## References


Aero, E. L., and Kuvshinskii, E. V. (1960). ''Fundamental equations of the theory of elastic materials with rotationally interacting particles.'' *Fiz. Tverd. Tela (S.-Peterburg),* 2, 1399–1409.

Armero, F. (1997). ''Localized anisotropic damage of brittle materials.'' *Computational plasticity: Fundamentals and applications*, D. R. J. Owen, E. Oñate, and E. Hinton, eds., International Center for Numerical Methods in Engineering, Barcelona, Spain, 635–640.

Armero, F., and Garikipati, K. (1995). ''Recent advances in the analysis and numerical simulation of strain localization in inelastic solids.'' *Computational plasticity: Fundamentals and applications*, D. R. J. Owen and E. Oñate, eds., International Center for Numerical Methods in Engineering, Barcelona, Spain, 547–561.

Barenblatt, G. I. (1962). ''The mathematical theory of equilibrium of cracks in brittle fracture.'' *Adv. Appl. Mech.,* 7, 55–129.

Bažant, Z. P. (1971). ''Micropolar medium as a model for buckling of grid frameworks. *Developments in mechanics*.'' *Proc., 12th Midwestern Mechanics Conference,* Univ. of Notre Dame, 587–593.

Bažant, Z. P. (1976). ''Instability, ductility, and size effect in strain-softening concrete.'' *J. Eng. Mech. Div.,* 102(2), 331–344.

Bažant, Z. P. (1982). ''Crack band model for fracture of geomaterials.'' *Proc., 4th Int. Conf. on Numerical Methods in Geomechanics*, Z. Eisenstein, ed., Univ. of Alberta, Edmonton, 3, 1137–1152.

Bažant, Z. P. (1984a). ''Imbricate continuum and its variational derivation.'' *J. Eng. Mech.,* 110(12), 1693–1712.

Bažant, Z. P. (1984b). ''Size effect in blunt fracture: Concrete, rock, and metal.'' *J. Eng. Mech.,* 110(4), 518–535.

Bažant, Z. P. (1985a). ''Fracture in concrete and reinforced concrete.'' *Mechanics of geomaterials: Rocks, concretes, and soils*, Z. P. Bažant, ed., Wiley, London, 259–303.

Bažant, Z. P. (1985b). ''Mechanics of fracture and progressive cracking concrete structures.'' *Fracture mechanics of concrete: Structural application and numerical calculation*, G. C. Sih and A. DiTommaso, eds., Martinus Nijhoff, Dordrecht, The Netherlands, 1, 1–94.

Bažant, Z. P. (1986). ''Mechanics of distributed cracking.'' *Appl. Mech. Rev.,* 39, 675–705.

Bažant, Z. P. (1987). ''Why continuum damage is nonlocal: Justification by quasiperiodic microcrack array.'' *Mech. Res. Commun.,* 14, 407–419.

Bažant, Z. P. (1991). ''Why continuum damage is nonlocal: Micromechanics arguments.'' *J. Eng. Mech.,* 117(5), 1070–1087.

Bažant, Z. P. (1994). ''Nonlocal damage theory based on micromechanics of crack interactions.'' *J. Eng. Mech.,* 120(3), 593–617.

Bažant, Z. P. (2001). ''Concrete fracture models: Testing and practice.'' *Eng. Fract. Mech.,* 69, 165–206.

Bažant, Z. P. (2002a). ''Reminiscences on four decades of struggle and progress in softening damage and size effect.'' *Concr. J. (Japan Concr. Inst.),* 40, 16–28.

Bažant, Z. P. (2002b). *Scaling of structural strength*, Hermes-Penton, London.

Bažant, Z. P., and Belytschko, T. B. (1985). ''Wave propagation in a strain-softening bar: Exact solution.'' *J. Eng. Mech.,* 111(3), 381–389.

Bažant, Z. P., Belytschko, T. B., and Chang, T.-P. (1984). ''Continuum model for strain softening.'' *J. Eng. Mech.,* 110(12), 1666–1692.

Bažant, Z. P., Caner, F. C., Carol, I., Adley, M. D., and Akers, S. (2000). ''Microplane model M4 for concrete: I. Formulation with work-conjugate deviatoric stress.'' *J. Eng. Mech.,* 126(9), 944–953.

Bažant, Z. P., and Cedolin, L. (1979). ''Blunt crack band propagation in finite element analysis.'' *J. Eng. Mech.,* 105(2), 297–315.

Bažant, Z. P., and Cedolin, L. (1991). *Stability of structures*, Oxford University Press, New York, Chap. 10.

Bažant, Z. P., Červenka, J., and Wierer, M. (2001). ''Equivalent localization element for crack band model and as alternative to elements with embedded discontinuities.'' *Fracture mechanics of concrete structures*, R. de Borst et al., ed., Balkema, Lisse, The Netherlands, 765–772.

Bažant, Z. P., and Chang, T.-P. (1984). ''Instability of nonlocal continuum and strain averaging.'' *J. Eng. Mech.,* 110(10), 1441–1450.

Bažant, Z. P., and Christensen, M. (1972a). ''Analogy between micropolar continuum and grid frameworks under initial stress.'' *Int. J. Solids Struct.,* 8, 327–346.

Bažant, Z. P., and Christensen, M. (1972b). ''Long-wave extensional buckling of large regular frames.'' *J. Struct. Div., ASCE,* 98(10), 2269–2289.

Bažant, Z. P., and Jirásek, M. (1994). ''Nonlocal model based on crack interactions: A localization study.'' *J. Eng. Mater. Technol.,* 116, 256–259.

Bažant, Z. P., and Li, Y.-N. (1995). ''Stability of cohesive crack model.'' *J. Appl. Mech.,* 62, 959–969.

Bažant, Z. P., and Li, Y.-N. (1997). ''Cohesive crack model with rate-dependent crack opening and viscoelasticity: I. Mathematical model and scaling.'' *Int. J. Fract.* 86, 247–265.

Bažant, Z. P., and Lin, F.-B. (1988a). ''Nonlocal smeared cracking model for concrete fracture.'' *J. Struct. Eng.,* 114(11), 2493–2510.

Bažant, Z. P., and Lin, F.-B. (1988b). ''Nonlocal yield-limit degradation.'' *Int. J. Numer. Methods Eng.,* 26, 1805–1823.

Bažant, Z. P., and Novák, D. (2000a). ''Energetic-statistical size effect in quasibrittle failure at crack initiation.'' *ACI Mater. J.,* 97, 381–392.

Bažant, Z. P., and Novák, D. (2000b). ''Probabilistic nonlocal theory for quasibrittle fracture initiation and size effect: I. Theory.'' *J. Eng. Mech.,* 126(2), 166–174.

Bažant, Z. P., and Novák, D. (2000c). ''Probabilistic nonlocal theory for quasibrittle fracture initiation and size effect: II. Application.'' *J. Eng. Mech.,* 126(2), 175–185.

Bažant, Z. P., and Oh, B.-H. (1983). ''Crack band theory for fracture of concrete.'' *Mater. Struct.,* 16, 155–177.

Bažant, Z. P., and Ohtsubo, H. (1977). ''Stability conditions for propagation of a system of cracks in a brittle solid.'' *Mech. Res. Commun.,* 4, 353–366.

Bažant, Z. P., and Ožbolt, J. (1990). ''Nonlocal microplane model for fracture, damage, and size effect in structures.'' *J. Eng. Mech.,* 116(11), 2485–2505.

Bažant, Z. P., and Pfeiffer, P. A. (1987). ''Determination of fracture energy from size effect and brittleness number.'' *ACI Mater. J.,* 84, 463–480.

Bažant, Z. P., and Pijaudier-Cabot, G. (1988). ''Nonlocal continuum damage, localization instability and convergence.'' *J. Appl. Mech.,* 55, 287–293.

Bažant, Z. P., and Pijaudier-Cabot, G. (1989). ''Measurement of characteristic length of nonlocal continuum.'' *J. Eng. Mech.,* 115(4), 755–767.

Bažant, Z. P., and Planas, J. (1998). *Fracture and size effect in concrete and other quasibrittle materials*, CRC, Boca Raton, Fla.

Bažant, Z. P., Tabbara, M. R., Kazemi, M. T., and Pijaudier-Cabot, G. (1990). ''Random particle model for fracture of aggregate or fiber composites.'' *J. Eng. Mech.,* 116(8), 1686–1705.

Bažant, Z. P., and Xi, Y. (1991). ''Statistical size effect in quasibrittle structures: II. Nonlocal theory.'' *J. Eng. Mech.,* 117(11), 2623–2640.

Bažant, Z. P., Xiang, Y., Adley, M., Prat, P. C., and Akers, S. A. (1996). ''Microplane model for concrete: II: Data delocalization and verification.'' *J. Eng. Mech.,* 122(3), 255–262.

Belytschko, T., Fish, J., and Engelmann, B. E. (1988). ''A finite-element with embedded localization zones.'' *Comput. Methods Appl. Mech. Eng.,* 70, 59–89.

Benvenuti, E., Borino, G., and Tralli, A. (2000). ''A thermodynamically consistent nonlocal formulation for elastodamaging materials: Theory and computations.'' *Proc., ECCOMAS 2000*, Barcelona, Spain, CD-ROM.

Besdo, D. (1974). ''Ein Beitrag zur nichtlinearen Theorie des Cosserat Kontinuums.'' *Acta Mech.,* 20, 105–131.

Borino, G., and Failla, B. (2000). ''Thermodynamically consistent plasticity models with local and nonlocal internal variables.'' *Proc., European Congr. on Computational Methods in Applied Sciences and Engineering, ECCOMAS*, Barcelona, Spain, CD-ROM.

Borino, G., Failla, B., and Parrinello, F. (2002). ''A symmetric formulation for nonlocal damage models.'' H. A. Mang, F. G. Rammerstorfer, and J. Eberhardsteiner, eds., *Proc., 5th World Congress on Computa-*

*tional Mechanics (WCCM V)*, Vienna Univ. of Technology, Vienna, Austria, ISBN 3-9501554-0-6, ⟨http://wccm.tuwien.ac.at⟩.
Borino, G., Fuschi, P., and Polizzotto, C. (1999). "A thermodynamic approach to nonlocal plasticity and related variational approaches." *J. Appl. Mech.*, 66, 952–963.
Borino, G., and Polizzotto, C. (1999). "Comments on 'Nonlocal bar revisited by Christer Nilsson'." *Int. J. Solids Struct.*, 36, 3085–3091.
Boudon-Cussac, D., Hild, F., and Pijaudier-Cabot, G. (1999). "Tensile damage in concrete: Analysis of experimental technique." *J. Eng. Mech.*, 125(8), 906–913.
Camacho, G. T., and Ortiz, M. (1996). "Computational modeling of impact damage in brittle materials." *Int. J. Solids Struct.*, 33, 2899–2938.
Cangemi, L., Cocu, M., and Raous, M. (1996). "Adhesion and friction model for the fiber/matrix interface of a composite." *Proc., 1996 Engineering Systems Design and Analysis Conf.*, 157–163.
Carmeliet, J. (1999). "Optimal estimation of gradient damage parameters from localization phenomena in quasibrittle materials." *Mech. Cohesive-Frict. Mater.*, 4, 1–16.
Carol, I., and Bažant, Z. P. (1997). "Damage and plasticity in microplane theory." *Int. J. Solids Struct.*, 34, 3807–3835.
Carol, I., and Prat, P. C. (1990). "A statically constrained microplane model for the smeared analysis of concrete cracking." in *Computer aided analysis and design of concrete structures*, N. Bićanić and H. Mang, eds., Pineridge, Swansea, U.K., 919–930.
Cedolin, L., and Bažant, Z. P. (1980). "Effect of finite-element choice in blunt crack band analysis." *Comput. Methods Appl. Mech. Eng.*, 24, 305–316.
Červenka, J. (1994). "Discrete crack modeling in concrete structures." PhD thesis, Univ. of Colorado, Boulder, Colo.
Červenka, J., Bažant, Z. P., and Wierer, M. (2002). "Equivalent localization element for crack band approach to mesh-size sensitivity in microplane model." *Int. J. Numer. Methods Eng.*, in press.
Chaboche, J. L., Girard, R., and Levasseur, P. (1997a). "On the interface debonding models." *Int. J. Damage Mech.*, 6, 220–257.
Chaboche, J. L., Girard, R., and Schaff, A. (1997b). "Numerical analysis of composite systems by using interphase/interface models." *Computational Mech., Berlin*, 20, 3–11.
Chandrasekhar, S. (1950). *Radiation transfer*, Oxford University Press, London.
Comi, C. (2001). "A nonlocal model with tension and compression damage mechanisms." *Eur. J. Mech. A/Solids*, 20, 1–22.
Comi, C., and Perego, U. (2001). "Numerical aspects of nonlocal damage analyses." *Rev. Europ. Elements Finis*, 10, 227–242.
Cope, R. J., Rao, P. V., Clark, L. A., and Norris, P. (1980). "Modeling of reinforced concrete behavior for finite-element analysis of bridge slabs." *Numerical methods for nonlinear problems*, Pineridge, Swansea, 1, 457–470.
Cordebois, J. P., and Sidoroff, F. (1979). "Anisotropie élastique induite par endommagement." *Comportement mécanique des solides anisotropes, No. 295, Colloques internationaux du CNRS*, Editions du CNRS, Grenoble, France, 761–774.
Cosserat, E., and Cosserat, F. (1909). *Théorie des corps déformables*, Herrman, Paris.
Daux, C., Moës, N., Dolbow, J., Sukumar, N., and Belytschko, T. (2000). "Arbitrary branched and intersecting cracks with the extended finite-element method." *Int. J. Numer. Methods Eng.*, 48, 1741–1760.
de Borst, R. (1986). "Nonlinear analysis of frictional materials." PhD thesis, Delft Univ. of Technology, Delft, The Netherlands.
Denarié, E., Saouma, V. E., Iocco, A., and Varelas, D. (2001). "Concrete fracture process zone characterization with fiber optics." *J. Eng. Mech.*, 127(5), 494–502.
di Prisco, M., and Mazars, J. (1996). "Crush–crack: A nonlocal damage model for concrete." *J. Mech. Cohesive Frict. Mater.*, 1, 321–347.
Díez, P., Egozcue, J. J., and Huerta, A. (1998). "A posteriori error estimation for standard finite-element analysis." *Comput. Methods Appl. Mech. Eng.*, 163, 141–157.
Drugan, W. J., and Willis, J. R. (1996). "A micromechanics-based nonlocal constitutive equation and estimates of representative volume element size for elastic composites." *J. Mech. Phys. Solids*, 44, 497–524.
Duarte, C. A., and Oden, J. T. (1996). "H–p clouds—An h–p meshless method." *Numer. Methods Partial Diff. Eq.*, 12, 673–705.
Dugdale, D. S. (1960). "Yielding of steel sheets containing slits." *J. Mech. Phys. Solids*, 8, 100–108.
Duhem, P. (1893). "Le potentiel thermodynamique et la pression hydrostatique." *Ann. Sci. Ec. Normale Super.*, 10, 183–230.
Dvorkin, E. N., Cuitiño, A. M., and Gioia, G. (1990). "Finite elements with displacement interpolated embedded localization lines insensitive to mesh size and distortions." *Comput. Methods Appl. Mech. Eng.*, 90, 829–844.
Edelen, D. G. B. (1969). "Protoelastic bodies with large deformation." *Arch. Ration. Mech. Anal.*, 34, 283–300.
Edelen, D. G. B., Green, A. E., and Laws, N. (1971). "Nonlocal continuum mechanics." *Arch. Ration. Mech. Anal.*, 43, 36–44.
Edelen, D. G. B., and Laws, N. (1971). "On the thermodynamics of systems with nonlocality." *Arch. Ration. Mech. Anal.*, 43, 24–35.
Engelen, R. A. B., Geers, M. G. D., and Baaijens, F. P. T. (2002). "Nonlocal implicit gradient-enhanced elastoplasticity for the modeling of softening behavior." *Int. J. Plast.*, in press.
Ericksen, J. L. (1960). "Anisotropic fluids." *Arch. Ration. Mech. Anal.*, 4, 231–237.
Eringen, A. C. (1964). "Simple microfluids." *Int. J. Eng. Sci.*, 2, 205–217.
Eringen, A. C. (1966a). "Linear theory of micropolar elasticity." *J. Math. Mech.*, 15, 909–924.
Eringen, A. C. (1966b). "Mechanics of micromorphic materials." *Proc., 11th Int. Congr. of Applied Mechanics*, Springer, Berlin, 131–138.
Eringen, A. C. (1966c). "A unified theory of thermomechanical materials." *Int. J. Eng. Sci.*, 4, 179–202.
Eringen, A. C. (1972). "Linear theory of nonlocal elasticity and dispersion of plane waves." *Int. J. Eng. Sci.*, 10, 425–435.
Eringen, A. C. (1981). "On nonlocal plasticity." *Int. J. Eng. Sci.*, 19, 1461–1474.
Eringen, A. C. (1983). "Theories of nonlocal plasticity." *Int. J. Eng. Sci.*, 21, 741–751.
Eringen, A. C., and Ari, N. (1983). "Nonlocal stress field at a Griffith crack." *Cryst. Lattice Defects Amorphous Mater.*, 10, 33–38.
Eringen, A. C., and Edelen, D. G. B. (1972). "On nonlocal elasticity." *Int. J. Eng. Sci.*, 10, 233–248.
Eringen, A. C., and Kim, B. S. (1974). "Stress concentration at the tip of a crack." *Mech. Res. Commun.*, 1, 233–237.
Eringen, A. C., Speziale, C. G., and Kim, B. S. (1977). "Crack-tip problem in nonlocal elasticity." *J. Mech. Phys. Solids*, 25, 339–355.
Eringen, A. C., and Suhubi, E. S. (1964). "Nonlinear theory of simple microelastic solids." *Int. J. Eng. Sci.*, 2, 189–203, 389–404.
Etse, G., and Willam, K. J. (1994). "Fracture-energy formulation for inelastic behavior of plain concrete." *J. Eng. Mech.*, 120(9), 1983–2011.
Fabrikant, V. I. (1990). "Complete solutions to some mixed boundary value problems in elasticity." *Adv. Appl. Mech.*, 27, 153–223.
Ferrara, L. (1998). "A contribution to the modeling of mixed-mode fracture and shear transfer in plain and reinforced concrete." PhD thesis, Politecnico di Milano, Milano, Italy.
Fischer, R. A., and Tippett, L. H. C. (1928). "Limiting forms of the frequency distribution of the largest and smallest member of a sample." *Proc. Cambridge Philos. Soc.*, 24, 180–190.
Fish, J., and Yu, Q. (2001). "Multiscale damage modeling for composite materials: Theory and computational framework." *Int. J. Numer. Methods Eng.*, 52, 161–191.
Fleck, N. A., and Hutchinson, J. W. (1993). "A phenomenological theory for strain gradient effects in plasticity." *J. Mech. Phys. Solids*, 41, 1825–1857.
Fleck, N. A., and Hutchinson, J. W. (2001). "A reformulation of strain gradient plasticity." *J. Mech. Phys. Solids*, 49, 2245–2271.
Fleck, N. A., Muller, G. M., Ashby, M. F., and Hutchinson, J. W. (1994). "Strain gradient plasticity: Theory and experiment." *Acta Metall. Mater.*, 42, 475–487.

Gao, H., and Huang, Y. (2001). ''Taylor-based nonlocal theory of plasticity.'' *Int. J. Solids Struct.,* 38, 2615–2637.
Gao, H., Huang, Y., Nix, W. D., and Hutchinson, J. W. (1999). ''Mechanism-based strain gradient plasticity. I. Theory.'' *J. Mech. Phys. Solids,* 47, 1239–1263.
Geers, M. G. D., de Borst, R., and Brekelmans, W. A. M. (1996a). ''Computing strain fields from discrete displacement fields in 2D solids.'' *Int. J. Solids Struct.,* 33, 4293–4307.
Geers, M. G. D., Engelen, R. A. B., and Ubachs, R. J. M. (2001). ''On the numerical modeling of ductile damage with an implicit gradient-enhanced formulation.'' *Rev. Euro. Eléments finis*, 10, 173–191.
Geers, M. G. D., Peijs, T., Brekelmans, W. A. M., and de Borst, R. (1996b). ''Experimental monitoring of strain localization and failure behavior of composite materials.'' *Compos. Sci. Technol.,* 56, 1283–1290.
Green, A. E. (1965). ''Micromaterials and multipolar continuum mechanics.'' *Int. J. Eng. Sci.,* 3, 533–537.
Green, A. E., and Naghdi, P. M. (1965). ''A general theory of an elastic–plastic continuum.'' *Arch. Ration. Mech. Anal.,* 18, 251–281.
Green, A. E., and Rivlin, R. S. (1964a). ''Multipolar continuum mechanics.'' *Arch. Ration. Mech. Anal.,* 17, 113–147.
Green, A. E., and Rivlin, R. S. (1964b). ''Simple force and stress multipoles.'' *Arch. Ration. Mech. Anal.,* 16, 325–353.
Grioli, G. (1960). ''Elasticità asimmetrica.'' *Annali di matematica pura ed applicata*, Ser. IV, 50, 389–417.
Günther, W. (1958). ''Zur Statik und Kinematik des Cosseratschen Kontinuum.'' *Abh. Braunschweigischen Wissenschaftlichen Gesellschaft,* 10, 195–213.
Gupta, A. K., and Akbar, H. (1984). ''Cracking in reinforced concrete analysis.'' *J. Struct. Eng.,* 110(8), 1735–1746.
Hadamard, J. (1903). *Leçons sur la propagation des ondes*, Hermann, Paris.
Halphen, B., and Nguyen, Q. S. (1975). ''Sur les matériaux standards généralisés.'' *J. Mec.,* 14, 39–63.
Hillerborg, A., Modéer, M., and Peterson, P. E. (1976). ''Analysis of crack propagation and crack growth in concrete by means of fracture mechanics and finite elements.'' *Cem. Concr. Res.,* 6, 773–782.
Hodgkin, A. L. (1964). *The conduction of nervous impulse*, Thomas, Springfield, Ill.
Hu, X. Z., and Wittmann, F. H. (2000). ''Size effect on toughness induced by cracks close to free surface.'' *Eng. Fract. Mech.,* 65, 209–211.
Huang, Y., Gao, H., Nix, W. D., and Hutchinson, J. W. (2000). ''Mechanism-based strain gradient plasticity. II. Analysis.'' *J. Mech. Phys. Solids,* 48, 99–128.
Huerta, A., Díez, P., Rodríguez-Ferran, A., and Pijaudier-Cabot, G. (1998). ''Error estimation and finite-element analysis of softening solids.'' *Advances in adaptive computational methods in mechanics*, P. Ladevèze and J. T. Oden, eds., Elsevier, Oxford, U.K., 333–348.
Huerta, A., and Pijaudier-Cabot, G. (1994). ''Discretization influence on regularization by two localisation limiters.'' *J. Eng. Mech.,* 120(6), 1198–1218.
Irwin, G. R. (1958). ''Fracture.'' in *Handbuch der Physik*, S. Flügge, ed., Springer, Berlin, VI, 551–590.
Jirásek, M. (1998a). ''Finite elements with embedded cracks.'' *LSC Internal Rep. No. 98/01*, Swiss Federal Institute of Technology, Lausanne, Switzerland.
Jirásek, M. (1998b). ''Nonlocal models for damage and fracture: Comparison of approaches.'' *Int. J. Solids Struct.,* 35, 4133–4145.
Jirásek, M. (1999). ''Comments on microplane theory.'' *Mechanics of quasibrittle materials and structures*, G. Pijaudier-Cabot, Z. Bittnar, and B. Gérard, eds., Hermès Science, Paris, 55–77.
Jirásek, M. (2000a). ''Comparative study on finite elements with embedded cracks.'' *Comput. Methods Appl. Mech. Eng.,* 188, 307–330.
Jirásek, M. (2000b). ''Conditions of uniqueness for finite elements with embedded cracks.'' *Proc., 6th Int. Conf. on Computational Plasticity*, Barcelona. CD-ROM.
Jirásek, M., and Bažant, Z. P. (1994). ''Localization analysis of nonlocal model based on crack interactions.'' *J. Eng. Mech.,* 120(7), 1521–1542.
Jirásek, M., and Bažant, Z. P. (1995). ''Macroscopic fracture characteristics of random particle systems.'' *Int. J. Fract.,* 69, 201–228.
Jirásek, M., and Bažant, Z. P. (2001). *Inelastic analysis of structures*, Wiley, Chichester, U.K.
Jirásek, M., and Patzák, B. (2002). ''Consistent tangent stiffness for nonlocal damage models.'' *Comput. Struct.*, in press.
Jirásek, M., and Rolshoven, S. (2002). ''Comparison of integral-type nonlocal plasticity models for strain-softening materials.'' *Int. J. Eng. Sci.*, in press.
Jirásek, M., and Zimmermann, T. (1997). ''Rotating crack model with transition to scalar damage: I. Local formulation, II. Nonlocal formulation and adaptivity.'' *LSC Internal Rep. No. 97/01*, Swiss Federal Institute of Technology, Lausanne, Switzerland.
Jirásek, M., and Zimmermann, T. (1998a). ''Analysis of rotating crack model.'' *J. Eng. Mech.,* 124(8), 842–851.
Jirásek, M., and Zimmermann, T. (1998b). ''Rotating crack model with transition to scalar damage.'' *J. Eng. Mech.,* 124(3), 277–284.
Jirásek, M., and Zimmermann, T. (2001). ''Embedded crack model: I. Basic formulation.'' *Int. J. Numer. Methods Eng.,* 50, 1269–1290.
Kachanov, M. (1985). ''A simple technique of stress analysis in elastic solids with many cracks.'' *Int. J. Fract.,* 28, R11–R19.
Kachanov, M. (1987). ''Elastic solids with many cracks: A simple method of analysis.'' *Int. J. Solids Struct.,* 23, 23–43.
Kang, H. D. (1997). ''Triaxial constitutive model for plain and reinforced concrete behavior.'' PhD thesis, Univ. of Colorado, Boulder, Colo.
Kennedy, T. C., and Nahan, M. F. (1996). ''A simple nonlocal damage model for predicting failure of notched laminates.'' *Compos. Struct.,* 35, 229–236.
Kennedy, T. C., and Nahan, M. F. (1997). ''A simple nonlocal damage model for predicting failure in a composite shell containing a crack.'' *Compos. Struct.,* 39, 85–91.
Klisinski, M., Runesson, K., and Sture, S. (1991). ''Finite element with inner softening band.'' *J. Eng. Mech.,* 117(3), 575–587.
Koiter, W. T. (1964). ''Couple stresses in the theory of elasticity.'' *Proc. K. Ned. Akad. Wet., Ser. B: Phys. Sci.,* 67, 17–44.
Kröner, E. (1966). ''Continuum mechanics and range of atomic cohesion forces.'' *Proc., 1st Int. Conf. on Fracture*, T. Yokobori, T. Kawasaki, and J. Swedlow, eds., Japanese Society for Strength and Fracture of Materials, Sendai, Japan, 27.
Kröner, E. (1967). ''Elasticity theory of materials with long range cohesive forces.'' *Int. J. Solids Struct.,* 3, 731–742.
Kröner, E., and Datta, B. K. (1966). ''Nichtlokale Elastostatik: Ableitung aus der Gittertheorie.'' *Z. Phys.,* 196, 203–211.
Krumhansl, J. A. (1965). ''Generalized continuum field representation for lattice vibrations.'' *Lattice dynamics*, R. F. Wallis, ed., Pergamon, London, 627–634.
Kunin, I. A. (1966a). ''Model of elastic medium simple structure with spatial dispersion.'' *Prikl. Mat. Mekh.,* 30, 942 (in Russian).
Kunin, I. A. (1966b). ''Theory of elasticity with spatial dispersion.'' *Prikl. Mat. Mekh.,* 30, 866 (in Russian).
Kunin, I. A. (1968). ''The theory of elastic media with microstructure and the theory of dislocations.'' *Mechanics of generalized continua*, E. Kröner, ed., Springer, Heidelberg, 321–329.
Labuz, J. F., Cattaneo, S., and Chen, L.-H. (2001). ''Acoustic emission at failure in quasibrittle materials.'' *Constr. Build. Mater.,* 15, 225–233.
Ladevèze, P. (1992). ''A damage computational method for composite structures.'' *Comput. Struct.,* 44, 79–87.
Ladevèze, P. (1995). ''A damage computational method for composites: Basic aspects and micromechanical relations.'' *Computational Mech., Berlin,* 17, 142–150.
Ladevèze, P., Allix, O., Deü, J.-F., and Lévêque, D. (2000). ''A mesomodel for localization and damage computation in laminates.'' *Comput. Methods Appl. Mech. Eng.,* 183, 105–122.
Lakes, R. S. (1986). ''Experimental microelasticity of two porous solids.'' *Int. J. Solids Struct.,* 22, 55–63.
Landis, E. N. (1999). ''Micro–macro-fracture relationships and acoustic emissions in concrete.'' *Constr. Build. Mater.,* 13, 65–72.
Larsson, R., Runesson, K., and Sture, S. (1996). ''Embedded localization band in undrained soil based on regularized strong discontinuity—

Theory and FE analysis.'' *Int. J. Solids Struct.,* 33, 3081–3101.
Lemaitre, J., and Chaboche, J.-L. (1990). *Mechanics of solid materials*, Cambridge University Press, Cambridge, U.K.
Lippmann, H. (1969). ''Eine Cosserat Theorie des plastischen Fließens,'' *Acta Mech.,* 8, 255–284.
Lotfi, H. R., and Shing, P. B. (1995). ''Embedded representation of fracture in concrete with mixed finite elements.'' *Int. J. Numer. Methods Eng.,* 38, 1307–1325.
Luciano, R., and Willis, J. R. (2001). ''Nonlocal constitutive response of a random laminate subjected to configuration-dependent body force.'' *J. Mech. Phys. Solids,* 49, 431–444.
Ma, Q., and Clarke, D. R. (1995). ''Size-dependent hardness in silver single crystals.'' *J. Mater. Res.,* 10, 853–863.
Mariotte, E. (1686). *Traité du mouvement des eaux*, M. de la Hire, ed., English translation by J. T. Desvaguliers, London (1718).
Mazars, J. (1984). ''Application de la mécanique de l'endommagement au comportement nonlinéaire et à la rupture du béton de structure.'' *Thèse de Doctorat d'Etat*, Univ. Paris VI, France.
Mazars, J., and Berthaud, Y. (1989). ''Une technique expérimentale appliquée au béton pour créer un endommagement diffus et mettre en évidence son caractère unilateral.'' *Compt. Rendus Acad. Sci., Paris,* 308, 579–584.
Mazars, J., Berthaud, Y., and Ramtani, S. (1990). ''The unilateral behavior of damaged concrete.'' *Eng. Fract. Mech.,* 35, 629–635.
Melenk, J. M., and Babuška, I. (1996). ''The partition of unity finite-element method: Basic theory and applications.'' *Comput. Methods Appl. Mech. Eng.,* 39, 289–314.
Mihashi, H., and Nomura, N. (1996). ''Correlation between characteristics of fracture process zone and tension-softening properties of concrete.'' *Nucl. Eng. Des.,* 165, 359–376.
Mindlin, R. D. (1964). ''Microstructure in linear elasticity.'' *Arch. Ration. Mech. Anal.,* 16, 51–78.
Mindlin, R. D. (1965). ''Second gradient of strain and surface tension in linear elasticity.'' *Int. J. Solids Struct.,* 1, 417–438.
Mindlin, R. D., and Tiersten, H. F. (1962). ''Effects of couple stresses in linear elasticity.'' *Arch. Ration. Mech. Anal.,* 11, 415–448.
Moës, N., and Belytschko, T. (2002). ''Extended finite-element method for cohesive crack growth.'' *Eng. Fract. Mech.,* 69, 813–833.
Moës, N., Dolbow, J., and Belytschko, T. (1999). ''A finite-element method for crack growth without remeshing.'' *Int. J. Numer. Methods Eng.,* 46, 131–150.
Morrison, J. L. M. (1939). ''The yield of mild steel with particular reference to the effect of size of specimen.'' *Proc. Inst. Mech. Eng.,* 142, 193–223.
Naghdi, P. M., and Trapp, J. A. (1975). ''The significance of formulating plasticity theory with reference to loading surfaces in strain space.'' *Int. J. Eng. Sci.,* 13, 785–797.
Needleman, A. (1987). ''A continuum model for void nucleation by inclusion debonding.'' *J. Appl. Mech.,* 54, 525–531.
Needleman, A. (1988). ''Material rate dependence and mesh sensitivity in localization problems.'' *Comput. Methods Appl. Mech. Eng.,* 67, 69–85.
Nilsson, C. (1994). ''On nonlocal plasticity, strain softening, and localization.'' *Rep. No. TVSM-1007*, Division of Structural Mechanics, Lund Institute of Technology, Lund, Sweden.
Nilsson, C. (1997). ''Nonlocal strain softening bar revisited.'' *Int. J. Solids Struct.,* 34, 4399–4419.
Nilsson, C. (1999). ''Author's closure.'' *Int. J. Solids Struct.,* 36, 3093–3100.
Nix, W. D. (1989). ''Mechanical properties of thin films.'' *Metall. Trans. A,* 20A, 2217–2245.
Noll, W. (1972). ''A new mathematical theory of simple materials.'' *Arch. Ration. Mech. Anal.,* 48, 1–50.
Nooru-Mohamed, M. B. (1992). ''Mixed-mode fracture of concrete: An experimental approach.'' PhD thesis, Delft Univ. of Technology, Delft, The Netherlands.
Ohlsson, U., and Olofsson, T. (1997). ''Mixed-mode fracture and anchor bolts in concrete: Analysis with inner softening bands.'' *J. Eng. Mech.,* 123(10), 1027–1033.
Oliver, J. (1996). ''Modeling strong discontinuities in solid mechanics via strain softening constitutive equations. Part 1: Fundamentals. Part 2: Numerical simulation.'' *Int. J. Numer. Methods Eng.,* 39, 3575–3624.
Oliver, J., Cervera, M., and Manzoli, O. (1998). ''On the use of strain-softening models for the simulation of strong discontinuities in solids.'' *Material instabilities in solids*, R. de Borst and E. van der Giessen, eds., Wiley, Chichester, U.K., 107–123.
Olofsson, T., Klisinski, M., and Nedar, P. (1994). ''Inner softening bands: A new approach to localization in finite elements.'' *Computational modeling of concrete structures*, H. Mang, N. Bićanić, and R. de Borst, eds., Pineridge, Swansea, U.K., 373–382.
Ortiz, M. (1988). ''Microcrack coalescence and macroscopic crack growth initiation in brittle solids.'' *Int. J. Solids Struct.,* 24, 231–250.
Ortiz, M., Leroy, Y., and Needleman, A. (1987). ''A finite-element method for localized failure analysis.'' *Comput. Methods Appl. Mech. Eng.,* 61, 189–214.
Ortiz, M., and Pandolfi, A. (1999). ''Finite-deformation irreversible cohesive elements for three-dimensional crack-propagation analysis.'' *Int. J. Numer. Methods Eng.,* 44, 1267–1282.
Oseen, C. W. (1933). ''The theory of liquid crystals.'' *Trans. Faraday Soc.,* 29, 883–899.
Otsuka, K., and Date, H. (2000). ''Fracture process zone in concrete tension specimen.'' *Eng. Fract. Mech.,* 65, 111–131.
Ožbolt, J., and Bažant, Z. P. (1996). ''Numerical smeared fracture analysis: Nonlocal microcrack interaction approach.'' *Int. J. Numer. Methods Eng.,* 39, 635–661.
Pandolfi, A., Krysl, P., and Ortiz, M. (1999). ''Finite-element simulation of ring expansion and fragmentation: The capturing of length and time scales through cohesive models of fracture.'' *Int. J. Fract.,* 95, 279–297.
Peerlings, R. H. J., de Borst, R., Brekelmans, W. A. M., and de Vree, J. H. P. (1996). ''Gradient-enhanced damage for quasibrittle materials.'' *Int. J. Numer. Methods Eng.,* 39, 3391–3403.
Pietruszczak, S., and Mróz, Z. (1981). ''Finite-element analysis of deformation of strain-softening materials.'' *Int. J. Numer. Methods Eng.,* 17, 327–334.
Pijaudier-Cabot, G., and Bažant, Z. P. (1987). ''Nonlocal damage theory.'' *J. Eng. Mech.,* 113(10), 1512–1533.
Planas, J., and Elices, M. (1992). ''Asymptotic analysis of a cohesive crack: 1. Theoretical background.'' *Int. J. Fract.,* 55, 153–177.
Planas, J., and Elices, M. (1993). ''Asymptotic analysis of a cohesive crack: 2. Influence of the softening curve.'' *Int. J. Fract.,* 64, 221–237.
Planas, J., Elices, M., and Guinea, G. V. (1993). ''Cohesive cracks versus nonlocal models: Closing the gap.'' *Int. J. Fract.,* 63, 173–187.
Planas, J., Guinea, G. V., and Elices, M. (1996). ''Basic issues on nonlocal models: Uniaxial modeling.'' *Tech. Rep. No. 96-jp03*, Departamento de Ciencia de Materiales, ETS de Ingenieros de Caminos, Univ. Politécnica de Madrid, Ciudad Univ. sn., 28040 Madrid, Spain.
Polizzotto, C. (2001). ''Nonlocal elasticity and related variational principles.'' *Int. J. Solids Struct.,* 38, 7359–7380.
Polizzotto, C. (2002). ''Remarks on some aspects of nonlocal theories in solid mechanics.'' *Proc., 6th National Congr. SIMAI*, Chia Laguna, Italy, CD-ROM.
Polizzotto, C., Borino, G., and Fuschi, P. (1998). ''A thermodynamic consistent formulation of nonlocal and gradient plasticity.'' *Mech. Res. Commun.,* 25, 75–82.
Poole, W. J., Ashby, M. F., and Fleck, N. A. (1996). ''Microhardness of annealed and work-hardened copper polycrystals.'' *Scr. Mater.,* 34, 559–564.
Rajagopal, E. S. (1960). ''The existence of interfacial couples in infinitesimal elasticity.'' *Ann. Phys. (Leipzig)*, 6, 192–201.
Rashid, Y. R. (1968). ''Analysis of prestressed concrete pressure vessels.'' *Nucl. Eng. Des.,* 7, 334–344.
Rayleigh, O. M. (1918). ''Notes on the theory of lubrication.'' *Philos. Mag.,* 35, 1–12.
Read, H. E., and Hegemier, G. A. (1984). ''Strain softening of rock, soil, and concrete—A review article.'' *Mech. Mater.,* 3, 271–294.

Richards, C. W. (1958). ‘‘Effects of size on the yielding of mild steel beams.’’ *Proc. Am. Soc. Test. Mater.,* 58, 955–970.

Rodríguez-Ferran, A., and Huerta, A. (2000). ‘‘Error estimation and adaptivity for nonlocal damage models.’’ *Int. J. Solids Struct.,* 37, 7501–7528.

Rogula, D. (1965). ‘‘Influence of spatial acoustic dispersion on dynamical properties of dislocations. I.’’ *Bull. Acad. Pol. Sci., Ser. Sci. Tech.,* 13, 337–343.

Rogula, D. (1982). ‘‘Introduction to nonlocal theory of material media.’’ *Nonlocal theory of material media, CISM courses and lectures*, D. Rogula, ed., Springer, Wien, 268, 125–222.

Rolshoven, S., and Jirásek, M. (2001). ‘‘On regularized plasticity models for strain-softening materials.’’ *Fracture mechanics of concrete structures*, R. de Borst, J. Mazars, G. Pijaudier-Cabot, and J. G. M. van Mier, eds., Balkema, Lisse, 617–624.

Rossi, P., and Wu, X. (1992). ‘‘Probabilistic model for material behavior analysis and appraisement of concrete structures.’’ *Mag. Concrete Res.,* 44, 271–280.

Rots, J. G. (1988). ‘‘Computational modeling of concrete fracture.’’ PhD thesis, Delft Univ. of Technol., Delft, The Netherlands.

Sandler, I. S. (1984). ‘‘Strain-softening for static and dynamic problems.’’ *Proc., Symp. on Constitutive Equations*, ASME, Winter Annual Meeting, New Orleans, 217–231.

Saouridis, C. (1988). ‘‘Identification et numérisation objectives des comportements adoucissants: Une approche multiéchelle de l'endommagement du béton.’’ PhD thesis, Univ. Paris VI.

Saouridis, C., and Mazars, J. (1992). ‘‘Prediction of the failure and size effect in concrete via a biscale damage approach.’’ *Eng. Comput.,* 9, 329–344.

Schlangen, E. (1993). ‘‘Experimental and numerical analysis of fracture processes in concrete.’’ PhD thesis, Delft Univ. of Technology, Delft, The Netherlands.

Schlangen, E., and van Mier, J. G. M. (1992). ‘‘Simple lattice model for numerical simulation of fracture of concrete materials and structures.’’ *Mater. Struct.,* 25, 534–542.

Shi, G. H. (1991). ‘‘Manifold method of material analysis.’’ *Trans., 9th Army Conf. on Applied Mathematics and Computing,* Minneapolis, 57–76.

Simo, J. C., and Oliver, J. (1994). ‘‘A new approach to the analysis and simulation of strain softening in solids.’’ *Fracture and damage in quasibrittle structures*, Z. P. Bažant, Z. Bittnar, M. Jirásek, and J. Mazars, eds., E&FN Spon, London, 25–39.

Simo, J. C., Oliver, J., and Armero, F. (1993). ‘‘An analysis of strong discontinuities induced by strain softening in rate-independent inelastic solids.’’ *Computational Mech., Berlin,* 12, 277–296.

Sluys, L. J. (1997). ‘‘Discontinuous modeling of shear banding.’’ *Computational plasticity: Fundamentals and applications,* D. R. J. Owen, E. Oñate, and E. Hinton, eds., Int. Center for Numerical Methods in Eng., Barcelona, Spain, 735–744.

Stolken, J. S., and Evans, A. G. (1998). ‘‘A microbend test method for measuring the plasticity length scale.’’ *Acta Mater.,* 46, 5109–5115.

Strömberg, L., and Ristinmaa, M. (1996). ‘‘FE formulation of a nonlocal plasticity theory.’’ *Comput. Methods Appl. Mech. Eng.,* 136, 127–144.

Sukumar, N., Moës, N., Moran, B., and Belytschko, T. (2000). ‘‘Extended finite-element method for three-dimensional crack modeling.’’ *Int. J. Numer. Methods Eng.,* 48, 1549–1570.

Svedberg, T. (1996). ‘‘A thermodynamically consistent theory of gradient-regularized plasticity coupled to damage.’’ *Licentiate thesis*, Chalmers Univ. of Technology.

Svedberg, T., and Runesson, K. (1998). ‘‘Thermodynamically consistent nonlocal and gradient formulations of plasticity.’’ *Nonlocal aspects in solid mechanics*, A. Brillard and J. F. Ganghoffer, eds., EUROMECH Colloquium 378, Mulhouse, France, 32–37.

Thomas, T. Y. (1961). *Plastic flow and fracture of solids*, Academic, New York.

Toupin, R. A. (1962). ‘‘Elastic materials with couple stresses.’’ *Arch. Ration. Mech. Anal.,* 11, 385–414.

Toupin, R. A. (1964). ‘‘Theories of elasticity with couple stress.’’ *Arch. Ration. Mech. Anal.,* 17, 85–112.

Truesdell, C., and Toupin, R. A. (1960). ‘‘Classical field theories of mechanics.’’ *Handbuch der physik*, Springer, Berlin, III, 1.

Tvergaard, V., and Hutchinson, J. W. (1992). ‘‘The relation between crack growth resistance and fracture process parameters in elastic–plastic solids.’’ *J. Mech. Phys. Solids,* 40, 1377–1397.

Tvergaard, V., and Hutchinson, J. W. (1993). ‘‘The influence of plasticity on mixed mode interface toughness.’’ *J. Mech. Phys. Solids,* 41, 1119–1135.

Valanis, K. C. (1990). ‘‘A theory of damage in brittle materials.’’ *Eng. Fract. Mech.,* 36, 403–416.

Valanis, K. C. (1991). ‘‘A global damage theory and the hyperbolicity of the wave problem.’’ *J. Appl. Mech.,* 58, 311–316.

van Gils, M. A. J. (1997). ‘‘Quasibrittle fracture of ceramics.’’ PhD thesis, Eindhoven Univ. of Technology, The Netherlands.

van Mier, J. G. M. (1997). *Fracture processes of concrete*, CRC, Boca Raton, Fla.

Vermeer, P. A., and Brinkgreve, R. B. J. (1994). ‘‘A new effective nonlocal strain measure for softening plasticity.’’ *Localization and bifurcation theory for soils and rocks*, R. Chambon, J. Desrues, and I. Vardoulakis, eds., Balkema, Rotterdam, The Netherlands, 89–100.

Voigt, W. (1887). ‘‘Theoretische Studien über die Elastizitätsverhältnisse der Krystalle.’’ Abh. Königlichen Gesellschaft Wiss. Göttingen.

Voigt, W. (1894). ‘‘Über Medien ohne innere Kräfte und eine durch sie gelieferte mechanische Deutung der Maxwell-Hertzschen Gleichungen.’’ *Abh. Königlichen Gesellschaft Wiss. Göttingen,* 40, 72–79.

Walsh, P. F. (1972). ‘‘Fracture of plain concrete.’’ *Indian Concr. J.,* 46, 469–470 and 476.

Walsh, P. F. (1976). ‘‘Crack initiation in plain concrete.’’ *Mag. Concrete Res.,* 28, 37–41.

Wei, Y., and Hutchinson, J. W. (1999). ‘‘Models of interface separation accompanied by plastic dissipation at multiple scales.’’ *Int. J. Fract.,* 95, 1–17.

Weibull, W. (1939). ‘‘The phenomenon of rupture in solids.’’ *Proc. R. Swedish Inst. Eng. Res. (Ing. Akad. Handl.),* 153, 1–55.

Wells, G. N. (2001). ‘‘Discontinuous modeling of strain localization and failure.’’ PhD thesis, Delft Univ. of Technol., Delft, The Netherlands.

Willam, K., Bićanić, N., and Sture, S. (1986). ‘‘Composite fracture model for strain-softening and localized failure of concrete.’’ *Computational modeling of reinforced concrete structures*, E. Hinton and D. R. J. Owen, eds., Pineridge, Swansea, 122–153.

Xu, X.-P., and Needleman, A. (1994). ‘‘Numerical simulations of fast crack growth in brittle solids.’’ *J. Mech. Phys. Solids,* 42, 1397–1434.