LINEAR CREEP PROBLEMS SOLVED BY A SUCCESSION OF GENERALIZED THERMOELASTICITY PROBLEMS

ZDENĚK P. BAŽANT

The usual approximate methods for integral equations and ordinary differential equations are generalized to creep problems in time which represent simultaneously a boundary value problem in space. The integration in time is reduced to subsequent elastic analyses with fictitious enforced internal strains or prestresses, the effect of which is further reduced to volume and surface loads. The method is useful for time-invariant as well as transient creep laws. Moreover an analogy between time-variant creep and elasticity is presented. The use for multi-layered plates and redundant structures with bars of composite section is demonstrated.

A general method for solving linear creep problems is known so far only for the time-invariant creep law and is based on the elastic-viscoelastic analogy and the use of the Laplace transform. For a transient creep law, e.g. for polymers at variable temperature or for aging concrete, no general method, to the author's knowledge, has been presented. In this paper, in agreement with the tendency of modern applied mathematics towards convergent numerical algorithms for computers rather than to analytical methods, general numerical integration procedures in time will be presented. Particularly for concrete creep (eq. 9b) they were derived by the author already in [3]–[5]. Moreover, an analogy of time-variant creep with elasticity will be shown, allowing to obtain pertinent equations for creep from corresponding elastic equations.

LINEAR CREEP LAWS

Using the principle of superposition in time, the linear creep law is obtained in the form

$$\varepsilon(t) = E_{cr}^{-1}(t, t_0)\, \sigma(t_0) + \int_{t_0}^{t} E_{cr}^{-1}(t, \tau)\, [\partial\sigma(\tau)/\partial\tau]\, \mathrm{d}\tau \tag{1}$$

where ε is the strain, σ the stress, t time, t_0 instant of first loading. $E_{cr}(t, \tau)$ is the creep modulus. The creep compliance E_{cr}^{-1} represents the strain in time t caused by a constant unit stress applied in time τ. Integrating by parts, eq. (1) becomes:

$$\varepsilon(t) = \sigma(t)/E(t) + \int_{t_0}^{t} \sigma(\tau)\, L(t, \tau)\, d\tau \tag{2}$$

where

$$L(t, \tau) = -\partial E_{cr}^{-1}(t, \tau)/\partial \tau\,. \tag{3}$$

On the other hand, the creep law may be written in the form

$$\sigma(t) = E(t)\, \varepsilon(t) + \int_{t_0}^{t} \varepsilon(\tau)\, R(t, \tau)\, d\tau\,, \tag{4}$$

where

$$R(t, \tau) = -\partial E_{rel}(t, \tau)/\partial \tau\,. \tag{5}$$

Here $E_{rel}(t, \tau)$ is the relaxation modulus representing the stress in time t caused by a constant unit strain introduced in time τ. The creep laws (1) or (2) and (4) may be written in the form:

$$\sigma = E^{-1}\varepsilon \quad \text{or} \quad \varepsilon = E\sigma\,, \tag{6}$$

where E^{-1} and E are linear integral operators defined by (1) or (2) and (4).

For special types of the function $E_{cr}^{-1}(t, \tau)$ or $E_{rel}(t, \tau)$ etc., the linear creep law may be also written in a differential form, where

$$E^{-1} = \left[a_0(t) + a_1(t)\frac{\partial}{\partial t} + \ldots + a_n(t)\frac{\partial^n}{\partial t^n}\right]\left[b_0(t) + b_1(t)\frac{\partial}{\partial t} + \ldots + b_n(t)\frac{\partial^n}{\partial t^n}\right]^{-1} \tag{7}$$

with initial conditions in time t_0 given by differential operators of order $n - 1$, $n - 2$, etc. For the time-invariant creep law, $a_0, b_0, a_1, \ldots$ are constants.

For polymers, the functions $E_{cr}^{-1}(t, \tau)$, $L(t, \tau)$, $E_{rel}(t, \tau), \ldots$ may be determined approximately from creep or relaxation tests at constant temperature, at which creep is time-invariant. Then E_{cr}, E_{rel}, L are functions of only one variable $(t - \tau)$. According to the hypothesis of time reduction with the reduction coefficient $a_T(t)$ for the temperature $T(t)$ [17], [18]:

$$E_{cr} = \bar{E}_{cr}\left(\int_{\tau}^{t} dt'/a_T(t')\right) \tag{8}$$

etc.

For concrete with respect to its aging, the most simple form is that of Dischinger-Whitney

$$E_{cr}^{-1} = \frac{1}{E(\tau)} + \frac{\varphi(t) - \varphi(\tau)}{E_0}\,, \tag{9a}$$

where $\varphi(t) \approx 1 - e^{-bt}$. More accurately, it is necessary to consider the law of Arutyunyan-Maslov

$$E_{cr}^{-1} = \frac{1}{E(\tau)} + \frac{\varphi_\infty(\tau)}{E(\tau)}\left(1 - e^{-\gamma(t-\tau)}\right), \tag{9b}$$

where $\varphi_\infty(\tau) \approx C + Ae^{-b\tau}$. Eq. (9a) allows to represent the creep law in the form

$$\partial\varepsilon/\partial\varphi = [1/E(t)]\, \partial\sigma/\partial\varphi + \sigma/E_0\,, \tag{9}$$

eq. (9b) in the form of a differential equation of second order with variable coefficients [4], [6], [7].

Similarly, the creep law for multiaxial stress and strain may be introduced:

$$\varepsilon_{ij} = C_{ijkl}\sigma_{kl} \quad \text{or} \quad \sigma_{ij} = E_{ijkl}\varepsilon_{kl}\,, \tag{10}$$

where C_{ijkl} or E_{ijkl} are operators which have the same form as E^{-1} or E with the kernels L_{ijkl} and R_{ijkl}; σ_{ij} and ε_{ij} are the components of the stress and strain tensors in Cartesian coordinates $x_1 = x$, $x_2 = y$, $x_3 = z$. The summation convention for repeated latine subscripts is considered, except for x, y, z.

SYSTEM OF EQUATIONS FOR CREEP PROBLEM

Besides the creep law (10), the solution of σ_{ij} and ε_{ij} must fulfil in each point of the given body the equations of equilibrium and the geometric equations:

$$\sigma_{ij,j} + f_i(t) = 0\,, \tag{11}$$

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{kjj})\,, \tag{12}$$

where $f_i(t)$ are volume forces and u_i displacements. At the part Γ_1 of the surface let the loads $p_i(t)$ and at the remaining part Γ_2 the displacements $b_i(t)$ be given as

$$n_j\sigma_{ij} = p_i(t) \text{ on } \Gamma_1\,, \quad u_i = b_i(t) \text{ on } \Gamma_2\,. \tag{13}$$

Here n_j is the unit normal vector of the surface. Instead of (12) for small strains six compatibility conditions have to be considered:

$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{ik,jl} - \varepsilon_{jl,ik} = 0\,. \tag{14}$$

(Note: In a linearized continuum stability problem it is necessary to substitute $b_{ij} + S_{jk}u_{i,k}$ for b_j in (11), (13) where S_{jk} is the initial equilibrium state of stress, and σ_{ij}, u_i and ε_{ij} are incremental values, x_i Lagrange's coordinates.)

The well-known elastic-viscoelastic analogy [9], [12], [17], [18], has been generalized in a certain sense for creep with transient conditions by the author [7], [6], [4]. It is based on the fact that for the integral or differential creep operators $\mathbf{E}^{-1}$ etc. according to (1), (2), (4), (7) etc., the principal rules of algebra are still valid (the associative and commutative rules for addition and multiplication and the distribution rule for multiplication of a sum). Therefore, the derivation of the equations for some linear creep problem would be, in principle, identical with the derivation in the theory of elasticity, since eqs. (6) or (10) are analogous to Hooke's law. The equations for time-variant creep are directly obtained by introducing the corresponding creep operators $\mathbf{E}, \ldots$ in the place of elastic constants $E, \ldots$ The condition is that the creep law is linear. The geometric relations (and, eventually the equilibrium equations), however, may be also nonlinear (finite strain). Care should be taken whether in deriving the elastic equation the elastic constants were not eliminated (because. $[(\partial/\partial t)/(\partial/\partial t)]\, y \neq y$ but $y +$ const.).

a) To demonstrate this, the equations for the problem of the bending of a nonhomogeneous restrained straight beam with the axis x and a small deflections $w = w(x\ t)$ on the assumption of homogeneous and perpendicular plane sections will be given. Analogously to the elastic equation $EJ(\partial^2 w/\partial x^2) = M$, where $M = M(x, t)$ is the bending moment and J is the inertia moment of the cross-section, by introduction of $\mathbf{E}$ according to (2), we obtain:

$$\frac{\partial^2 w(x,t)}{\partial x^2} = -\frac{M(x,t)}{E(x,t)\,J(x)} - \int_{t_0}^{t} \frac{M(x,\tau)}{J(x)}\, L(x; t, \tau)\, \mathrm{d}\tau\,. \tag{15}$$

Together with the equilibrium equation $\partial^2 M/\partial x^2 = -p$ and the boundary conditions of the restraint $w = \partial w/\partial x = 0$ for $x = 0$ and $x = l$, this problem in two unknown functions w, M is mathematically formulated.

b) As another example consider the bending of a nonhomogeneous redundar framed structure with homogenous section. In analogy with the elastic equations of the force method, the conditions for the redundants $X_1, \ldots, X_n$ are that

$$\sum_{j=1}^{n} X_j(t) \int_{(x)} \frac{M^{(i)}M^{(j)}}{\mathbf{E}J}\, \mathrm{d}x + \int_{(x)} \frac{M^{(i)}\, M^{(0)}(t)}{\mathbf{E}J}\, \mathrm{d}x + a_i(t) = 0\,, \tag{16}$$

where $M^{(k)}(x)$ are bending moments when $X_k = 1$ and $X_i = 0$ for all $i \neq k$ with zero loading, $M^{(0)}(t, x)$ are moments for $X_i \equiv 0$ and given loading, and $a_k(t)$ are the enforced deformations for $X_i \equiv 0$ in the sense of X_k. Introduction of $\mathbf{E}$ according to (2) gives a system of n Volterra's integral equarions for $X_i(t)$. If the structure consists of bars of composite sections, instead of $\mathbf{E}J$ in (16) we have to write $\mathbf{E}_1 J_1 + \mathbf{E}_2 J_2$, and (16) will no more have a form of Volterra's integral equations in time.

c) Further let us consider a homogenous orthotropic rectangular plate of thickness h, freely supported at the boundaries $x = 0$, $x = a$, $y = 0$, $y = b$. If the load is expressed in terms of a Fourier series $p = \sum_{m,n=1}^{\infty} p_{mn}(t) \sin \alpha_m x \sin \beta_n y$, where $\alpha_m = m\pi/a$, $\beta_n = n\pi/b$, and the deflections assumed as $w = \sum_{m,n=1}^{\infty} w_{mn}(t) \sin \alpha_m x \sin \beta_n y$, we may directly write, interchanging in the known elastic solution for the coefficients w_{mn} in ([16] (p. 406)), the elastic constants with creep operators

$$\tag{17} \tfrac{1}{12}\left[\alpha_m^4 \boldsymbol{E}_{xx} + 2\alpha_m^2\beta_n^2(\boldsymbol{E}_{xy} + 2\boldsymbol{G}_{xy}) + \beta_n^4 \boldsymbol{E}_{yy}\right] w_{mn}(t) = p_{mn}(t) \,.$$

Introducing, for instance, $\boldsymbol{E}_{xx}, \ldots$ in the form (4), these equations represent independent Volterra's integral equations for $w_{mn}(t)$. Similarly, also the equation for creep of composite beams, plates, sandwich plates, shells, thick-walled cylinders, etc. may be determined [8], [4].

Eqs. (16) or (17), (17a), etc., which represent equations only in time, can be solved, in general, by any approximate numerical method for ordinary differential or integral equations, but the integral form is more suitable. To solve the creep problem in time it is necessary to reduce it to differential or integral equations which correspond to algebraic equations in elasticity. For creep at boundary value problems in space it is necessary, therefore, first to introduce a certain approximate method for integration in space, resulting in algebraic equations for elastic problem, e.g. Fourier series, variational methods or finite difference methods (at which, however, usual interation procedures in space coordinates x_i are in this formulation not usable).

Therefore, it would be of advantage to generalize the integration method in time in such a way that it would be independent of the integration method in space.

APPROXIMATE METHODS OF INTEGRATION IN TIME

1. The simplest idea is to replace the integral in the creep law (2) or (4) by finite sum, e.g.

$$\tag{18} \int_{t_0}^{t} \sigma(\tau)\, L(t, \tau)\, \mathrm{d}\tau \approx \Delta t \sum_{r=0}^{m} c_{(r)} \sigma_{(r)}\, L(t_{(m)}, t_{(r)}) \,,$$

where e.g. for the trapezoid rule $c_{(0)} = c_{(m)} = \frac{1}{2}$, $c_{(1)} = c_{(2)} = \ldots = c_{(m-1)} = 1$. For the case steady loading in which the final values for $t \to \infty$ are to be found it is more convenient first to substitute a new integration variable ϑ, e.g. $\vartheta = 1 - \mathrm{e}^{-(t-t_0)/\tau_R}$ and use constant portions $\Delta\vartheta$. Denoting by $c'_{(r)}$ the coefficients of numerical integration in ϑ, it is necessary only to replace:

$$\tag{18a} \Delta t c_{(r)} = \Delta\vartheta c'_{(r)} \tau_R / (1 - \vartheta_{(r)}) \,.$$

The anisotropic creep law (10) may then be written in the form:

$$\tag{19a} \varepsilon_{ij_{(m)}} = C'_{ijkl_{(m)}} \sigma_{kl_{(m)}} + \varepsilon^0_{ij_{(m-1)}} \,,$$

or

(19b) $$\sigma_{ij(m)} = E'_{ijkl(m)}\varepsilon_{kl(m)} + \sigma^0_{ij(m-1)},$$

where

(20a) $$C'_{ijkl(m)} = C_{ijkl(m)} + \Delta t\, c_{(m)}L_{ijkl}(t_{(m)}, t_{(m)}),$$

(20b) $$\varepsilon^0_{ij(m-1)} = \Delta t \sum_{r=0}^{m-1} c_{(r)}\sigma_{kl(r)}L_{ijkl}(t_{(m)}, t_{(r)}),$$

or

$$E'_{ijkl(m)} = E_{ijkl(m)} + \Delta t\, c_{(m)}\, R_{ijkl}(t_{(m)}, t_{(r)}),$$

$$\sigma^0_{ij} = \Delta t \sum_{r=0}^{m-1} c_{(r)}\varepsilon_{kl(r)}\, R_{ijkl}(t_{(m)}, t_{(r)}).$$

Obviously, eqs. (19a, b) are recurrent equations. The values for $t_{(0)} = t_0, t_{(1)}, \ldots$ $\ldots, t_{(m-1)}$ are already known, while the values for $t_{(m)}$ are sought for, and the term $\varepsilon^0_{ij(m-1)}$ or $\sigma^0_{ij(m-1)}$ is also known.

Physically, eq. (19a) represents an elastic stress-strain law with (fictitious) enforced internal strains ε^0_{ij}. In a one-dimensional case it may be regarded as a given temperature dilatation $\varepsilon^0 = \alpha\,\Delta T$ caused by the temperature change ΔT (or as a given shrinkage or swell), and eq. (19) as a stress-strain law of thermoelasticity. For multiaxial stress, however, this is a generalized stress-strain law of thermoelasticity, since $\varepsilon^0_{xx}, \varepsilon^0_{yy}, \varepsilon^0_{zz}$ may be different and, further, also enforced shears $\varepsilon^0_{xy}, \ldots$ may occur. Together with (11)–(14) we have thus a problem of generalized thermoelasticity.

Eq. (19b) represents an elastic stress-strain law with (fictitious) internal prestress $-\sigma^0_{ij}$.

With respect to unchangen eqs. (11)–(14), the equations (19a, b) allow to express the numerical integration by the following statement: The stresses and strains in time $t_{(m)}$ may be computed as for an elastic body with changed elastic constants $C'_{ijkl(m)}$ or $E'_{ijkl(m)}$ according to (20a, b) under internal enforced strains or internal prestresses determined from the preceding values of stresses or strains according to eqs. (20a, b), given loads in time $t_{(m)}$, prescribed displacements, thermal dilatations and shrinkages from t_0 to $t_{(m)}$.

2. A modified method, using directly the creep modulus E_{cr}, may be based on the creep law (1). Evaluating the integral in it by the trapezoid rule and calculating the difference $\Delta\varepsilon_{(m)} = \varepsilon_{(m)} - \varepsilon_{(m-1)}$ it leads for uniaxial stress to:

(21) $$\Delta\varepsilon_{(m)} = \Delta\sigma_{(m)}/E'_{(m)} + \varepsilon^0_{(m-1)},$$

where

(21a) $$\varepsilon^0_{(m-1)} = \sigma_{(0)}\,\Delta E^{-1}_{cr(m,0)} + \tfrac{1}{2}\sum_{r=1}^{m-1}\Delta\sigma_{(r)}\left[E^{-1}_{cr(m,\,r-1)} + \Delta E^{-1}_{cr(m,r)}\right],$$

$$\Delta E^{-1}_{cr(m,r)} = E^{-1}_{cr}(t_{(m)}, t_{(r)}) - E^{-1}_{cr}(t_{(m-1)}, t_{(r)}),\quad E'^{-1}_{(m)} = \tfrac{1}{2}\left[E^{-1}_{cr}(t_{(m)}, t_{(m-1)}) + E^{-1}_{(m)}\right].$$

The statement for the integration procedure is similar. The same may also be done for the relaxation modulus E_{rel}.

3. The generalization of the well-known successive approximations (iterations) for Volterra's integral equations is not so evident. Eqs. (2) or (4) are not integral equations, because in general neither $\varepsilon(t)$ nor $\sigma(t)$ are given, but are bound by further conditions (11)–(14) to the given loads. If an algebraic auxiliary condition $F[\varepsilon(t), \sigma(t), p(t)] = 0$ would be given, the successive approximations $\sigma_{(r)}(t)$, $\varepsilon_{(r)}(t)$ for the eq. (2) would result, evidently, from the equation:

$$\varepsilon_{(r)}(t) = \sigma_{(r)}(t)/E(t) + \varepsilon^0_{(r-1)}(t)\,, \tag{22}$$

where

$$\varepsilon^0_{(r-1)}(t) = \int_{t_0}^{t} \sigma_{(r-1)}(\tau)\, L(t, \tau)\, \mathrm{d}\tau \tag{22a}$$

and $F[\varepsilon_{(r)}(t), \sigma_{(r)}(t), p(t)] = 0$. Using some approximate method, e.g. the finite difference method, for integration in space, the eqs. (11)–(14) are transformed into algebraic auxiliary conditions. This allows to extend the validity also for the problem (11)–(14) (or (15)) and to compute the sucessive approximations according to the equation of the type (22) with the equations (11)–(14) written for $\varepsilon_{(r)}$ and $\sigma_{(r)}$. As a first approximation the initial values or the elastic solution may be introduced. In this system of equations time has the role of an independent parameter and, in view of (22), we have again an elastic problem with an internal (fictitious) enforced strain $\varepsilon^0_{(r-1)}(t)$, eq. (22a). The statement for the computation procedure is similar to that ad 1.

The same may also be done for the creep law (4). It yields the elastic law

$$\sigma_{(r)}(t) = E(t)\, \varepsilon_{(r)}(t) + \sigma_{(r-1)}(t)\,, \tag{23}$$

with the internal prestress

$$\sigma^0_{(r-1)}(t) = \int_{t_0}^{t} \varepsilon_{(r-1)}(\tau)\, R(t, \tau)\, \mathrm{d}\tau\,. \tag{23a}$$

4. A modified method od successive approximations may be based on the derivative of eq. (1), having the character of Volterra's equation between $\partial\varepsilon/\partial t$ and $\partial\sigma/\partial t$ (see [8]).

5. Similarly also the methods for creep laws in differential form may be built up ([3], [4], 1964).

a) Thus, for instance the creep law (9b) for concrete according to the most simple finite difference method may be replaced by

$$\Delta\varepsilon_{(m)} = \Delta\sigma_{(m)}/E'_{(m)} + \varepsilon^0_{(m-1)}\,, \tag{24}$$

where

$$\text{(24a)} \qquad \varepsilon^0_{(m-1)} = \Delta\varphi\, \sigma_{(m-1)}/E_0\,, \quad E'^{-1}_{(m)} = \tfrac{1}{2}\bigl(E^{-1}_{(m-1)} + E^{-1}_{(m)} + \Delta\varphi/E\bigr)\,.$$

The corresponding calculation procedure is yet obvious and is the same as that according to (21), (21a) if (9a) is substituted.

b) High accuracy is attained by the Runge-Kutta method. It has been derived for the creep law (9b) [3], [4] that

$$\text{(25)} \qquad \begin{aligned} \Delta\varepsilon_{ij(m)} &= \tfrac{1}{6}\Delta\varphi\bigl({}^1\varepsilon_{ij(m)} + 2\,{}^2\varepsilon_{ij(m)} + 2\,{}^3\varepsilon_{ij(m)} + {}^4\varepsilon_{ij(m)}\bigr)\,,\\ \Delta\sigma_{ij(m)} &= \tfrac{1}{6}\Delta\varphi\bigl({}^1\sigma_{ij(m)} + 2\,{}^2\sigma_{ij(m)} + 2\,{}^3\sigma_{ij(m)} + {}^4\sigma_{ij(m)}\bigr)\,, \end{aligned}$$

where for the auxiliary values ${}^1\varepsilon_{ij}$, ${}^1\sigma_{ij}$, ${}^2\varepsilon_{ij}$, ..., we have, successively, the elastic laws with elastic constants in the times which correspond to $\varphi_{(m-1)}$, $\varphi_{(m)} - \frac{1}{2}\Delta\varphi$, $\varphi_{(m)} - \frac{1}{2}\Delta\varphi$, $\varphi_{(m)}$ and with enforced internal strains:

$$\text{(25a)} \qquad \begin{aligned} &{}^1\varepsilon^0_{(m)} = \sigma_{(m-1)}/E_0\,, \quad {}^2\varepsilon^0_{(m)} = \bigl(\sigma_{(m-1)} + \tfrac{1}{2}\Delta\varphi\, {}^1\sigma_{(m)}\bigr)/E_0\,,\\ &{}^3\varepsilon^0_{(m)} = \bigl(\sigma_{(m-1)} + \tfrac{1}{2}\Delta\varphi\, {}^2\sigma_{(m)}\bigr)/E_0\,, \quad {}^4\varepsilon^0_{(m)} = \bigl(\sigma_{(m-1)} + \Delta\varphi\, {}^3\sigma_{(m)}\bigr)/E_0\,, \end{aligned}$$

while the derivatives of given loads, prescribed displacements etc. in time corresponding to $\varphi_{(m-1)}$, $\varphi_{(m)} - \frac{1}{2}\Delta\varphi$, $\varphi_{(m)} - \frac{1}{2}\Delta\varphi$ and $\varphi_{(m)}$ have to be considered in place of external loads etc.

Similar methods may also be developed for second-order differential law corresponding to (9c) [8] and for creep according to standard model etc. [8].

Power series representation of σ_{ij}, ε_{ij} in time leads also to elastic laws between the coefficients of the series [8].

Another way of derivation of all these methods may start from the known approximate methods for a system of equations (16) for redundants, because an elastic continuum may be idealized with a framework consisting of bars [14], [15], [11], [13], [1], the number of which increases to infinity. The significance of approximate solution of (16) is, namely, the same as that obtained above, and it is independent of the selection and number of redundants. For instance, the successive approximations of (16) with E according to (2) are given by the following recurrent algebraic equations with t as a parameter:

$$\text{(26)} \qquad \sum_j X_{j(r)}(t) \int_{(x)} \frac{M^{(i)}M^{(j)}}{E(t)\,J}\,\mathrm{d}x + \int \frac{M^{(i)}\,M^{(0)}(t)}{E(t)\,J}\,\mathrm{d}x + a_i(t) +$$

$$+ \left\{\sum_j \int_{t_0}^{t} X_{j(r-1)}(\tau) \left[\int_{(x)} \frac{M^{(i)}M^{(j)}}{J} L(t,\tau)\,\mathrm{d}x + \int_{(x)} \frac{M^{(i)}\,M^0_{(r-1)}(\tau)}{J}\,L(t,\tau)\,\mathrm{d}x\right]\mathrm{d}\tau\right\} = 0\,,$$

where the term $\{\ldots\}$ represents the deformation due to $\varepsilon^0_{(r-1)}$ on the primary system $(X_i \equiv 0)$ in the sense of X_i.

All these methods may also be interpreted purely mathematically. Thus, for instance, the problem (15) is integrated in time by succesive aproximations ad 3, solving successively the recurrent boundary value problems in x:

(26a) $$\frac{\partial^2 w_{(r)}(x, t)}{\partial x^2} = -\frac{M_{(r)}(x, t)}{E(x, t)\, J(x)} - \int_{t_0}^{t} \frac{M_{(r-1)}(x, \tau)}{J(x)} L(x; t, \tau)\, \mathrm{d}\tau$$

with $\partial^2 M_{(r)}/\partial x^2 = p$ and for $x = 0$ and $x = l$, with $w_{(r)} = \partial w_{(r=0)}/\partial x$. For the general problem (10), (11)–(14) the formulation is similar.

The proof of convergence with possible estimation of error, however, has not yet been given, although in certain practical cases the convergence is physically obvious and was verified by numerical computation [4].

REPLACEMENT OF INTERNAL ENFORCED STRAINS OR PRESTRESSES BY EXTERNAL LOADS

Similarly as in thermoelastic problems [10], the effect of enforced strains or prestresses will now be reduced to equivalent external loads. The effect of fictitious internal enforced strains ε^0_{ij} in the law of type (19a) is equivalent to the effect of fictitious internal prestresses $=\sigma^0_{ij}$ in the law (19b), determined from the relationships:

(27) $$\varepsilon^0_{ij} = C'_{ijkl}\, \sigma^0_{kl} \quad \text{or} \quad \sigma^0_{ij} = E_{ijkl}\, \varepsilon^0_{kl}\,.$$

The solution of the generalized thermoelastic problem (19a, b), (11)–(14) will be sought for in the form:

(28) $$\sigma_{ij} = -\sigma^0_{ij} + \sigma^{\mathrm{I}}_{ij}\,, \quad \varepsilon_{ij} = \varepsilon^{\mathrm{I}}_{ij}\,, \quad u_i = u^{\mathrm{I}}_i\,.$$

For the prestresses $-\sigma^0_{ij}$ we determine the volume forces $-f^0_i$ and surface forces $-p^0_i$ necessary to fulfil the equilibrium:

(29) $$f^0_i = -\sigma_{ij,j}\,, \quad p^0_i = n_j \sigma^0_{ij}\,.$$

If the creep parameters are discontinuous through a certain internal surface Γ^0_i, with unit normal vectors n^0_i, then also σ^0_{ij} are obtained discontinuous on Γ^0, and the loads $-q^0_i = -n^0_j(\sigma^0_{ij(+)} - \sigma^0_{ij(-)})$ must be added on Γ^0. Substitution of (29) into (19a, b) and (10)–(13) yields with respect to (27):

(30) $$\varepsilon^{\mathrm{I}}_{ij} = C'_{ijkl}\sigma^{\mathrm{I}}_{kl} \quad \text{or} \quad \sigma^{\mathrm{I}}_{ij} = E'_{ijkl}\varepsilon^{\mathrm{I}}_{kl}\,,$$

(31) $$\sigma^{\mathrm{I}}_{ij,j} + f_i + f^0_i = 0\,, \quad \varepsilon^{\mathrm{I}}_{ij} = \tfrac{1}{2}(u^{\mathrm{I}}_{i,j} + u^{\mathrm{I}}_{j,i} + u^{\mathrm{I}}_{k,i}u^{\mathrm{I}}_{k,j})$$

with the boundary conditions

(32) $$n_j\sigma^{\mathrm{I}}_{ij} = p_i + p^0_i \ (\text{on } \Gamma_1)\,, \quad u^{\mathrm{I}}_i = b_i \ (\text{on } \Gamma_2)\,,$$

and eventually, according to the condition $n^0_j(\sigma_{ij(+)} - \sigma_{ij(-)}) = 0$, with $n^0_j(\sigma^{\mathrm{I}}_{ij(+)} - \sigma^{\mathrm{I}}_{ij(-)}) = q^0_i$ (on Γ^0).

Thus we have an elastic problem. The deformations, small as well as finite, caused by internal enforced strains or prestresses, are equal to the deformations caused by volume and surface loads and, accasionaly, internal surface loads, which are in equilibrium with the internal prestresses. For obtaining the stresses we must subtract the initial prestresses. The same would result for linearized stability problems, as the equilibrium equation is still linear.

In a region with homogeneous creep, i.e. proportional to elastic deformation, we obtain $\varepsilon^0_{ij(m-1)} \sim \varepsilon_{ij(m-1)}, \sigma^0_{ij} \sim \sigma_{ij(m-1)}$, so that $f^0_i \sim f_i$, $p^0_i \sim p_i$, $q^0_i = 0$.

PLATES AND STRUCTURES OF BARS. PRACTICAL CALCULATION

Engineer's solutions of bars, plates, shells, sandwich shells, etc. are based on various simplifying deformation hypotheses. As the structures are solved directly by means of integral internal forces in the cross-sections and deformations of these sections, also the enforced deformations may be considered integrally for the entire section provided that the creep is homogeneous in the section. Moreover, in these structures there is no difference between volume and surface loads.

a) To demonstrate this, consider firstly a plane framed structure consisting of bars with composite section which remain plane and perpendicular after deformation. In each section for each part (ν) for elastic deformation, the well-known equations $w_{,xx} = M^\nu/E^\nu J^\nu$ and $\varepsilon^\nu = N^\nu/E^\nu F^\nu$ are valid. Here M^ν, N^ν are the bending moment and the normal force in the part (ν) with respect to its centroid, J^ν is the inertia moment to the centroid of the part (ν), F^ν is the area, $w_{,xx}$ is the curvature of the bar, equal for all parts, and ε^ν is the elongation in the centroidal fibre of the part (ν). If each part (ν) is homogeneous, we may directly introduce fictitious interal enforced curvatures, using e.g. the method ad 1., eq. (20a):

$$(33) \qquad \begin{aligned} w^{0\nu}_{,xx(m-1)} &= \Delta t\,(J^\nu)^{-1} \sum_{r=0}^{m-1} c_{(r)} M^\nu_{(r)}\, L^\nu(t_{(m)}, t_{(r)})\,, \\ \varepsilon^{0\nu}_{(m-1)} &= \Delta t\,(F^\nu)^{-1} \sum_{r=0}^{m-1} c_{(r)} N^\nu_{(r)}\, L^\nu(t_{(m)}, t_{(r)})\,. \end{aligned}$$

According to (27), the equivalent prestressing moments $M^{0\nu}$ and normal forces $N^{0\nu}$ are:

$$(34) \qquad M^{0\nu} = E^\nu J^\nu w^{0\nu}_{,xx}\,, \quad N^{0\nu} = E^\nu F^\nu \varepsilon^{0\nu}\,.$$

We may also directly calculate, (eq. (20b)):

$$(35) \qquad \begin{aligned} M^{0\nu}_{(m-1)} &= \Delta t\, J^\nu \sum_{r=0}^{m-1} c_{(r)}\, w_{,xx(r)}\, R^\nu(t_{(m)}, t_{(r)})\,, \\ N^{0\nu}_{(m-1)} &= \Delta t\, F^\nu \sum_{r=0}^{m-1} c_{(r)}\, \varepsilon^\nu_{(r)}\, R^\nu(t_{(m)}, t_{(r)})\,. \end{aligned}$$

The equivalent composite prestressing moments M and normal forces N with respect to the axis x of the bar are

$$(36) \qquad M^0 = \sum_{(v)}(M^{0v} + e^v N^{0v}), \quad N^0 = \sum_{(v)} N^{0v},$$

where e^v are the distances of centroids of the parts (v) from the x-axis. Now we determine from the well-known equilibrium conditions the equivalent external transverse loads p and longitudinal loads n^0 (to the forst order):

$$(37) \qquad p^0 = -\mathrm{d}^2 M^0/\mathrm{d}x^2, \quad n^0 = -\mathrm{d}N^0/\mathrm{d}x$$

with loading moments, normal forces and shear forces equal to M^0, N^0 and $V^0 = \mathrm{d}M^0/\mathrm{d}x$ at the ends of the bars and, possibly, with difference loads $M^0(x^+) - M^0(x^-)$, $N^0(x+) - N^0(x^-)$, $V^0(x^+) - V^0(x^-)$ in the sections of discontinuity of $M^0, \ldots$.

The solution for the distribution of internal forces in a composite section of a statically determinate bar is a special case of this, at which the loads p^0, n^0 etc. need not to be determined. The solution of structure with homogeneous sections is a special case, in which eq. (36) fall out and $M^v = M$ etc.

b) Secondly, consider a thin multilayered orthotropic plate in a plane (x, y), the normals of which remain straight and perpendicular after deformation. The stress-strain law let be:

$$(38) \qquad \sigma_{xx} = E_{xx}\varepsilon_{xx} + E_{xy}\varepsilon_{yy}, \quad \sigma_{yy} = \ldots, \quad \sigma_{xy} = 2G_{xy}\varepsilon_{xy},$$

or converselíy

$$\varepsilon_{xx} = C_{xx}\sigma_{xx} + C_{xy}\sigma_{yy}, \quad \varepsilon_{yy} = \ldots, \quad 2\varepsilon_{xy} = G_{xy}^{-1}\sigma_{xy},$$

where the operators E_{xx}, $E_{xy} = E_{yx}$, G_{xy}, ... or C_{xx}, C_{xy}, C_{xy}, ... have the same form as E or E^{-1}, eq. (2) or (4), with kernels R_{xx}, R_{xy}, R_{yy}, Q_{xy}, or L_{xx}, L_{yy}, L_{xy}, K_{xy}. According to the well-known elastic relations between the second derivatives w_{xx}, w_{xy}, w_{yy} of a (small) deflection w, the strains ε_{xx}^v, ε_{yy}^v, ε_{xy}^v at the middle of the layer (v), the bending and torsional moments M_x^v, M_y^v, M_{xy}^v and normal and shear forces N_x^v, N_y^v, N_{xy}^v in the plane (x, y) for each layer (v), we may write, using e.g. the method ad 1., eq. (20a):

$$(38) \qquad w^{0v}_{,xx(m-1)} = 12(d^v)^{-3}\,\Delta t \sum_{r=0}^{m-1} c_{(r)}\left[M^v_{x(r)} L^v_{xx}(t_{(m)}, t_{(r)}) + M^v_{y(r)} L^v_{xy}(t_{(m)}, t_{(r)})\right],$$

$$w^{0v}_{,yy(m-1)} = \ldots,$$

$$w^{0v}_{,xy(m-1)} = 12(d^v)^{-3}\,\Delta t \sum_{r=0}^{m-1} c_{(r)} M^v_{xy(r)} K^v_{xy}(t_{(m)}, t_{(r)}),$$

$$\varepsilon^{0v}_{xx(m-1)} = (d^v)^{-1}\,\Delta t \sum_{r=0}^{m-1} c_{(r)}\left[N^v_{x(r)} L^v_{xx}(t_{(m)}, t_{(r)}) + N^v_{y(r)} L^v_{xy}(t_{(m)}, t_{(r)})\right],$$

$$\varepsilon^{0v}_{yy(m-1)} = \ldots, \quad 2\varepsilon^{0v}_{xy(m-1)} = (d^v)^{-1}\,\Delta t \sum_{r=0}^{m-1} c_{(r)} N^v_{xy(r)} K^v_{xy}(t_{(m)}, t_{(r)}).$$

According to (27), the equivalent prestresses are:

$$(39)\qquad \begin{aligned} M_x^{0v} &= \tfrac{1}{12}(d^v)^3\,(E_{xx}^v w_{,xx}^{0v} + E_{xy} w_{,yy}^{0v}), \quad M_y^{0v} = \dots, \\ M_{xy}^{0v} &= \tfrac{1}{12}(d^v)^3\, G_{xy}^v w_{,xy}^{0v}, \\ N_x^{0\varepsilon} &= d^v(E_{xx}^v \varepsilon_{xx}^{0v} + E_{xy}^v \varepsilon_{yy}^{0v}), \dots, N_{xy}^{0v} = 2d^v G_{xy}^v \varepsilon_{xy}^{0v}. \end{aligned}$$

They can also be calculated directly:

$$(40)\qquad \begin{aligned} M_{x(m-1)}^{0v} &= \tfrac{1}{12}(d^v)^3\,\Delta t \sum_{r=0}^{m-1} c_{(r)}\big[R_{xx}^v(t_{(m)}, t_{(r)})\, w_{,xx_{(r)}} + R_{xy}^v(t_{(m)}, t_{(r)})\, w_{,xy_{(r)}}\big], \\ M_{y(m-1)}^{0v} &= \dots, M_{xy(m-1)}^{0v} = \tfrac{1}{12}(d^v)^3\,\Delta t \sum_{r=0}^{m-1} c_{(r)} Q_{xy}^v(t_{(m)}, t_{(r)})\, w_{,xy_{(r)}}, \\ N_{x(m-1)}^{0v} &= d^v\,\Delta t \sum_{r=0}^{m-1} c_{(r)}\big[R_{xx}^v(t_{(m)}, t_{(r)})\, \varepsilon_{xx(r)}^v + R_{xy}^v(t_{(m)}, t_{(r)})\, \varepsilon_{yy(r)}^v\big], \\ N_{y(m-1)}^{0v} &= \dots, \quad N_{xy}^{0v} = d^v\,\Delta t \sum_{r=0}^{m-1} c_{(r)} \cdot 2Q_{xy}^v(t_{(m)}, t_{(r)})\, \varepsilon_{xy(r)}^v. \end{aligned}$$

The equivalent composite internal moments and forces are:

$$(41)\qquad \begin{aligned} M_x^0 &= \sum_{(v)}(M_x^{0v} + e^v N_x^{0v}), \dots, M_{xy}^0 = \sum_{(v)}(M_{xy}^{0v} + e^v N_{xy}^{0v}), \\ N_x^0 &= \sum_{(v)} N_x^{0v}, \dots, \quad N_{xy}^0 = \sum_{(v)} N_{xy}^{0v}, \end{aligned}$$

where e^v are the distances of the middle of the layers (v) from the middle plane (x, y). Now, the equivalent transverse load p_z^0 and inplane loads p_x^0, p_y^0 (to the first order terms) are:

$$(42)\qquad \begin{aligned} p_z^0 &= \partial^2 M_x^0/\partial x^2 + 2\partial^2 M_{xy}/\partial x\,\partial y + \partial^2 M_y^0/\partial y^2, \\ p_x^0 &= -\partial N_x^0/\partial x - \partial N_{xy}^0/\partial y, \quad p_y^0 = \dots \end{aligned}$$

At the boundary it is necessary to add the loading moments and forces equal to $M_x^0, M_{xy}^0, \dots, N_{xy}^0$ and, eventually, in the lines of discontinuity of $M_x^0, \dots$, the difference loads: $M_{x(+)}^0 - M_{x(-)}^0, \dots, N_{xy(+)}^0 - N_{xy(-)}^0$ with the shear forces: $V_{x(+)}^0 - V_{x(-)}^0$, $V_{y(+)}^0 - V_{y(-)}^0$, where $V_x^0 = \partial M_x^0/\partial x + \partial M_{xy}^0/\partial y$, etc.

For the bending of a symmetric-layered plate it is possible to consider directly the moments carried by a symmetric pair of layers, the forces $N_x^v, \dots$ being unnecessary. A thin plate, homogeneous in each normal, is a special case, in which eq. (41) falls out and $M_x^v = M_x$ etc.

Thin-walled bars, shear-deforming bars, space frames and especially sandwich plates and shells may be solved in a similar way (see [8]).

In analyses of structures solved in terms of displacements, such as plates, e.g., it is less laborious to use the operators for the history of deformation, i.e. $\boldsymbol{E}_{xx}$ etc. In analyses by the force method, the use of operators for the history of stress, i.e. $\boldsymbol{E}^{-1}$, $\boldsymbol{C}_{xx}$, ... is more efficient. In practical calculation, the finite difference method in space is frequently used. The loads equivalent to prestresses are than concentrated loads in the nodes of the network. Sometimes it is advantageous if influence lines, influence surfaces or fields of the structure are available.

CONCLUSION

The method presented herein permits to solve any creep problem (solvable in elasticity) with the linear creep law by means of a succession of corresponding elastic problems, i.e. by coupling in a sequence the known computer programs for elastic problems. This method is useful for time-invariant creep as well as for the creep law with ageing. The presented formulation of the analogy of creep with elasticity allows to obtain from any equation for the elastic problem the corresponding equations in the case of creep.

References

[1] Ang A. H. S., Newmark, N. M.: A numerical procedure for the analysis of continuous plates, 2nd Conference on Electronic Computation, ASCE, p. 379, Sept. 1960, Pittsbourgh.

[2] Arutyunyan, N. Ch.: Some questions of the creep theory (in Russian), Tekhteorizdat, Moskva 1952.

[3] Bažant, Z. P.: Approximate methods of solving creep and shrinkage of nonhomogenous concrete structures and the use of computers (in Czech), Stavebnícky časopis, Slovak Acad. Sci. XII, 1964, No 7, p. 414.

[4] Bažant, Z. P.: Concrete creep in structural analysis (in Czech), State publ. House Techn. Lit., Prague 1966.

[5] Bažant, Z. P.: Die Berechnung des Kriechens und Schwindens nicht homogener Betonkonstruktionen, 7. Kongress IVBH, Rio de Janeiro 1964, Vorbericht, 887–98.

[6] Bažant, Z. P.: Influence of creep on long-time stability and buckling strength of concrete columns (in Czech), Proc. 5th Nat. Conf. on Prestressed Concrete in Žilina, Nov. 1965, House of Technology VTS, Bratislava (Czechoslovakia) 1965, p. 115.

[7] Bažant, Z. P.: Phenomenological theories for creep of concrete based on rheological models, Acta Technica Cz. Acad. Sci., 1966, No 1, p. 82–109.

[8] Bažant, Z. P.: Analysis of time-variations of stress and strain for nonhomogeneous linear creep (in Czech), Research Report No 75/66, Building Research Institute, Techn. Univ., Prague, 1966.

[9] Bland, D. H.: The theory of linear viscoelasticity, Pergamon Press, Oxford 1960.

[10] Boley, B. A., Weiner, J. H.: Theory of thermal stresses, Wiley, New York 1960.

[11] Clough, R. W.: The finite element method in plane stress analysis, 2nd Conf. on Electronic Computation, ASCE, p. 345, Sept. 1960, Pittsbourgh.

[12] Freudenthal, A. M., Geiringer, H.: The mathematical theories of the inelastic continuum, Handbuch der Physik, Band VI, Springer, Berlin 1958.

[13] Havelka, K.: Bridge plates of arbitrary shape and stiffness (in Slovak), Stavebnícky časopis, Slovak Acad. Sci., XI, 1963, No 1, p. 5.
[14] Hrennikoff, A.: Solution of Problems of Elasticity by the framework method, J. Appl. Mech., Dec. 1941.
[15] McHenry, D.: A lattice analogy for the solution of stress problems, Jour. Institution of Civ. Engrs., Dec. 1943, pp. 59–82.
[16] Timoshenko, S., Woinowski-Krieger, S.: Theory of plates and shells, 2nd ed., McGraw-Hill, New York 1959.
[17] Williams, M. Z.: The structural analysis of viscoelastic materials, AIAA Summer Meeting, Los Angeles 1963.
[18] Engineering desing for plastics, ed. by E. Baer (Chpt. 3, Viscoelastic behaviour and time-temperature relationships, by E. Passaglia, J. R.: Knox; Chpt. 4, Viscoelastic analysis, by H. H. Hilton, Chpt. 14; Reinforced plastics, by H. T. Cornet), New York, Reinhold 1964.
[19] Handbuch der Physik, Band I.: Mathematische Methoden, ed. S. Flügge, Springer, Berlin 1956.

[Received February 15, 1967]

Ing. Zdeněk P. Bažant, CSc., Building Research Institute, Technical University, Šolínova 7, Prague 6