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OBSAH — СОДЕРЖАНИЕ — INHALTVERZEICHNIS — CONTENTS — SOMMAIRE

ACTA POLYTECHNICA — Práce ČVUT v Praze, I, 3, 1967

ON INSTABILITY OF THREE-DIMENSIONAL BODIES

ZDENĚK P. BAŽANT*)

1 Introduction

The ever continued incorporation of plastic materials as load carrying structural elements makes it imperative — in view of the material low modulus of elasticity — to initiate intensive studies of the problem of stability. Traditional engineering analyses of the stability of bars, thin plates and shells will no longer serve a useful purpose in such studies as the problems involved are also those of the stability of three-dimensional bodies (Fig. 1) which have to be solved when e. g. examinations are made of the local, surface and overall stability of thick slabs, sandwich plates; with all probability, they also play an important role in the failure of laminated materials by de-lamination under compressive stresses. In geology similar phenomena are the essential nature of the process of strata folding.

Many researchers have studied the problem of the stability of three-dimensional bodies, starting with Bryan in 1889 [8]. Particular notice should be taken in this connection of the work by Pearson [8], Biot [1] and Neuber [7]. The problem may be formulated either as an energy one (equation [5]) or that of the existence of another adjacent equilibrium state of deformation. We shall use the energy method for deriving the variational and differential conditions of instability. We shall start from Pearson's [8], [9] criterion of stability for small changes of deformation and deduce it ourselves in a simple, straightforward manner. Let us note at this point that the equations arrived at in our paper differ from those of Biot [1] which he has derived from the conditions of equilibrium for an adjacent equilibrium state, as well as from those due to Neuber [6], [7], [4]. This circumstance is due to different definitions of stress and strain and of the law of elasticity at a finite change of strain.

2 The energy condition of stability

We may write the condition of stability deduced by Pearson for small changes of strain of an elastic anisotropic body, i. e. the condition that no small change of strain from the equilibrium state of strain of the body be possible, for a load

*) ZDENĚK P. BAŽANT, CSc., Building Research Institute, Technical University of Prague, (*Stavební ústav ČVUT v Praze, Šolínova, 7, Praha 6 - Dejvice, Czechoslovakia*).

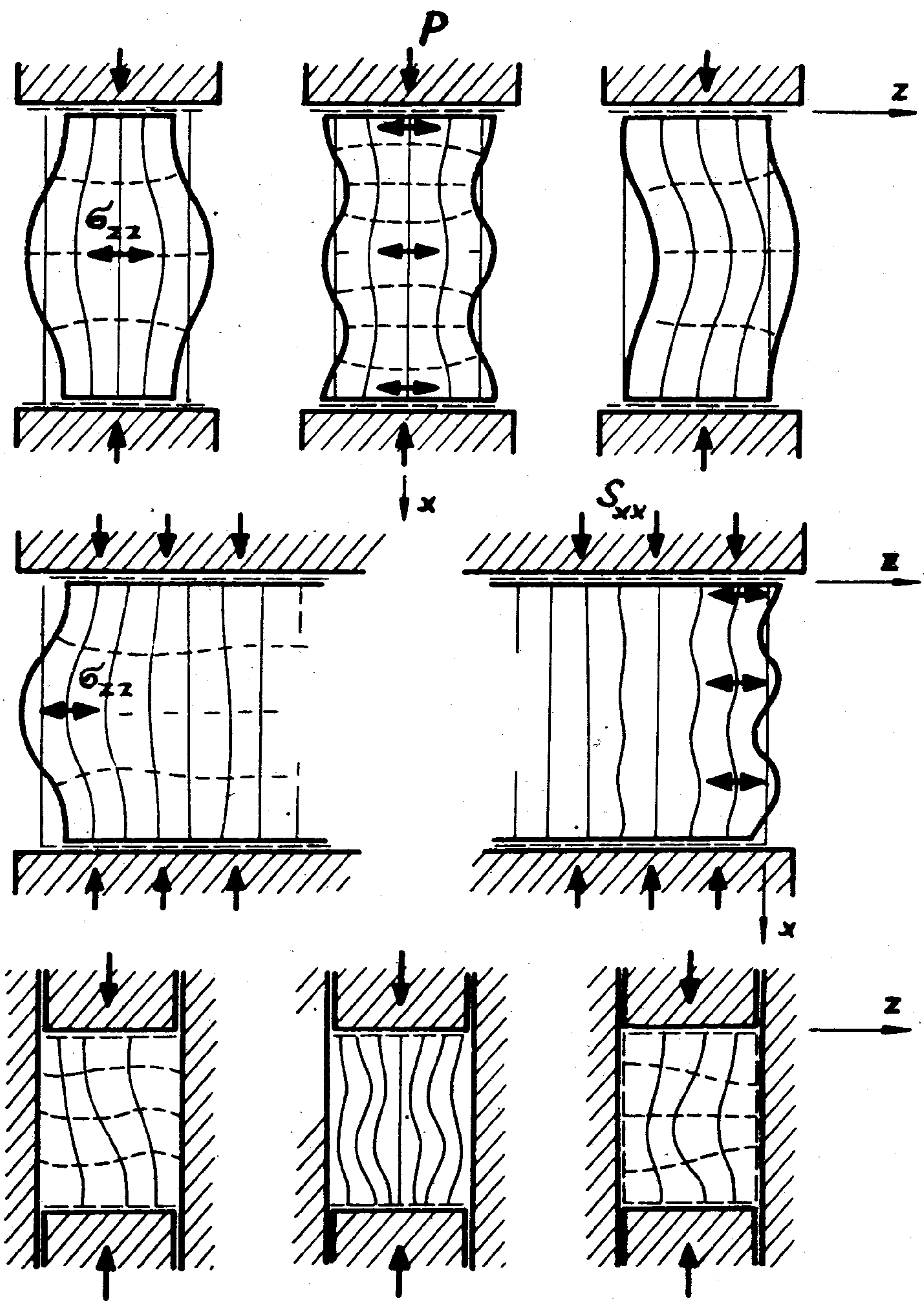

Fig. 1 Some possible forms ot loss of stability of a compressed thick slab and a half-space

of a given direction and magnitude, in the Cartesian coordinates x_1, x_2, x_3 in the following form ([8], eq. /13/, [9], eq. /X. 4. 23/:

$$U_1 + U_2 > 0 \tag{1}$$

where

$$U_1 = \int_V \frac{1}{2} C_{ijkl}\, e_{ij}\, e_{kl}\, \mathrm{d}V\,, \tag{2a}$$

$$U_2 = \int_V \frac{1}{2} S_{ij}\, u_{k,i}\, u_{k,j}\, \mathrm{d}V\,, \tag{2b}$$

V = volume of the body; u_i = increments of the displacement against the initial equilibrium state in which a point with coordinates x_i (state II) passes to a point with coordinates x_i' (state II'), $x_i' = x_i + u_i$; e_{ij} = components of a small change of strain with respect to the initial state (II); S_{ij} = components of the Euler tensor of initial stresses in the initial equilibrium state of strain (II); C_{ijkl} = incremental elastic constants of the anisotropic law of elasticity; $\sigma_{ij} = C_{ijkl} e_{kl}$ for small changes of strain e_{ij} and small changes of stress σ_{ij} against the initial state. The static interpretation of σ_{ij} will not be introduced. In the above as in the forthcoming discussion we always consider the summation rule for twice repeated subscripts (with the exception of x, y, z). Cubscripts separated by a comma denote a derivative, viz. $u_{k,i} = \frac{\partial u_k}{\partial x_i}$.

For an isotropic material it is especially:

$$U_1 = \int_V \frac{1}{2} \left(2G e_{ij} + \delta_{ij} \frac{2G\nu}{1 - 2\nu} e_{kk}\right) e_{ij}\, \mathrm{d}V\,, \tag{2c}$$

where G is the shear modulus, ν — Poisson's ratio, and δ_{ij} Kronecker's symbol equal to 1 for $i = j$ and 0 for $i \neq j$. Expression U_1 represents the potential energy of a change of strain against the initial equilibrium state. Expression U_2 represents a change of potential energy corresponding to the initial stresses S_{ij} and the difference between a finite and a small (i. e. variation) change of strain since the (Lagrange) tensor of a finite change of strain ε_{ij} in coordinates x_i is

$$\varepsilon_{ij} = e_{ij} + \frac{1}{2} u_{k,i} u_{k,j}\,, \tag{4}$$

where the tensor of a small change of strain or a variation of strain is

$$e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \tag{4a}$$

Let us first indicate a simple direct derivation of condition (1). The energy condition of stability can be stated thus: A body is a stable whenever for each infinitesimal change of strain u_i, e_{ij} satisfying the given boundary conditions, the increment of the potential energy of the internal forces ΔU is larger than work ΔW consumed by the external loads. In other words, change $\Delta U - \Delta W$ of the total potential energy $U - W$ is positive, viz.

$$U - \Delta W > 0 \tag{5}$$

We shall have to include in this change also the terms of the second order, i. e. consider the tensor of the finite strain (4) since the first variation $\delta(U - W)$ in the equilibrium state is zero. The increment of the internal energy is

$$\Delta U = \int_V \left(S_{ij} e_{ij} + \frac{1}{2} \sigma_{ij} e_{ij} \right) \mathrm{d}V =$$

$$= \int_V S_{ij} \frac{1}{2} (u_{i,j} + u_{j,i}) \mathrm{d}V + \int_V S_{ij} \frac{1}{2} u_{k,i} u_{k,j} \mathrm{d}V + \int_V \frac{1}{2} C_{ijkl} e_{ij} e_{kl} \mathrm{d}V \, . \tag{6}$$

Denoting by P_i the load on surface S, which changes neither its magnitude nor its direction at a change of strain (a "dead" load; this does not hold true for e. g. the hydrostatic pressure), and by F_i the volume forces, we get further that

$$\Delta W = \int_V F_i u_i \, \mathrm{d}W + \int_S P_i u_i \, \mathrm{d}S. \tag{7}$$

According to the conditions of equilibrium of the initial state of stress (a consequence of the variation condition $\delta (U - W) = 0$), it is

$$S_{ij,j} + F_i = 0, \quad \text{(in volume } V\text{)}, \tag{8}$$

$$n_j S_{ij} = P_i \quad \text{(on surface } S\text{)} \tag{9}$$

where n_i is the unit vector of the outward normal to element $\mathrm{d}S$; thus we get from the above that

$$\Delta W = - \int_V S_{ij,j} u_i \, \mathrm{d}V + \int_S n_j S_{ij} u_i \, \mathrm{d}S.$$

Because of the continuity of functions S_{ij} and u_i, we may change the second integral according to the Gauss theorem [9], [3] to the volume one, and successively carry out the following rearrangements

$$\Delta W = \int_V S_{ij,j} u_i \, \mathrm{d}V + \int_V \frac{\partial}{\partial x_j} (S_{ij} u_i) \, \mathrm{d}V =$$

$$= \int_V S_{ij} u_{i,j} \, \mathrm{d}V = \int_V S_{ij} \frac{1}{2} (u_{i,j} + u_{j,i}) \, \mathrm{d}V \tag{10}$$

in which we have taken advantage of the symmetry of the stress tensor, $S_{ij} = S_{ji}$. We find easily that $\Delta U - \Delta W = U_1 + U_2$, and in view of [5], obtain the condition of stability [1]. The mathematical transition from expression [7] to [10] can be interpreted physically as follows: The work of the external forces expended on the change of displacement equals the work of the internal forces expended on a small change i. e. variation of strain (4a). As this is actually as expression of the principle of virtual work, we could also write relation (10) directly.

3 The variational condition of instability

Let us now have given a certain equilibrium state of stress S^0_{ij} of the body, and see for what multiples μS^0_{ij} of this state of stress the body continues to be stable, and for what value μ_{cr} in becomes unstable. Exchanging in (3) S_{ij} for μS^0_{ij}, U_2 for μU^0_2, we obtain from equation (1) the condition of stability

$$U_1 + \mu U^0_2 > 0$$

or

$$\mu > -U_1/U^0_2 . \tag{11}$$

The lowest critical value μ_{cr} of factor μ is, therefore,

$$\mu_{cr} = \min(-U_1/U_2) . \tag{12}$$

The condition of the minimum is the zero value of variation U_1/U^0_2, i. e.

$$\delta\left(\frac{U_1}{U^0_2}\right) = \frac{U^0_2\delta U_1 - U_1\delta U^0_2}{(U^0_2)^2} = \frac{\delta U_1 + \mu_{cr}\delta U^0_2}{U^0_2}$$

or

$$\delta U_1 + \mu_{cr}\delta U^0_2 = \delta(U_1 + \mu_{cr}U^0_2) = 0 .$$

The condition of instability is thus as follows:

$$\delta(U_1 + U_2) = 0 \tag{13}$$

Otherwise we may also write — according to (6) and (10) that

$$\delta(\Delta U - \Delta W) = 0 \quad \text{or} \quad \delta^2(U - W) = 0 .$$

This agrees with the fact that the total potential energy $U - W$ of the internal and external forces is not a local minimum in the case of instability.

4 The differential equation of instability

Expression $U_1 + U_2$ may be considered a functional of functions $u_i(x_j)$. Denote the integrand of this functional by $J = J_1 + J_2$ where J_1, J_2 are the integrands of (2a) and (2b). Rearrange the condition of the minimum (13) as follows ($\partial J/\partial u_i = 0$):

$$\delta\int_V J(u_i, u_{i,j})\,\mathrm{d}V = \int_V \frac{\partial J}{\partial u_{i,j}}\,\delta(u_{i,j})\,\mathrm{d}V =$$

$$= -\int_V \frac{\partial}{\partial x_j}\left(\frac{\partial J}{\partial u_{i,j}}\right)\delta u_i\,\mathrm{d}V + \int_V \frac{\partial}{\partial x_j}\left(\frac{\partial J}{\partial u_{i,j}}\,\delta u_i\right)\mathrm{d}V =$$

$$= -\int_V \frac{\partial}{\partial x_j}\left(\frac{\partial J}{\partial u_{i,j}}\right)\delta u_i\,\mathrm{d}V + \int_S n_j\,\frac{\partial J}{\partial u_{i,j}}\,\delta u_i\,\mathrm{d}S = 0 . \tag{13a}$$

In the above we have made use of the commutativity of variation δ with integration as well as with differentiation, $\delta(u_{i,j}) = \partial(\delta u_i)/\partial x_j$, and of the Gauss theorem [9]. Since equation (13a) must be fulfilled for any arbitrary function $u_i(x_j)$, in must hold in all points of the body (the basic lemma of the calculus of variations) that

$$\frac{\partial}{\partial x_j}\left(\frac{\partial J}{\partial u_{i,j}}\right) = 0 \tag{14}$$

(Euler's conditions), while on the surface it must be either

$$n_j \frac{\partial J}{\partial u_{i,j}} = 0 \tag{14a}$$

or $\delta u_i = 0$. It is sufficient to consider in expression J_1 here a small strain (4a) only. We may introduce to (14), (14a):

$$\frac{\partial J_1}{\partial u_{i,j}} = \frac{\partial J_1}{\partial e_{ij}} = C_{ijkl} e_{kl} = \sigma_{ij}\,; \tag{15a}$$

$$\frac{\partial J_2}{\partial u_{i,j}} = S_{pq} \frac{1}{2} \frac{\partial}{\partial u_{i,j}} (u_{k,p} u_{kq}) = S_{jk} u_{i,k}\,. \tag{15b}$$

In the above, we made use of the symmetry $C_{ijkl} = C_{klij}$, and $S_{ij} = S_{ji}$. Because of (8) we thus get that the equation of equilibrium in the coordinates of the initial state x_i must be satisfied in all points of the body,

$$\boxed{\sigma_{ij,j} + S_{jk} u_{i,jk} - F_k u_{i,k} = 0} \tag{16}$$

and because of (9) (for a "dead" load), it must be either

$$\boxed{n_j \sigma_{ij} + P_j u_{i,j} = 0} \tag{16a}$$

or $u_i = 0$.
For zero volume forces, $F_k = 0$,

$$\sigma_{ij,j} + S_{jk} u_{i,jk} = 0\,. \tag{16b}$$

On a free unloaded surface it must hold ($P_j = 0$) that

$$n_j \sigma_{ij} = 0 \tag{16c}$$

which is in the form of the surface condition of equilibrium of the theory of the first order. Equations (16) are formally identical with the equations of equilibrium in the theory of the first order if we consider the volume forces F_i as dependent on strain:

$$F_i^* = S_{jk} u_{i,jk} - F_k u_{i,k}\,. \tag{16d}$$

Note: According ot Biot [1], in contrast to (16c), $F_i^* = S_{jk} u_{i,jk} - \frac{1}{2} S_{jk} e_{ik,j} - \frac{1}{2} S_{ik} e_{jk,j}$ (for $F_i = 0$). For bars and plates, equations (33), (34), this term is the same as in (16b).
When solving for the loss of stability u_i, σ_{ij} in (10) is expressed by the intermediary of u_i according to the law of elasticity and the geometric equations. We thus get equations

$$\frac{1}{2} C_{ijkl}(u_{k,lj} + u_{l,kj}) + S_{jk} u_{i,jk} - F_k u_{i,k} = 0 \tag{17}$$

with the boundary conditions (16a) whose form with respect to the displacements is evident. For an isotropic body in particular, they are of the form

$$G u_{i,jj} + \frac{G}{1 - 2\nu} u_{j,ij} + S_{jk} u_{i,jk} - F_k u_{i,k} = 0 \tag{17a}$$

and we see that except for the last two terms representing the volume force F_i, they are identical with the well-known Lamé's equations of the classical theory of elasticity ([3], p. 184, [9], eq. /VII. 4. 13/).

Let us also write the differential equations of the loss of stability for an orthotropic material following the law of elasticity

$$\left.\begin{aligned} \sigma_{xx} &= E_{xx}e_{xx} + E_{xy}e_{yy} + E_{xz}e_{zz}\,, \ldots \\ \tau_{xy} &= 2G_{xy}e_{xy}\,, \ldots \end{aligned}\right\} \tag{18}$$

(we shall write neither here nor hereafter the equations that can be set up by a cyclic exchange of subscripts x, y, z) ($E_{xy} = E_{yx}, \ldots$). In the above we have introduced the following new notation $x = x_1$, $y = x_2$, $z = x_3$, $\sigma_{xx} = \sigma_{11}$, $\tau_{xy} = \sigma_{12}$, $e_{xx} = e_{11}$, $e_{xy} = e_{12}, \ldots$ Furthermore, let us write that $u_1 = u$, $u_2 = v$, $u_3 = w$. On introducing the geometric expression of small strain (4a) to (18) and thence for stresses to the equations of equilibrium (17) we obtain for $F_i = 0$ the following three equations:

$$\frac{\partial}{\partial x}\left(E_{xx}\frac{\partial u}{\partial x} + E_{xy}\frac{\partial v}{\partial y} + E_{xz}\frac{\partial w}{\partial z}\right) + \frac{\partial}{\partial y}\left[G_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] +$$
$$+ \frac{\partial}{\partial x}\left[G_{xz}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\right] + S_x\frac{\partial^2 u}{\partial x^2} + S_y\frac{\partial^2 u}{\partial y^2} + S_z\frac{\partial^2 u}{\partial z^2} + 2S_{xy}\frac{\partial^2 u}{\partial x\,\partial y} +$$
$$+ 2S_{yz}\frac{\partial^2 u}{\partial y\partial z} + 2S_{xz}\frac{\partial^2 u}{\partial x\partial z} = 0\,, \tag{19}$$

where $S_x = S_{11}$, $S_{xy} = S_{12}, \ldots$ As we can easily prove, the same equations would be obtained from condition (14) in which we have introduced for the orthotropic material

$$U_1 = \iiint \frac{1}{2}\left(E_{xx}\frac{\partial u}{\partial x} + E_{xy}\frac{\partial v}{\partial y} + E_{xz}\frac{\partial w}{\partial z}\right)\frac{\partial u}{\partial x} + \ldots$$
$$\ldots + G_{xy}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)^2 + \ldots\Big\}\, dx\, dy\, dz\,. \qquad))$$

5 Some forms of the loss of stability of three-dimensional bodies

The general solution of the eigenvalue problem defined is difficult. Let us now demonstrate that equations (19) can be solved straightforward for a plane periodic loss of stability (periodic in planes x, y) under the sole action of compressive stresses S_x, S_y, S_z ($S_{xy} = S_{yz} = S_{xz} = 0$) which we let be variable in dependence on z only. Let the material be homogeneous, i.e. $\partial E_{xx}/\partial x = 0$, etc. Let us look for a loss of stability in the form (with waves of lengths π/α, π/β:

$$\left.\begin{aligned} u &= \xi(z)\sin\alpha x\cos\beta y\,, \\ v &= \eta(z)\cos\alpha x\sin\beta y\,, \\ w &= \zeta(z)\cos\alpha x\cos\beta y\,, \end{aligned}\right\} \tag{21}$$

(which can satisfy the boundary conditions for $x = x_1$, $y = y_1$ on Fig. 1). On introducing the above expressions to the three equations (19), we find that they can be identically fulfilled if

$$
\left.
\begin{aligned}
& -(G_{xz}+S_{xx})\frac{\mathrm{d}^2\xi}{\mathrm{d}z^2}+[\alpha^2(E_{xx}+S_{xx})+\beta^2(G_{xy}+S_{yy})]\,\xi+\\
& +\alpha\beta(E_{xy}+G_{xy})\,\eta+\alpha(E_{xz}+G_{xz})\frac{\mathrm{d}\zeta}{\mathrm{d}z}=0\,,\\
& \alpha\beta(E_{xy}+G_{xy})\xi-(G_{yz}+S_{zz})\frac{\mathrm{d}^2\eta}{\mathrm{d}z^2}+\\
& +[\alpha^2(G_{xy}+S_{xx})+\beta^2(E_{yy}+S_{yy})]\eta+\beta(E_{yz}+G_{yz})\frac{\mathrm{d}\zeta}{\mathrm{d}z}=0\,,\\
& \alpha(E_{xz}+G_{xz})\frac{\mathrm{d}\xi}{\mathrm{d}z}+\beta(E_{yz}+G_{yz})\frac{\mathrm{d}\eta}{\mathrm{d}z}+(E_{zz}+S_{zz})\frac{\mathrm{d}^2\zeta}{\mathrm{d}z^2}-\\
& -[\alpha^2(G_{xz}+S_{xz})+\beta^2(G_{yz}+S_{yy})]\zeta=0\,.
\end{aligned}
\right\}
\tag{22}
$$

In this way we have obtained a system of three ordinary homogeneous linear differential equations fo $\xi(z)$, $\eta(z)$, $\zeta(z)$. After the formulation of the boundary conditions on boundaries $z = z_1$ and $z = z_2$, we can solve the eigenvalue problem by some of the well-known methods (e. g. the d'Alembert substitution

$$\xi = k_1 e^{\lambda z}, \quad \eta = k_2 e^{\lambda z}, \quad \zeta = k_3 e^{\lambda z}).$$

According to (16a) for $P_z = 0$ ($S_{zz} \equiv 0$), $P_x = P_y = 0$, the boundary conditions on surface $z = z_1$ are $\sigma_{zz} = 0$, $\sigma_{xz} = 0$, $\sigma_{yz} = 0$. On introducing to (18) we get

$$E_{xx}\frac{\partial u}{\partial x}+E_{yz}\frac{\partial v}{\partial y}+E_{zz}\frac{\partial w}{\partial z}=0\,,$$

$$G_{xz}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)=0\,,\qquad G_{yz}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)=0$$

and according to (21) we have the boundary conditions for $z = z_1$:

$$\alpha E_{xz}\xi+\beta E_{yz}\eta+E_{zz}\frac{\mathrm{d}\zeta}{\mathrm{d}z}=0\,,\quad \frac{\mathrm{d}\xi}{\mathrm{d}z}=\alpha\zeta\,,\quad \frac{\mathrm{d}\eta}{\mathrm{d}z}=\beta\zeta\,. \tag{23}$$

When analysing the stability of a thick plate bounded by planes $z = \pm d/2$, we would consider the above conditions on both surfaces. When solving the stability of a compressed half-space $z \geqq 0$, we would consider those conditions both for $z = 0$ and for $z \to \infty$ as well, so long as it is only the surface loss of stability we are interested in (the case is similar to that of surface Rayleigh's elastic waves /3/) which gives $\lim\limits_{z\to\infty}\xi = \lim\limits_{z\to\infty}\eta = \lim\limits_{z\to\infty}\zeta = \lim\limits_{z\to\infty}\frac{\mathrm{d}\xi}{\mathrm{d}z} = \ldots = 0$. Equations (22) can identically be fulfilled for an omnidirectionally periodic loss of stability if we choose

$$\xi = a\cos\gamma z\,,\quad \eta = b\cos\gamma z\,,\quad \zeta = c\sin\gamma z\,, \tag{24}$$

of course, so long as the stress is constant within the whole body. By introducing to (22) we namely obtain a system of three homogeneous algebraic equations for a, b, c:

$$
\left.
\begin{aligned}
& a[\alpha^2(E_{xx} + S_{xx}) + \beta^2(G_{xy} + S_{yy}) + \gamma^2(G_{xz} + S_{zz})] + \\
& + b\alpha\beta(E_{xy} + G_{xy}) + c\alpha\gamma(E_{xz} + G_{xz}) = 0\,, \\
& a\alpha\beta(E_{xy} + G_{xy}) + b[\alpha^2(G_{xy} + S_{xx}) + \beta^2(E_{yy} + S_{yy}) + \\
& + \gamma^2(G_{yz} + S_{zz})] + c\beta\gamma(E_{yz} + G_{yz}) = 0\,, \\
& a\alpha\gamma(E_{xz} + G_{xz}) + b\beta\gamma(E_{yz} + G_{yz}) + \\
& + c[\alpha^2(G_{xz} + S_{xx}) + \beta^2(G_{yz} + S_{yy}) + \gamma^2(E_{zz} + S_{zz})] = 0\,.
\end{aligned}
\right\} \quad (25)
$$

On introducing moreover (24) to the suface conditions (23), we obtain the conditions

$$
\left.
\begin{aligned}
& (\alpha E_{xz} a + \beta E_{yz} b + \gamma E_{zz} c) \cos \gamma z_1 = 0\,, \\
& (\gamma a + \alpha c) \sin \gamma z_1 = 0\,, \\
& (\gamma b + \beta c) \sin \gamma z_1 = 0\,.
\end{aligned}
\right\} \quad (26)
$$

w... may be fulfilled, however, only in elementary cases.

In general, the condition of zero value of the determinant of the system of homogeneous equations (25) yields an algebraic equation for the eigenvalues S_{xx}, S_{yy}, S_{zz}. In the boundary conditions (23) we must then substitute the general solution, i. e. a linear combination of corresponding functions (24). The method of solution is well-known and its application to practical cases is not an object of this paper.

The loss of stability is not possible for a zero stress. Accordingly, the determinant of system (25) is larger than zero. So long as $E_{xx} + S_{xx} > 0$, $G_{xz} + S_{xx} > 0$ etc., we may continue to consider (25) the equations of deformation of a material with changed elastic constants $E_{xx} + S_x$ instead of E_{xx}, ... at a zero stress, and instability is not possible. For the loss of stability to occur, one compressive stress at least must be in its absolute value of a magnitude comparable with some of the elastic constants E_{xx}, E_{yy}, E_{zz}, G_{xy}, G_{yz}, G_{xz} (provided that it is not $\gamma \to 0$, unlike for thin walls or bars). Thus, so far as the isotropic structural materials are concerned, the compressive stresses for the instability of a continuum would have to be enormous, and such cases do not come into consideration. But there exist, of course, orthotropic structural materials with one of the elastic constants very small, such as E_{zz} in transverse tension owing to defects, of laminated materials. In cases of that sort the loss of stability is apt to be dangerous, and if it leads to failure, is manifested by de-lamination. Yet the actual stress at failure might be — in view of the initial inequalities — considerably less than E_{zz}.

The plane loss of stability in plane (x, z) is a special case for $v = 0$, i. e. for $\beta = 0$. Then the second equation of system (22) drops off, and the resultant system contains but two ordinary equations for $\xi(z)$, $\zeta(z)$

$$
\left.
\begin{aligned}
& -(G_{xz} + S_{zz})\frac{d^2\xi}{dz^2} + \alpha^2(E_{xx} + S_{xx})\xi + \alpha(E_{xz} + G_{xz})\frac{d\zeta}{dz} = 0\,, \\
& \alpha(E_{xz} + G_{xz})\frac{d\xi}{dz} + (E_{zz} + S_{zz})\frac{d^2\zeta}{dz^2} - \alpha^2(G_{xz} + S_{xx})\zeta = 0\,.
\end{aligned}
\right\} \quad (27)
$$

Similarly in (23) the third boundary condition drops off, and we have the boundary conditions for $z = z_1$ as follows:

$$\alpha E_{xz}\xi + E_{zz}\frac{d\zeta}{dx} = 0\,, \quad \frac{d\xi}{dz} = \alpha\zeta\,, \tag{28}$$

For a two-directionally periodic plane loss of stability $\xi = a\cos\gamma z$, $\zeta = c\sin\gamma z$, we get in place of (25)

$$\left.\begin{aligned} &a[\alpha^2(E_{xx} + S_{xx}) + \gamma^2(G_{xz} + S_{zz})] + c\alpha\gamma(E_{xz} + G_{xz}) = 0\,,\\ &a\alpha\gamma(E_{xz} + G_{xz}) + c[\alpha^2(G_{xz} + S_{xx}) + \gamma^2(E_{zz} + S_{zz})] = 0 \end{aligned}\right\} \tag{29}$$

and in place of (26), two surface conditions

$$(\alpha E_{xz}a + \gamma E_{zz}c)\cos\gamma z_1 = 0\,, \quad (\gamma a + \alpha c)\sin\gamma z_1 = 0 \tag{30}$$

one of which can be satisfied (on both surfaces for a plate) by a suitable choice of γ and of the origin of the coordinates $z = 0$.

6 Direct variational solution

In complex cases when it is difficult to solve the differential equations (15) or (19) at the given boundary conditions, a solution may be obtained by the Ritz variational method from condition (13). In that condition we introduce U_1 and U_2 expressed according to (20) or (2a) or (2c) and (2a) for u_1, u_2, u_3 which are expressed for that purpose in the form of a linear combination of suitably chosen orthogonal functions $\varphi_\nu(x_i)$, etc. (satisfying at least the kinematic boundary conditions):

$$u = \sum_{\mu,\nu=1}^{\mu_1,\nu_1} a_{\mu\nu}\varphi_\nu(x, z)\,, \quad w = \sum_{\mu,\nu=1}^{\mu_1,\nu_1} a_{\mu\nu}\psi_\mu(x, z) \tag{31}$$

(we are considering a plane loss of stability, $v = 0$).

Note: When solving for the periodic loss of stability of a thick orthotropic plate (at zero Poisson's ratio, $E_{xz} = 0$) bounded by planes $z = \pm d/2$, a suitable choice is

$$\begin{aligned} \varphi_\nu &= \varphi_{rs} = \cos(2r - 1)\alpha x P_{2s}\frac{2z}{d}\,,\\ \psi_\mu &= \psi_{pq} = \sin(2p - 1)\alpha x \sin\frac{2q - 1}{d}\pi z\,, \end{aligned} \tag{32}$$

where $P_{2s}(\xi)$ are even Legendre's polynomials ([10], p. 601) which are orthogonal in the interval $-1 \leqq \xi \leqq 1$ and the only ones having the advantage that they satisfy even the integral conditions of equilibrium for the increments of stress σ_{xx} in a perpendicular section of the slab (even though they need not be satisfied by functions $\varphi_{\mu\nu}$:

$$\int_{d/2}^{-d/2} \sigma_{xx}\,dz = 0\,.$$

The conditions of the minimum of expression $U_1 + U_2$, i. e. the annuling of the partial derivatives with respect to a give us a system of homogeneous algebraic linear equations for $a_{\mu\nu}$.

7 Instability of a viscoelastic continuum

If we accept the equations of equilibrium (16) for the fundamental ones, all the differential equations deduced in the foregoing automatically apply also to the loss of stability of a viscoelastic continuum after we have exchanged in them the elastic constants $E_{xx}, \ldots$ for the operators of creep in time, $E_{xx}, \ldots$.

8 A comparison with the engineering solution of the stability of bars and plater

Let us now apply — for the purpose of a check of the correctness of the derived equations — expression (16d) to the stability of bars and plates.

Let us have a slender bar with axis $x_1 = x$ buckling under load P in the direction of axis $x_3 \doteq z$. The only non-zero initial stress is $S_x = S_{11} = -P/F$ where F is the section area. Equation (16d) thus gives ($u_3 = w$) the transverse volume load

$$F_z = -S_{xx} \frac{\partial^2 w}{\partial x^2} \tag{33}$$

and by integration for the whole section F load $P \frac{\partial^2 w}{\partial x^2}$ a well-known expression (11).

Consider further a thin wall in plane $(x, y) = (x_1, x_2)$ with the initial stresses $S_{11} = S_{xx}$, $S_{22} = S_{xy}$ and the other stress components equal to zero. It is then according to (16d)

$$F_x^* = -S_x \frac{\partial^2 w}{\partial x^2} - 2S_{xy} \frac{\partial^2 w}{\partial x \partial y} + S_y \frac{\partial^2 w}{\partial y^2} \tag{34}$$

again a well-known expression (cf. [11], p. 348, eq. /9 — 1/).

The agreement between his criterion (1) and the engineering solution was proved by Pearson himself [8]; in doing so he expressed relations (2a, b) for U_1, U_2 directly for the respective problem and sought the minimizing form of deformation. Using this procedure he has also shown that as agreement exists even for the stability of a cylindrical shell under external pressure and, in contrast to other theories, for buckling under flexure as well.

9 Hydrostatic load

At a hydrostatic load produced by gas or liquid, the vector of stress pn'_i changes its direction in the course of a change of deformation in the same way as the normal n'_j to surface S' rotates. For this reason we should consider in expression (7) for ΔW an additional term U_3 of a higher order which would then also appear on the left-hand side of Pearson's criterion (1) as well as in condition (13). (Expression ΔW must be equal to $p\Delta V$ where ΔV is a change of the body volume including the terms of a higher order). According to the idea of the gradual course of deformation u'_i in dependence on a single parameter

t, $u'_i = u_i t$, by integrating the elementary work $\mathrm{d}u'_i p\, n'_i \mathrm{d}S = \mathrm{d}t\, u_i\, p\, n'_i\, \mathrm{d}S'$ from $t = 0$ to $t = 1$, Pearson obtained ([8], eq. /26/)

$$U_3 = \int_S p(n_l u_l u_{k,k} - n_k u_l u_{k,l})\, \mathrm{d}S \tag{35}$$

which can also be written as

$$U_3 = \int_{S_p} \frac{1}{2}\, n_r \varepsilon_{prq} \varepsilon_{plk} u_l u_{k,q} P\, \mathrm{d}S_p \tag{35a}$$

since $\varepsilon_{plk}\varepsilon_{prq} = \delta_{lr}\delta_{kq} - \delta_{lq}\delta_{kr}$ [5], [9]; ε_{ijk} is the Levi-Civita's symbol (equal to 1 for even permutations of subscripts 123, -1 for odd permutations, otherwise 0), S_p is the part of the surface on which acts the hydrostatic pressure P.

It is then necessary to add in all forms of the left-hand side of equation (13a) also the variation of the surface integral (35) or (35a)

$$\delta \int_{S_p} J_3(u_i, u_{i,j})\, \mathrm{d}S_p = \int_{S_p} \frac{\partial J_3}{\partial u_i} \delta u_i\, \mathrm{d}S_p + \int_{S_p} \frac{\partial J_3}{\partial u_{i,j}} (\delta u_i)_{,j}\, \mathrm{d}S_p \tag{36}$$

where J_3 is the integrand in (35) or (35a). We can moreover compute

$$\frac{\partial J_3}{\partial u_i} = \frac{1}{2} P(n_i u_{k,k} - n_k u_{k,i}), \tag{37}$$

$$\frac{\partial J_3}{\partial u_{i,j}} = \frac{1}{2} n_r \varepsilon_{prj} \varepsilon_{pli} u_l P, \tag{38}$$

and with the aid of (38) express the last integral in (36) as

$$-\int_{S_p} \frac{1}{2} n_r \varepsilon_{prj} \varepsilon_{pli} (u_l P)_{,j} \delta u_i\, \mathrm{d}S_p + \int_{S_p} \frac{1}{2} n_r \varepsilon_{rjp} (\varepsilon_{pli} u_l P \delta u_i)_{,j}\, \mathrm{d}S_p$$

where the first integral canfurther be rearranged according to relation $\varepsilon_{pli}\varepsilon_{prj} = \delta_{lr}\delta_{ij} - \delta_{lj}\delta_{in}$, [5], [9] and according to Stokes theorem [5], [9] convert the second integral to an integral along curve L_P bounding the part S_p of the surface S, which is loaded by hydrostatic pressure P, i. e.

$$-\int_{S_p} \frac{1}{2} [n_l (u_l P)_{,i} - n_j (u_j P)_{,j}] \delta u_i\, \mathrm{d}S_p + \int_{L_p} \frac{1}{2} P \varepsilon_{ijk} u_i v_k \delta u_j\, \mathrm{d}L_p \tag{39}$$

where v_i is the unit tangential vector to curve L_p. On introducing (37) and (39) to (36) and on introducing variations (36) to (13a), we get in place of (16a), the surface condition for a surface loaded simultaneously by a "dead" load P_i and by hydrostatic pressure P:

$$n_j \sigma_{ij} + (P_j + P n_j) u_{i,j} + \left\{ P n_i u_{j,j} - P n_j u_{j,i} + \frac{1}{2} n_i u_j P_{,j} - \frac{1}{2} n_j u_j P_{,i} \right\} = 0 \tag{40}$$

where $\{\ldots\}$ represents against (16a) an additional term that can also be written in the form: $n_r \varepsilon_{pli} \varepsilon_{prj} \left(u_{l,j} P + \frac{1}{2} P_{,j} u_l \right)$. At the same time, surface S_P must be either continuous, closed, without the boundary curve L_p (the whole body is immersed in liquid or gas) or there must be $Pu_i v_k e_{ijk} = 0$ on the boundary curve L_p, i. e. either $P = 0$ (the liquid level), either u_i W v_i (parallel), or $u_i \doteq 0$. In practice, this is always fulfilled.

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