Numerical Analysis of Creep of an Indeterminate Composite Beam

Z. P. BAŽANT[1]

Approximating the hereditary integrals by finite sums, a creep problem may be reduced to a succession of elasticity problems with initial strains. It is shown how the well-known general relationship between initial strains and equivalent loads may be applied to a composite beam on statically indeterminate supports. A highly accurate numerical method for memory-type creep problems is demonstrated and its convergence studied by means of examples.

Introduction

Composite beam with statically indeterminate supports is a practically important type of structure whose creep cannot be exactly formulated with a finite system of differential or integral equations in time. If the creep properties are not viscoelastic, the Laplace transform cannot be applied. An approximate method of solution may be based on the following theorem:

Theorem 1

Let $\boldsymbol{\varepsilon}^0$ be the inelastic (initial) strain tensor and

$$\boldsymbol{\sigma} = \mathbf{C}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^0) \tag{1}$$

the linear stress-strain law, in which $\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}$ are the stress and strain tensors, and $\mathbf{C}$ is the tensor of elastic moduli. (The values of $\boldsymbol{\varepsilon}^0$ and $\mathbf{C}$ may vary with space coordinates.) Introduce the prestress tensor as follows:

$$\boldsymbol{\sigma}^0 = \mathbf{C}\boldsymbol{\varepsilon}^0 \tag{2}$$

and define $\mathbf{F}^1$ as a state of volume and surface loads which are in equilibrium with $\boldsymbol{\sigma}^0$. Then the solution of any problem of small elastic deformation of a body with the elastic law (1) at zero load is given by the stresses $\boldsymbol{\sigma}^1 - \boldsymbol{\sigma}^0$, the strains $\boldsymbol{\varepsilon}^1$ and the small displacements $\mathbf{u}^1$ where $\boldsymbol{\sigma}^1$, $\boldsymbol{\varepsilon}^1$, $\mathbf{u}^1$ is the solution of the same body for the given prescribed displacements, loads $\mathbf{F}^1$ and zero inelastic strains ($\boldsymbol{\varepsilon}^0 = 0$).

A special form of this theorem, in which $\boldsymbol{\varepsilon}^0$ are volumetric initial strains and material is isotropic, was presented by Duhamel and Neumann [7][2] and is known in thermoelasticity as Duhamel's analogy [7]. For deviatoric initial strains and an isotropic material, this theorem, relating $\mathbf{F}^1$ to $\boldsymbol{\varepsilon}^0$, has been shown first by Reissner in [9]. It was later independently derived by Eshelby [6] and, for a generally anisotropic material, by Bažant [2, 3] who applied this theorem to creep of aging concrete beams and frames of prestressed concrete bridges, and to concrete plates. First, application to creep (nonlinear creep of metallic plates) seems to be due to Lin [8]. A method of conversion of initial strains to nodal forces, equivalent to Theorem 1, has been independently developed within the finite-element method (see, e.g., [11]).

In the present Note,[3] creep of a steel-concrete composite beam will be solved, using Theorem 1. Steel will be considered as perfectly elastic. The uniaxial creep law of concrete will be considered as linear, in the following form:

$$\epsilon(t) = \frac{\sigma(t)}{E_b(t)} + \int_{t_0}^{t} \frac{\sigma(\tau)}{E_b(\tau)} L_b(t,\tau)d\tau = \mathbf{E}_b^{-1}\sigma(t) \tag{3}$$

where

$$\frac{L_b(t,\tau)}{E_b(\tau)} = -\frac{\partial}{\partial\tau}\left[\frac{1}{E_b(\tau)} + \frac{1}{E_b(\infty)}\left(0.6 + \frac{100}{\tau}\right)\frac{t-\tau}{t-\tau+60}\right] \tag{4}$$

$$E_b(\tau) = (1 - 0.6e^{-\tau/100}) \times 3 \times 10^6 \tag{5}$$

Here t, τ = time or age of concrete; σ, ϵ = normal stress and strain; E_b = Young's modulus, $L_b(t,\tau)$ = stress memory function.

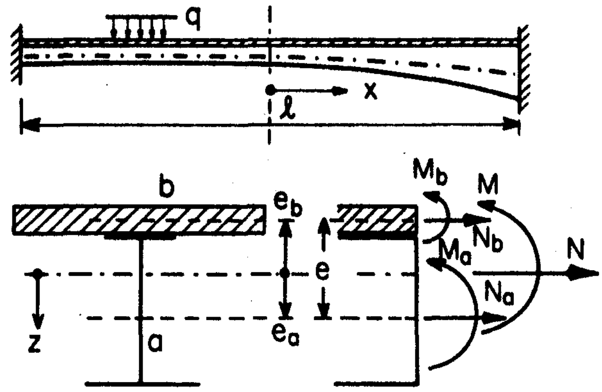

Fig. 1

Basic Relationships

The equilibrium relationships for a composite beam are the following:

$$M = M_a + M_b + N_a e_a + N_b e_b, \qquad N = N_a + N_b \tag{6}$$

$$\partial^2 M/\partial x^2 = -q, \qquad \partial N/\partial x = p \tag{7}$$

where

$$e_b = e/[1 + E_b F_b/(E_a F_a)], \qquad e_a = e - e_b \tag{8}$$

Notations: M, N = total bending moment and normal force; M_a, M_b and N_a, N_b = bending moments and normal forces about the centroidal axis of part a (steel) or b (concrete), Fig. 1; q and p = specific lateral and londitudinal loads; x = length coordinate of beam; E_a, E_b and F_a, F_b = Young's moduli and cross-sectional areas of part a and b; e_a, e_b = eccentricities of the centroid of part a or b with respect to the chosen axis of beam; e = distance between these centroids. For the numerical example, the beam will be considered to be fixed against rotation at the ends, but free in longitudinal displacement. The boundary conditions thus are

$$w = 0, \quad \partial w/\partial x = 0, \quad N = 0 \quad \text{for} \quad x = -l/2 \quad \text{and} \quad x = l/2 \tag{9}$$

where w = deflection, l = length of beam.

Elastic Problem

Assuming that the cross sections remain plane and perpendicular to the deflected axis of beam, and that their rotations are small, the deformation equations and the distribution of normal stresses σ_b are

$$[E_a(I_a + F_a e_a^2) + E_b(I_b + F_b e_b^2)](\partial^2 w/\partial x^2) = -M \tag{10}$$

$$(E_a F_a + E_b F_b)(\partial u/\partial x) = N \tag{11}$$

$$\sigma_b = (z - e_b)M_b/I_b + N_b/F_b \tag{12}$$

where I_a, I_b = centroidal moments of inertia of parts a and b; z = depth coordinate. As is well known, the solution by force method of the elastic beam with boundary conditions (9) leads to the equations

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[1] Associate Professor, Department of Civil Engineering, The Technological Institute, Northwestern University, Evanston, Ill.

[2] Numbers in brackets designate References at end of Note.

[3] This Note is based on author's Internal Research Report No. 68/2, "Approximate Analysis of Linear and Nonlinear Creep Problems—Initial Strain Method," Department of Civil Engineering, University of Toronto, Dec. 1968.

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$$M = M^{(0)} + \sum_{i=1}^{2} M^{(i)} X_i \tag{13}$$

$$\sum_{j=1}^{2} \delta_{ij} X_j + \delta_{i0} = 0 \qquad (i = 1, 2) \tag{14}$$

where

$$\delta_{ik} = \int_0^l \frac{M^{(i)} M^{(k)}\, dx}{E_a(I_a + F_a e_a^2) + E_b(I_b + F_b e_b^2)} \qquad (k = 0, 1, 2) \tag{15}$$

X_1, X_2 = statically indeterminate bending moments at the ends of the beam; $M^{(1)} = 0.5 - x/l = M$ for $q = 0$, $X_1 = 1$, $X_2 = 0$; $M^{(2)} = 0.5 + x/l = M$ for $q = 0$, $X_1 = 0$, $X_2 = 1$; $M^{(0)} = M$ for $X_1 = X_2 = 0$ and given $q(x)$. For a uniform load q, $M^{(0)} = q(l^2 - 4x^2)/8$.

Creep Problem and Its Conversion to a Sequence of Elasticity Problems

The mathematical formulation of problem is given by equations (10), (11), (9) or equations (13) and (14), in which E_b is replaced by the creep operator $\mathbf{E}_b$, the inverse of which is defined by equation (3). However, for the numerical solution, it is practical to solve directly the effect of the inelastic strains. Introduce a subdivision $t_{(0)}, t_{(1)}, \ldots t_{(n)}$ of the given time interval (t_0, t_1) into n equal subintervals Δt. The integral in equation (3) may be approximated by the finite sum

$$\sum_{s=0}^{r} c_{(s)}^{(r)} \sigma_{(s)} L_b(t_{(r)}, t_{(s)})/E_{b_{(s)}} \tag{16}$$

Subscript (s) stands for time $t_{(s)}$; $c_{(s)}^{(r)}$ are certain constants. The creep law (3) then becomes

$$\epsilon_{(r)} = \sigma_{(r)}/\bar{E}_{b_{(r)}} + \epsilon_{(r)}^0 \tag{17}$$

where

$$\bar{E}_{b_{(r)}} = E_{b_{(r)}}[1 + c_{(r)}^{(r)} L_b(t_{(r)}, t_{(r)})]^{-1} \tag{18}$$

$$\epsilon_{(r)}^0 = \sum_{s=0}^{r-1} c_{(s)}^{(r)} \sigma_{(s)} L_b(t_{(r)}, t_{(s)})/E_{b_{(s)}} \tag{19}$$

Assume that $\sigma_{(s)}$ has already been computed up to the time $t_{(r-1)}$. Then $\epsilon_{(r)}^0$, $\bar{E}_{b_{(r)}}$ can also be determined from (19) and (18). Thus equation (17) may formally be regarded as a fictitious elastic stress-strain law with a prescribed initial strain, the inelastic strain $\epsilon_{(r)}^0$. The fictitious modulus $\bar{E}_{b_{(r)}}$ is different from the actual modulus $E_{b_{(r)}}$. The integration of a creep problem is thus reduced to a sequence of elasticity problems with initial strains, each of which may be converted to an elasticity problem without initial strains according to Theorem 1.

Because of the linearity of creep law (3), the inelastic (creep) strains in part b, ϵ_b^0, are distributed linearly. On account of equation (2), the same applies for the corresponding prestresses σ_b^0 which are, according to (12) and (2),

$$\sigma_{b_{(r)}}^0 = (z - e_b) M_{b_{(r)}}^0 / I_b + N_{b_{(r)}}^0 / F_b = \bar{E}_{b_{(r)}} \epsilon_{b_{(r)}}^0 \tag{20}$$

where M_b^0, N_b^0 are certain internal forces which may be obtained if equation (12) is substituted for $\sigma_{(s)}$ in equation (19), and equation (19) into (20). Thus

$$M_{b_{(r)}}^0 = \bar{E}_{b_{(r)}} \sum_{s=0}^{r-1} c_{(s)}^{(r)} M_{b_{(s)}} L_b(t_{(r)}, t_{(s)})/E_{b_{(s)}} \tag{21}$$

$$N_{b_{(r)}}^0 = \bar{E}_{b_{(r)}} \sum_{s=0}^{r-1} c_{(s)}^{(r)} N_{b_{(s)}} L_b(t_{(r)}, t_{(s)})/E_{b_{(s)}} \tag{22}$$

The resultants of σ_b^0 over the entire composite cross section are

$$M_{(r)}^0 = M_{b(r)}^0 + N_{b(r)}^0 e_{b_{(r)}}, \qquad N_{(r)}^0 = N_{b(r)}^0 \tag{23}$$

The specific loads $p^0(x)$, $q^0(x)$, representing the loading state $\mathbf{F}^1$ in Theorem 1, are $p_{(r)}^1 = dN_{(r)}^0/dx$, $q_{(r)}^0 = -d^2M_{(r)}^0/dx^2$. However, if the force method is used, $p_{(r)}^1$ and $q_{(r)}^1$ need not be calculated because δ_{10} and δ_{20} may be determined directly from equation (15), substituting $M_{(r)}^0$ for $M^{(0)}$. Subsequently, the values $X_1 = X_{1_{(r)}}^1$, $X_2 = X_{2_{(r)}}^1$ and $M = M_{(r)}^1$ are calculated from equations (14) and (13). Because of the boundary condition $N = 0$, $N_{(r)}^1 = N_{(r)}^0$. The internal forces in part b, due to loading state $\mathbf{F}^1$ are

$$M_{b_{(r)}}^1 = \frac{M_{(r)}^1 \bar{E}_b}{\bar{E}_b(I_b + F_b \bar{e}_b^2) + E_a(I_a + F_a \bar{e}_a^2)} \tag{24}$$

$$N_{b_{(r)}}^1 = N_{(r)}^1 \bar{E}_b F_b / (\bar{E}_b F_b + E_a F_a) + M_{b_r}^1 \bar{e}_b F_b / I_b \tag{25}$$

where $\bar{e}_b$, $\bar{e}_a$ are the values according to (8), corresponding to $\bar{E}_b$ rather than E_b. Finally, according to Theorem 1, the internal forces and deflections in time $t_{(r)}$ are

$$\begin{aligned} M_{b_{(r)}} &= M_{b_{(r)}}^1 - M_{b_{(r)}}^0 + M_{b_{(r)}}^l \\ N_{b_{(r)}} &= N_{b_{(r)}}^1 - N_{b_{(r)}}^0 + N_{b_{(r)}}^l \\ w_{(r)} &= w_{(r)}^1 + w_{(r)}^l \end{aligned} \tag{26}$$

where $M_{b_{(r)}}^l$, $N_{b_{(r)}}^l$, $w_{(r)}^l$ is the elastic solution for the given load $q(x)$ in time $t_{(r)}$.

The algorithm of solution just described is repeated for each step Δt. In practical computations the functions M_b, N_b, M_b^0, N_b^0, M_b^1, N_b^1, $\bar{E}_b$, $\bar{e}_b$ of x were represented by their discrete values for a chosen subdivision of the length of beam and integrals (18) were approximately evaluated by the Simpson rule.

For the evaluation of the time integrals the coefficients $c_{(s)}^{(r)}$ were introduced according to the following formulas, valid for $r \geq 3$, with error of order Δt^4,

$$\int_{t_{(0)}}^{t_{(r)}} f(t)dt \doteq \frac{\Delta t}{24} \Big[9(f_{(0)} + f_{(r)}) + 19(f_{(1)} + f_{(r-1)}) - 5(f_{(2)} + f_{(r-2)}) + f_{(3)} + f_{(r-3)} + \sum_{s=1}^{r-2} (-f_{(s-1)} + 13 f_{(s)} + 13 f_{(s+1)} - f_{(s+2)}) \Big] \tag{27}$$

For $r = 2$, the Simpson rule was considered.

A certain inconvenience of the method described consists in the fact that the fictitious modulus $\bar{E}_b$ is distributed nonuniformly along the beam, even if $E_{b_{(r)}}(x)$ is constant; thus the values δ_{ij}, $\bar{e}_b$ and the coefficients in (24), (25) must be recalculated and equation (14) with a different matrix solved in each step. This deficiency may be suppressed by an iterative variant of this method as follows. Instead of equations (18), (19), (21), and (22), it can be also defined

$$\bar{E}_{b_{(r)}} = E_{b_{(r)}}, \qquad \epsilon_{(r)}^0 = \sum_{s=0}^{r} c_{(s)}^{(r)} \sigma_{(s)} L_b(t_{(r)}, t_{(s)})/E_{b(s)} \tag{28}$$

$$M_{b_{(r)}}^0 = E_{b_{(r)}} \sum_{s=0}^{r} c_{(s)}^{(r)} M_{b_{(s)}} L_b(t_{(r)}, t_{(s)})/E_{b_{(s)}} \tag{29}$$

Because the value $\sigma_{(r)}$ or $M_{b_{(r)}}$, $N_{b_{(r)}}$ in the rth step is an as yet unknown, it must be first estimated by extrapolation from the preceding values; an extrapolation formula with an error of order Δt^3 is $M_{b_{(r)}} \doteq M_{b_{(r-1)}} + 3M_{b_{(r-2)}} - M_{b_{(r-3)}}$. Further procedure is the same as before until $M_{b_{(r)}}$, $N_{b_{(r)}}$ is obtained. Then more exact values of $\epsilon_{(r)}^0$ or $M_{b_{(r)}}^0$, $N_{b_{(r)}}^0$ can be calculated from (29) and the whole procedure repeated until better values of $M_{b_{(r)}}$, $N_{b_{(r)}}$ are obtained, and so forth.

In the first three steps, if errors of lower order are to be avoided, a method of successive approximations must be used (in the in-

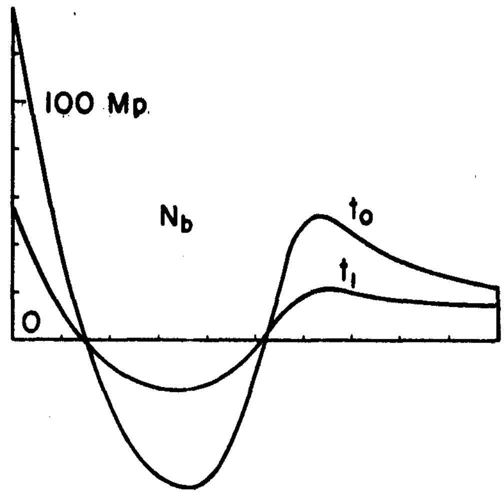

Fig. 2

terative variant as well as in the direct method). Suppose the approximate values $M_{b_{(0)}}, \ldots M_{b_{(3)}}$ are known. Then the values $M_{b_{(0.5)}}, N_{b_{(0.5)}}$ for the middle of the first interval $(t_{(0)}, t_{(1)})$ may be determined, using a 4th-order interpolation formula such as $M_{b_{(0.5)}} \doteq (5M_{b_{(0)}} + 15M_{b_{(1)}} - 5M_{b_{(2)}} + M_{b_{(3)}})/16$, etc. Then $M^0_{b_{(1)}}, M^0_{b_{(2)}}, M^0_{b_{(3)}}$ may be calculated from the values of $M_{b_{(r)}}$ in the preceding approximation. In the first step ($r = 1$), the value $M_{b_{(0.5)}}, \ldots$ allows the use of Simpson's rule. As a first approximation, one may simply choose: $M_{b_{(3)}} = M_{b_{(2)}} = M_{b_{(1)}} = M_{b_{(0)}}$.

Example

For verification, this solution has been programmed and a practical example analyzed (using computer IBM 7094). The following data were assumed (Mp = megapond = force kilogram $\times 10^6$):

$$l = 64m, \qquad F_b = 4.333m^2, \qquad I_b = 0.01167m^4,$$

for $x \geq 0$, $e = 1.02 + 2.5\xi^2$ where $\xi = x/(2l)$,

$$E_aF_a = (5.44 + 30.2\xi^2) \times 10^5\, Mp,$$

$$E_aI_a = (1.2 + 3.05\xi^2 + 9.58\xi^4 + 60.3\xi^6) \times 10^5 Mpm^2$$

for $x \leq 0$ $\qquad e, E_aF_a, E_aI_a = \text{const}$

A constant uniform load $0.8Mp/m$ is considered to be applied at the age $t_0 = 60$ days and the state of stress at the age $t_1 = 180$ days is to be calculated. The results of the analysis are given in Table 1 for different numbers of subdivisions of the time interval, n, and of the length of beam, N. It is seen that the convergence is very good and that both the direct method with variable $\bar{E}_b$ and the iterative method with constant E_b give very close results. The distribution of N_b for $t = t_0$ and $t = t_1$ is plotted in Fig. 2. The algorithm of the iterative variant of this method[4] is represented by the flow chart in Fig. 3.

Creep for a structure with a finite number of statically indeterminate quantities, such as a clamped beam of homogeneous cross section, may be formulated in terms of a system of Volterra's integral equations. It may be verified that the present method is then identical with the solution of these equations by replacement of integrals with finite sums [10].

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References

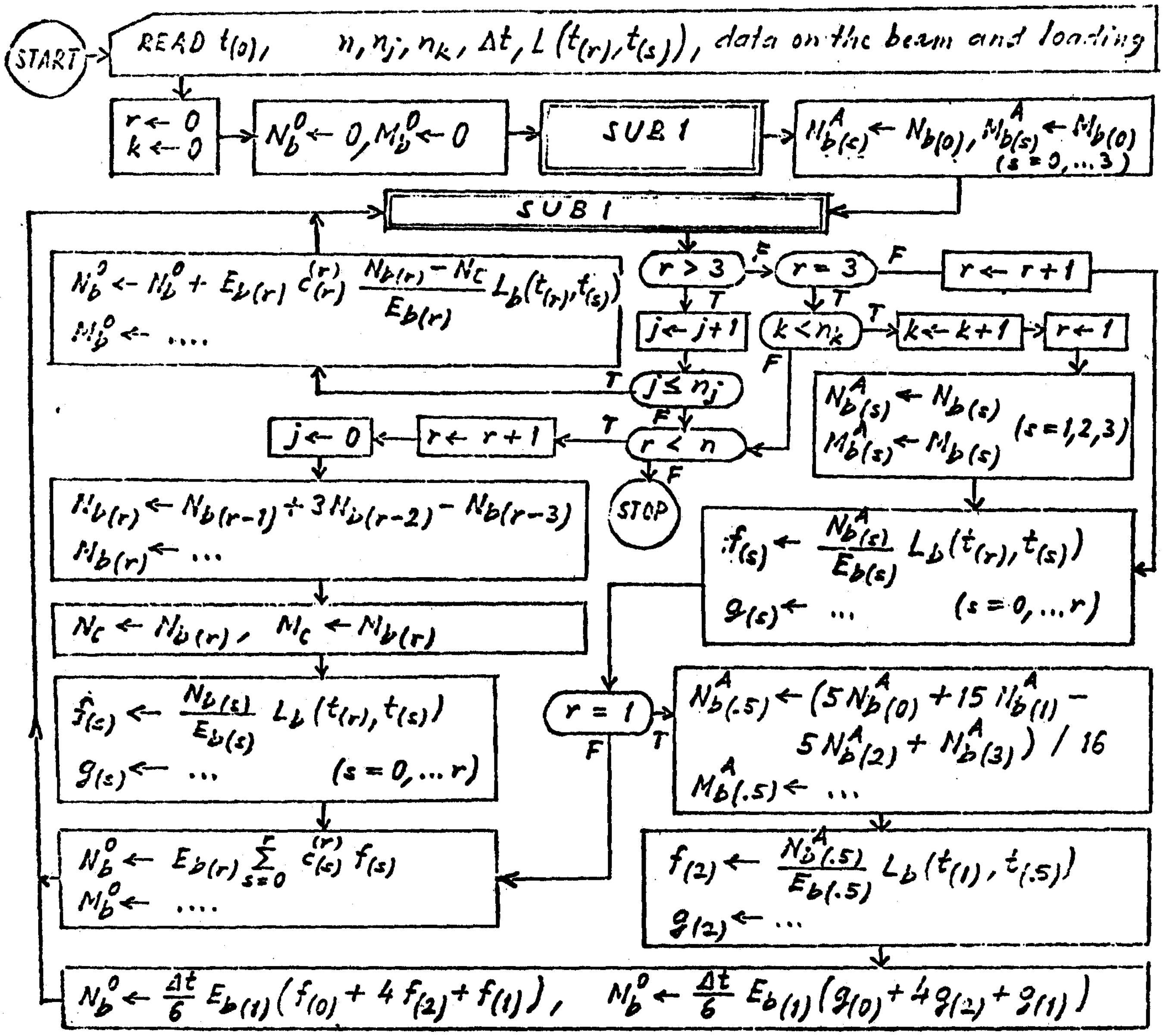
1 Arutyunian, N. Kh, "Some Problems in the Theory of Creep" (in Russian), *Techteorizdat*, Moscow, 1952, English translation, Pergamon Press, Oxford, 1966.

2 Bažant, Z. P., "Approximate Methods of Analysis of Creep and Shrinkage of Nonhomogeneous Structures and Use of Computers" (in Czech), *Stavebnícky Časopis*, SAV (Bratislava), 1964; see also

[4] This algorithm has also been found to converge well for multiaxial stress problems, such as reinforced plate bending, and for certain problems with nonlinear creep. A similar algorithm is also valid if creep depends on strain history, rather than stress history.

Table 1

Method	Subdivision N	n	$M(t_1)$ $x = -l/2$	$x = -l/4$	$x = 0$	$[Mpm]$ $x = l/2$	$N_b(t_1)[Mp]$ $x = -l/4$
Direct	4	2	−118.5	41.54	−3.2	−707.1	−21.6
	8	4	−123.9	47.50	14.1	−667.2	−23.3
	16	8	−122.5	47.35	12.4	−672.0	−19.6
	32	16	−123.1	47.32	12.9	−670.4	−18.2
	64	32	−123.29	47.332	13.16	−669.59	−17.615
Iterative (2 iter. per step)	4	2	−111.6	43.26	−6.7	−721.1	−29.3
	8	4	−123.9	46.97	13.1	−669.1	−22.4
	16	8	−122.4	47.25	12.1	−672.6	−19.7
	32	16	−123.0	47.28	12.8	−670.6	−18.3
	64	32	−123.28	47.315	13.11	−669.70	−17.620
	32	4	−122.0	47.28	11.7	−673.8	−25.4
	32	8	−122.6	47.29	12.4	−671.9	−19.6
	32	16	−123.1	47.32	12.9	−670.4	−18.2
	32	32	−123.30	47.333	13.16	−669.58	−17.615



SUB 1. For an elastic beam with the actual instantaneous moduli $E_{b_{(r)}}$ in time $t_{(r)}$ calculate $N_{b_{(r)}}$, $M_{b_{(r)}}$ due to initial strains represented by N_b^0, M_b^0 (equations (23), (15), (14), (13), (24), (25), (26)). Then add the values of N_b, M_b, w due to applied loads in time $t_{(r)}$. (Write: N_b, M_b, w, $t_{(r)}$.)

Fig. 3 Flow chart for the integration of creep problem by iterative method (with $\bar{E}_b = E_b$), valid for $n \geq 3$; n_j, n_k = number of iterations per step or number of approximations; T = true, F = false. (Subscript l for the subdivision of beam is not written.)

book in Czech: *Concrete Creep in Structural Analysis,*" SNTL (State Publishing House of Technical Literature), Prague, 1966.

3 Bažant, Z. P., "Linear Creep Solved by a Succession of Generalized Thermoelasticity Problems," *Acta Technica* (Czechoslovak Academy of Sciences, Prague), 1967, No. 5, pp. 581–594.

4 Bažant, Z. P., "Phenomenological Theories for Creep of Concrete Based on Rheological Models," *Acta Technica* (Czechoslovak Academy of Sciences, Prague), 1966, No. 1, pp. 82–109.

5 Boley, B. A., and Weiner, J. H., *Theory of Thermal Stresses,* Wiley, New York, 1960.

6 Eschelby, J. D., "The Determination of the Elastic Field of an Ellipsoidal Inclusion and Related Problems," *Proceedings of the Royal Society,* London, Series A, Vol. 241, 1957, p. 396.

7 Duhamel and Neumann, T. H., *Theory of Inelastic Structures,* Wiley, New York, 1968.

8 Lin, T. H., "Bending of a Plate With Nonlinear Strain-Hardening Creep," *Proceedings of the International Union of Theoretical and Applied Mechanics,* Colloquium on Creep in Structures, Springer, Berlin, 1962.

9 Reissner, H., "Eigenspannungen und Eigenspannungsquellen," *Zeitschrift für angewandte Mathematik und Mechanik,* Vol. 11, 1931, pp. 1–8.

10 Rektorys, K., et al., "Survey of Applicable Mathematics," *Iliffe,* London, 1969.

11 Zienkiewicz, O. C., and Cheung, Y. K., *The Finite-Element Method in Structural and Continuum Mechanics,* McGraw-Hill, New York, 1967.