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A Correlation Study of Formulations of Incremental Deformation and Stability of Continuous Bodies

In the past a number of different linearized mathematical formulations of the infinitesimal incremental deformations of continuous bodies under initial stress have been proposed. The best-known formulations are reviewed, tabulated, and subjected to a comparative study. It is demonstrated that they can be derived as special cases of a unified general formulation, and are all correct and mutually equivalent. In each formulation, the incremental elasticity constants and the incremental material stress tensor have a different significance. Their mutual relationships are established. Thus the analysis of a problem which has already been solved according to one formulation need not be repeated for another formulation. Furthermore, the connections to the various definitions of the objective stress rate are shown. The arbitrariness of choice between the infinitely many possible forms of incremental equilibrium equations corresponds to the arbitrariness in the definitions of (a) the finite strain tensor, (b) the material stress tensor, (c) the objective stress rates, (d) the stability criterion, and (e) the elastic material in finite strain. For demonstration of the differences, the problems of surface buckling of an orthotropic half space and a column with shear are studied. It is shown that the predicted buckling stresses can differ almost by a ratio of 1:2 if the proper distinction between various formulations is not made.

Introduction

The general problem of infinitesimal elastic stability of three-dimensional continuous bodies, as well as the closely related problem of small incremental deformations of a medium under initial stress, have been studied by many researchers. Various authors, however, have chosen different mathematical descriptions of the incremental deformation. In the stability theory this has led to various formulations which appear to be considerably different from each other.

The fact that stresses and strains may be defined in various ways is well known and has often been pointed out. It is also clear that such differences must be projected in the incremental equilibrium equations and other relationships, and that the various formulations must be equivalent. Nevertheless, it seems that no specific discussion of various formulations has been presented so far. Such a discussion is needed because different formulations continue to be used at the present time. Although in the theoretical literature only one formulation, namely, the formulation associated with the Cauchy-Green strain tensor [6, 10, 11, 13–17, 19, 21, 23, 26, 27, 29, 34–36][1] is now generally preferred, other formulations [5, 8, 18, 25, 30, 37] seem to prevail in the analysis of practical problems. The results of these analyses are often distrusted because it is not immediately obvious whether the differences are caused only by a choice of different mathematical formulation.

For the sake of simplicity, only rectangular Cartesian coordinates will be used. The coordinates of material particles in the initial stressed state under consideration will be denoted by x_i

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[1] Numbers in brackets designate References at end of paper.

ERRATA

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In equation (23) and the first line of column e in Table 1 the fraction $\frac{3}{2}$ should be changed to 2.

In the 4th line after equation (28), following the reference brackets, the last bracket should be followed by $F_{kl}F_{rl} = \delta_{kr} + 2\epsilon_{kr}^{a}$.

In the 5th line after equation (22d) following ∂t, the phrase "if coordinates x_i' are used,' should be inserted.

In equation (31) F_{pq} should be replaced by E_{pq}^{a}.

In the 3rd line of footnote 7, equation (7) should be (13).

In equation (22b), the last minus sign should be changed to a plus sign.

$(i = 1, 2, 3)$ and the coordinates of the same particle after the incremental displacement u_i will be designated by $x_i' = x_i + u_i$. No formal distinction will be made between the subscripts referring to the initial and the final configurations.[2] Subscripts preceded by a comma will denote partial derivatives with respect to x_i (not x_i'), e.g., $u_{i,j} = \partial u_i/\partial x_j$. Doubly repeated subscripts will imply summation over the range 1, 2, 3. To avoid lengthy discussions, attention will be restricted to the case of "dead" loads, i.e., loads which do not change direction and magnitude during the incremental deformation.

The best-known formulations of equilibrium equations for small incremental deformations which will be discussed here can all be written in the form

$$\tau_{ij,j} + \rho_0 f_i = 0 \qquad \text{(in the whole volume)} \tag{1}$$

$$n_j \tau_{ij} = p_i \qquad \text{(on stress boundary)} \tag{1a}$$

with the following expressions for τ_{ij}:

$$\tau_{ij} = \sigma_{ij}^a + T_{kj}^0 u_{i,k} \qquad \text{(Trefftz)} \tag{2a}$$

$$\tau_{ij} = \sigma_{ij}^b + T_{kj}^0 \omega_{ik} + \tfrac{1}{2} T_{kj}^0 e_{ik} - \tfrac{1}{2} T_{ki}^0 e_{jk} \qquad \text{(Biot)} \tag{2b}$$

$$\tau_{ij} = \sigma_{ij}^c + T_{kj}^0 \omega_{ik} - T_{ki}^0 e_{jk} \qquad \text{(Biezeno and Hencky)} \tag{2c}$$

$$\tau_{ij} = \sigma_{ij}^d + T_{kj}^0 \omega_{ik} - T_{ki}^0 e_{jk} + T_{ij}^0 e_{kk} \qquad \text{(Biot, Southwell)} \tag{2d}$$

Here ρ_0 is the initial mass density; n_i is the unit outward normal in the initial state; f_i and p_i are the increments of mass forces and surface loads (considered as "dead" loads); e_{ij} and ω_{ij} are the linearized incremental tensors of strain and rotation, i.e.,

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \qquad \omega_{ij} = \tfrac{1}{2}(u_{i,j} - u_{j,i}) \tag{3}$$

T_{ij}^0 is the stress tensor (of Cauchy) in the initial state; τ_{ij} is a certain nonsymmetric incremental stress tensor whose significance will be explained after equation (12). Tensors σ_{ij}^a, σ_{ij}^b, σ_{ij}^c, σ_{ij}^d represent certain incremental material stress tensors. They are labeled by different superscripts because each of them has a different significance as will be demonstrated later by equations (14*b*), (16*b*), (17*c*), and (20*a*). These tensors are all symmetric and are related to e_{ij} by linear relationships. For $p_i = f_i = 0$, equations (1) and (1*a*) represent conditions of neutral equilibrium.

Expression (2*a*) can be traced back to Trefftz [34, equation (29)]. The stability theories presented by Goodier and Plass [14, equation (10)], Pearson [27], Hill [13], Prager [29], Truesdell and Noll [35], Green and Adkins [15], Kappus [10], Novozhilov [26], Koiter [19], Nemat-Nasser [23], and others [1, 2, 6, 10, 16, 21, 36] are all based on expression (2*a*). A special type of stability criterion (for bodies with prescribed boundary displacements) presented by Hadamard [12] also implies equation (2*a*).

The equilibrium equations corresponding to expression (2*b*) are due to Biot [4, 5, equation (II.5.20)]. Identical equations were obtained, using different approaches, by Neuber [24, equations (61) and (50)] and by Prager [28, equations (17) and (18)].

The equilibrium equations corresponding to expression (2*c*) have been deduced by Biezeno and Hencky [3, equation (20) or equation (5)].

The equilibrium equations corresponding to expression (2*d*) were introduced by Biot [4, 5, equation (I.7.32)]. They were also used by Neuber [25, equations (12), (1), (3), and (15)]. It can be easily verified that for a uniform initial stress field, that is for $T_{ij,k}^0 = 0$, these equilibrium equations may also be written in the form

$$(\sigma_{ij}^c + T_{ki}^0 \omega_{jk} + T_{kj}^0 \omega_{ik})_{,j} + \rho_0 f_i = 0 \tag{3a}$$

which was used by Neuber [25, equation (16)]. A special form of equation (3*a*) in the coordinate system whose axes x_i coincide with the principal directions of initial stress was presented already by Southwell [31, equation (16)]. (It should be noted that the term in parentheses in equation (3*a*) does not have the significance of τ_{ij}, as will be later discussed, and may not be substituted for τ_{ij} in boundary condition (1*a*).)

Incremental Stresses and General Form of Incremental Equilibrium Equations

The work per unit initial volume which is done at the incremental deformation will be denoted by W. The variation of δW which corresponds to the variation δu_i of infinitesimal displacements u_i compatible with the boundary conditions of place may be expressed in either of the following two forms:

$$\delta W = S_{ij} \delta \epsilon_{ij} \quad \text{where} \quad S_{ij} = T_{ij}^0 + \sigma_{ij} \tag{4a}$$

$$\delta W = T_{ij}^* \delta u_{i,j} \quad \text{where} \quad T_{ij}^* = T_{ij}^0 + \tau_{ij} \tag{4b}$$

in which ϵ_{ij} = certain incremental finite strain tensor = a symmetric tensorial function of the displacement gradient $u_{i,j}$, such that $\delta W = 0$ when $u_{i,j}$ expresses a rotation. Equations (4*a*) and (4*b*) represent definitions of the incremental stress tensors σ_{ij} and τ_{ij}.

Tensor σ_{ij} may be assumed as symmetric since $\epsilon_{ij} = \epsilon_{ji}$. It vanishes for a rigid-body rotation because in this case $\delta \epsilon_{ij} =$

[2] The notation which does make such a distinction [10, 35] is, of course, superior. Here, however, such a notation is not suitable because it does not allow the introduction of the relative displacements u_i and other incremental quantities, which are convenient for the formulations that are associated with other than Cauchy-Green finite strain tensor.

Nomenclature

- C_{ijkl} = incremental elastic moduli
- e_{ij} = incremental linearized strain tensor
- f_i^0, f_i = initial body force per unit mass and its increment
- n_i^0 = unit outward normal at the surface
- p_i^0, p_i = initial surface load and its increment
- S = surface of body
- S_{ij} = objective material stress tensor referred to initial configuration
- T_{ij}, T_{ij}^0, ΔT_{ij} = Cauchy's stress tensor in final and initial states and their difference
- T_{ij}^* = mixed (Piola-Kirchhoff) stress tensor referred to the initial configuration, unsymmetric
- u_i = displacement increment = $x_i' - x_i$
- V = volume of the body
- x_i, x_i' = rectangular Cartesian coordinates of a particle in initial and final configurations
- ϵ_{ij} = incremental finite strain tensor
- ρ_0 = initial mass density
- $\sigma_{ij} = S_{ij} - T_{ij}^0$ = objective material stress increment referred to initial configuration (symmetric)
- $\tau_{ij} = T_{ij}^* - T_{ij}^0$ = mixed stress increment referred to the initial configuration, unsymmetric
- ω_{ij} = incremental linearized rotation

Superscripts

- a, b, c, d, e, f = individual formulations, Table 1

Superscript

- ^ = objective stress rate

$V = 0$. Owing to the invariance of the expression for work, tensor σ_{ij} is objective (i.e., invariant at any observer transformation [35, 10, 29, 16]). Thus σ_{ij} is a tensor which may be used in stress-strain relations. If there exists an incremental strain potential, $U(\epsilon_{ij})$, σ_{ij} may be also defined by the equation $S_{ij} = \partial U/\partial \epsilon_{ij}$ which leads to (4a) if U is identified with W. For adiabatic conditions U represents the total energy, while for isothermic conditions U is the Helmholtz free energy.

Tensor τ_{ij} is in general unsymmetric, unless $u_{i,j}$ represents pure (i.e., symmetric) deformation. From equation (4b) (as well as equation (13) derived later) it can be concluded that T^*_{ij} or $T^0_{ij} + \tau_{ij}$ must represent the total forces in the directions x_1, x_2, x_3 acting on the sides of a deformed element which was originally (in the initial state) a unit cube. Thus tensor T^*_{ij} is equivalent to the mixed (Piola-Kirchhoff) stress tensor (of the first kind) [35, 16, 11] which is, however, referred to the initial stressed state rather than a natural unstressed state as in the usual definition. Sometimes the mixed tensor is also called Lagrangian [29] or Boussinesq's [21] stress tensor, or "pseudostress."

The relationship between σ_{ij} and τ_{ij} may be derived by subtracting equations (4a) and (4b). Thus

$$\tau_{ij}\delta u_{i,j} = \sigma_{ij}\delta e_{ij} + T^0_{ij}\delta(\epsilon_{ij} - u_{i,j}) = \sigma_{ij}\delta u_{i,j} + T^0_{ij}\delta(\epsilon_{ij} - e_{ij})$$
$$= \left[\sigma_{ij} + T^0_{pq}\frac{\partial(\epsilon_{pq} - e_{pq})}{\partial u_{i,j}}\right]\delta u_{i,j} \quad (5)$$

where the relations $\sigma_{ij}\delta e_{ij} = \sigma_{ij}\delta u_{i,j}$ and $T^0_{ij}\delta u_{i,j} = T^0_{ij}\delta e_{ij}$ based on symmetry of σ_{ij} and T^0_{ij} have been utilized. Equation (5) is satisfied for any $u_{i,j}$ if, and only if,

$$\tau_{ij} = \sigma_{ij} + T^0_{pq}\frac{\partial(\epsilon_{pq} - e_{pq})}{\partial u_{i,j}} \quad (6)$$

For infinitesimal incremental deformation, $u_{i,j}$, σ_{ij}, and τ_{ij} are infinitely small of first order while T^0_{pq} is finite. Therefore it is sufficient if the finite strain tensor is expressed exactly only up to second order terms in $u_{i,j}$.

The stress tensor of Cauchy after the incremental deformation, T_{ij} (which is also called Eulerian stress [29] or "true stress" since it represents forces in the directions x_1, x_2, x_3 acting on a unit cube which has been cut out after the incremental deformation in point x_i'), may be expressed with the help of the mixed tensor T^*_{ij} (or τ_{ij}) as follows [29, equation (IX.4.3)] or [11, p. 439]:

$$T_{ij} = \frac{\partial x_j'}{\partial x_k} T^*_{ik} J^{-1} = (T^*_{ij} + T^*_{ik}u_{j,k})J^{-1} \quad (7)$$

where $J = \det(\partial x_i'/\partial x_j)$ = Jacobian of the transformation.[3]

[3] For reader's convenience let us show how equation (7) may be obtained. Imagine within the given body an arbitrary infinitesimal parallelogram of area dS and unit normal ν_i, defined by two line elements dx_i and δx_i. After the incremental deformation, dS, ν_i, dx_i, δx_i are transformed in dS', ν_i', dx_i', $\delta x_i'$. The force acting on dS from one side must be the same, whether it is determined from T_{ij} or T^*_{ij}. Hence

$$T_{ij}\nu_j' dS' = T^*_{ik}\nu_k dS \quad (7a)$$

Here

$$\nu_j' dS' = \bar{e}_{jmn} dx_m' \delta x_n' = \bar{e}_{jmn}\frac{\partial x_m'}{\partial x_p}\frac{\partial x_n'}{\partial x_q} dx_p \delta x_q$$

where $\bar{e}_{jmn}$ = permutation symbol. Furthermore,

$$\frac{\partial x_j'}{\partial x_k}\nu_j' dS' = J\bar{e}_{kpq} dx_p \delta x_q = J\nu_k dS \quad (7b)$$

because

$$\bar{e}_{jmn}\frac{\partial x_j'}{\partial x_k}\frac{\partial x_m'}{\partial x_p}\frac{\partial x_n'}{\partial x_q} = J\bar{e}_{kpq}$$

Expressing $\nu_k dS$ according to (7b), it is seen that (7a) is satisfied for any ν_j if, and only if, equation (7) is valid.

For small incremental deformations, $J^{-1} \doteq 1 - u_{k,k}$, so that

$$\Delta T_{ij} = T_{ij} - T^0_{ij} \doteq \tau_{ij} + T^0_{ik}u_{j,k} - T^0_{ij}u_{k,k} \quad (8)$$

It should be noted that the relationship between ΔT_{ij} and τ_{ij} is independent of the choice of the form of finite strain tensor ϵ_{ij}, while the relationship between σ_{ij} and τ_{ij} (or σ_{ij} and ΔT_{ij}) depends on this choice.

The equilibrium conditions may be obtained from the principle of virtual work. For the equilibrium state after the incremental deformation, this principle yields the equation

$$\int_{(V)} \delta W dV = \int_{(V)} \rho_0(f_i^0 + f_i)\delta u_i dV + \int_{(S)} (p_i^0 + p_i)\delta u_i dS \quad (9)$$

which must be satisfied for any variation δu_i compatible with the boundary conditions of place; f_i and p_i are the incremental forces per unit mass and the incremental surface loads (dead loads); V = volume, S = surface of the body in the initial state. The assumption that the initial state is an equilibrium state may be expressed by the principle of virtual work as follows:

$$\int_{(V)} T^0_{ij}\delta u_{i,j} dV = \int_{(V)} \rho_0 f_i^0 \delta u_i dV + \int_{(S)} p_i^0 \delta u_i dS \quad (9a)$$

If (9a) is subtracted from (9) and the relations $\sigma_{ij}\delta\epsilon_{ij} \approx \sigma_{ij}\delta u_{i,j}$ and (4a) are noted, the following equilibrium condition for the incremental deformation is obtained:

$$\int_{(V)} \tau_{ij}\delta u_{i,j} dV = \int_{(V)} \rho_0 f_i \delta u_i dV + \int_{(S)} p_i \delta u_i dS \quad (10)$$

Here the left-hand side may also be written in the form

$$\int_{(V)} \tau_{ij}\frac{\partial(\delta u_i)}{\partial x_j} dV \quad \text{or} \quad \int_{(V)} \frac{\partial}{\partial x_j}(\tau_{ij}\delta u_i) dV - \int_{(V)} \tau_{ij,j}\delta u_i dV \quad (11)$$

Applying the Gauss theorem [29, 16] to the first integral in the last expression, it is found that equation (10) is equivalent to the following condition:

$$\int_{(S)} n_j\tau_{ij}\delta u_i dS - \int_{(V)} \tau_{ij,j}\delta u_i dV = \int_{(V)} \rho_0 f_i \delta u_i dV + \int_{(S)} p_i \delta u_i dS \quad (12)$$

where n_j is the unit outward normal at the surface. To satisfy this condition for any δu_i, it is necessary and sufficient that

$$\tau_{ij,j} + \rho_0 f_i = 0 \qquad \text{(in volume V)} \quad (13)$$

$$n_j\tau_{ij} = p_i \qquad \text{(on stress boundary)} \quad (13a)$$

This is the general form of the differential equilibrium equations for incremental deformations. These equations also corroborate the physical significance of τ_{ij} as explained after (4b).

It should be noted that equations (4a)–(13a) are valid irrespective of the material properties as well as the choice of the form of ϵ_{ij}.

Special Forms of Incremental Equilibrium Equations

The common basis from which various special formulations may be obtained is equation (6), expressing the incremental mixed (Piola-Kirchhoff) tensor (of the first kind) as a function of the general incremental finite strain tensor ϵ_{ij}. Some of the infinitely many admissible forms of ϵ_{ij} will now be considered.

1 Substituting the classical Lagrangian (Green's) finite strain tensor

$$\epsilon_{ij}^a = e_{ij} + \tfrac{1}{2}u_{k,i}u_{k,j} \tag{14}$$

for ϵ_{ij} in equation (6), it can be verified that

$$\tau_{ij} = \sigma_{ij}^a + T_{kj}^0 u_{i,k} \tag{14a}$$

Superscript a is used here and in the sequel to denote quantities which are conjugated with ϵ_{ij}^a. The relationship between the objective material stress increment σ_{ij}^a and the increment ΔT_{ij} of the true stress follows from equation (8);

$$\sigma_{ij}^a = \Delta T_{ij} - T_{kj}^0 u_{i,k} - T_{ki}^0 u_{j,k} + T_{ij}^0 u_{k,k} \tag{14b}$$

It is seen that equation (14a) is identical with (2a). The formulation of stability theory used by Trefftz, Pearson, Hill, etc., is thus connected with the classical Lagrangian finite strain (or Cauchy-Green tensor). It is also noteworthy that, according to equation (14b) of (14a), the tensor

$$S_{ij}^a = T_{ij}^0 + \sigma_{ij}^a \tag{14c}$$

appears to be identical with the Piola-Kirchhoff tensor of the second kind [35, 16, 10, 29] except that it is referred to the initial stressed state under consideration rather than a natural unstressed state.

2 Considering the expression (4a) for work, it may be convenient to look for such a formulation that the work $T_{ij}^0\delta\epsilon_{ij}$ be expressed as the work done by the components of T_{ij}^0 on the displacements u_i as if T_{ij}^0 were forces kept constant during the incremental deformation. In the special case of a symmetric transformation $u_{i,j} = u_{j,i}$, called pure deformation, this work equals exactly $T_{ij}^0 u_{i,j}$ (per unit initial volume).[4] Consequently, $\epsilon_{ij} = u_{i,j}$. For a general, unsymmetric transformation, the expression for ϵ_{ij} may be obtained if the transformation $u_{i,j}$ is decomposed into a pure deformation ϵ_{ij} (at which the work equals $T_{ij}^0\epsilon_{ij}$), followed by a rotation (at which the work is zero). This decomposition is called polar decomposition [35, p. 52]. Up to the second-order terms in $u_{i,j}$, the finite strain tensor ϵ_{ij}^b defined by this decomposition is[5] [5]:

$$\epsilon_{ij}^b = e_{ij} + \tfrac{1}{2}u_{k,i}u_{k,j} - \tfrac{1}{2}e_{ki}e_{kj} \tag{15}$$

The incremental elastic moduli C_{ijkl} and the objective material stress increment σ_{ij} which is conjugate to ϵ_{ij}^b will be denoted by C_{ijkl}^b and σ_{ij}^b. Substituting $\epsilon_{ij} = \epsilon_{ij}^b$ in equations (6) and (8), it can be obtained

$$\tau_{ij} = \sigma_{ij}^b + T_{kj}^0 u_{i,k} - \tfrac{1}{2}(T_{kj}^0 e_{ik} + T_{ki}^0 e_{jk}) \tag{16a}$$

$$\sigma_{ij}^b = \Delta T_{ij} - T_{kj}^0 u_{i,k} - T_{ki}^0 u_{j,k} + T_{ij}^0 u_{k,k} + \tfrac{1}{2}(T_{kj}^0 e_{ik} + T_{ki}^0 e_{jk}) \tag{16b}$$

Obviously, equation (16a) coincides with equation (2b), obtained first by Biot [4]. Also it should be noticed that the objective material stress tensor

$$S_{ij}^b = T_{ij}^0 + \sigma_{ij}^b \tag{16c}$$

[4] Note that this is not equal $T_{ij}^0\epsilon_{ij}^a$; e.g., for a uniaxial extension, $\epsilon_{11}^a = u_{1,1} + \frac{1}{2}u_{1,1}^2$ while (assuming T_{11}^0 as constant) the work equals $T_{11}^0 u_{1,1}$.

[5] This second-order approximation may be simply obtained as follows. The transformation of any vector dx_i at the incremental deformation $u_{i,j}$, and its transformation at the pure deformation ϵ_{ij}^b defined by the foregoing decomposition, are given by the relationships

$$dx_i' = (\delta_{ij} + u_{i,j})dx_j, \qquad dx_i'' = (\delta_{ij} + \epsilon_{ij}^b)dx_j \tag{a}$$

where δ_i is Kronecker delta. For rotation $dx_k'dx_k' = dx_k''dx_k''$. Thus the following must hold

$$(\delta_{kj} + u_{k,j})dx_j(\delta_{ki} + u_{k,i})dx_i = (\delta_{kj} + \epsilon_{kj}^b)dx_j(\delta_{ki} + \epsilon_{ki}^b)dx_i \tag{b}$$

This equation is satisfied for any dx_i if, and only if,

$$\epsilon_{ij}^b + \tfrac{1}{2}\epsilon_{ki}^b\epsilon_{kj}^b = e_{ij} + \tfrac{1}{2}u_{k,i}u_{k,j} \tag{c}$$

Replacing ϵ_{ki}^b by e_{ki} in the second-order term, equation (15) is obtained.

is different from the Piola-Kirchhoff tensor of the second kind [35, 16, 10, 29] given by equation (14c).

3 In general, any tensor polynomial in ϵ_{ij}^a or ϵ_{ij}^b whose first-order term equals e_{ij} might be adopted for representation of ϵ_{ij}. Thus the tensor.

$$\epsilon_{ij} = e_{ij} + \tfrac{1}{2}u_{k,i}u_{k,j} + ae_{ki}e_{kj} + be_{ij}e_{kk} + c\delta_{ij}e_{kl}e_{kl} + d\delta_{ij}e_{kk}e_{ll} \tag{17}$$

where a, b, c, d are arbitrary constants and δ_{ij} is Kronecker delta, is the most general expression for an admissible second-order approximation to a certain finite strain tensor ϵ_{ij}.

It is interesting to consider the finite strain tensor

$$\epsilon_{ij}^c = e_{ij} + \tfrac{1}{2}u_{k,i}u_{k,j} - e_{ki}e_{kj} \tag{17a}$$

which represents the so-called logarithmic strain because $\epsilon_{11}^c = u_{1,1} - \frac{1}{2}u_{1,1}^2 =$ the second-order approximation to ln $(1 + e_{11})$. The substitution of ϵ_{ij}^c for ϵ_{ij} in equations (8) and (13a) leads to the relationships

$$\tau_{ij} = \sigma_{ij}^c + T_{kj}^0 u_{i,k} - T_{kj}^0 e_{ik} - T_{ki}^0 e_{jk} \tag{17b}$$

$$\sigma_{ij}^c = \Delta T_{ij} - T_{ki}^0\omega_{jk} - T_{kj}^0\omega_{ik} + T_{ij}^0 u_{k,k} \tag{17c}$$

In this manner one could obtain infinitely many forms of equilibrium equations, in each of which σ_{ij} would be defined in a different manner. It may be verified, however, that no admissible expression for ϵ_{ij} leads to equation (2d) used by Biot and Neuber (as well as to equation (3a)). The closest admissible expression is (2c) which differs from (2d) by the term $T_{ij}^0 u_{k,k}$. For materials of small volume compressibility (nonporous materials) this difference is, of course, negligible.

Relationships Between Various Definitions of Incremental Elastic Moduli

All of the equations introduced so far are valid without regard to the material properties. Let us now consider that the material behaves in the infinitesimal incremental deformation as elastic (while the initial stressed state may still include other than elastic strains, e.g., plastic strains). Then

$$\begin{aligned} \sigma_{ij}^a &= C_{ijkl}^a e_{kl}, \\ \sigma_{ij}^b &= C_{ijkl}^b e_{kl}, \\ \sigma_{ij}^c &= C_{ijkl}^c e_{kl}, \\ \sigma_{ij}^d &= C_{ijkl}^d e_{kl} \end{aligned} \tag{18}$$

where C_{ijkl}^a, ... are the incremental elastic moduli associated with σ_{ij}^a, etc. It is easy to verify by substitution that equations (14), (16a), (17b), and (2d) (or equation (4a) for the various forms of τ_{ij}) are mutually equivalent and represent the same material if the following relationships between the incremental elastic moduli hold:

$$C_{ijkl}^a = C_{ijkl}^b - \tfrac{1}{4}(\delta_{il}T_{jk}^0 + \delta_{jl}T_{ik}^0 + \delta_{ik}T_{jl}^0 + \delta_{jk}T_{il}^0) \tag{19a}$$

$$C_{ijkl}^c = C_{ijkl}^b + \tfrac{1}{4}(\delta_{il}T_{jk}^0 + \delta_{jl}T_{ik}^0 + \delta_{ik}T_{jl}^0 + \delta_{jk}T_{il}^0) \tag{19b}$$

$$C_{ijk}^d = C_{ijkl}^c - \delta_{kl}T_{ij}^0 \tag{19c}$$

Another relationship which correlates C_{ijkl}^d and C_{ijk}^b has been shown by Biot [5, equation (II.4.25)].

Obviously all of these relationships satisfy the necessary conditions of symmetry for the incremental elastic moduli, that is, $C_{ijkl} = C_{jikl} = C_{ijlk}$.

In addition, relationships (19a) and (19b) also preserve the symmetry condition $C_{ijkl} = C_{klij}$ which must be fulfilled by C_{ijkl}^a, C_{ijkl}^b, and C_{ijkl}^c if a potential for infinitesimal incremental deformations exists.

Relationship (19c), however, does not satisfy the latter symmetry condition, and if the potential exists

$$C_{ijkl}^d \neq C_{klij}^d \tag{20}$$

This asymmetry, which is an inconvenient formal feature of equation (2*d*) devised by Biot and Neuber (or equation (3*a*) obtained by Southwell), disappears only for an incompressible material.

In the sequel an expression for the incremental material stress tensor σ_{ij}^d, corresponding to equation (2*d*), will be also needed. Equations (2*d*) and (8) yield

$$\sigma_{ij}^d = C_{ijkl}^d e_{kl} = \Delta T_{ij} - T_{ki}^0 \omega_{jk} - T_{kj}^0 \omega_{ik} \qquad (20a)$$

The relationship of the material stress increments σ_{ij}^b and σ_{ij}^d is, according to (16*b*) and (20*a*),

$$\sigma_{ij}^d = \sigma_{ij}^b + \tfrac{1}{2}(T_{ik}^0 e_{jk} + T_{jk}^0 e_{ik}) - T_{ij}^0 e_{kk} \qquad (20b)$$

which is a relationship known from Biot's theory [15, equation (II.2.22)][6].

With the help of boundary conditions (1*a*) or (11*a*) it is possible to determine how the incremental elastic moduli can be measured. This topic has been dealt with in detail by Biot [5] and a relationship of moduli C_{ijkl}^d and C_{ijkl}^b to the "measurable slide moduli" [5] has been derived. Using this relationship and equations (19*a*, *b*, *c*), the moduli C_{ijkl}^a and C_{ijkl}^c may be also determined from measurements.

Objective Stress Rates

If the increment of deformation is associated with interval Δt of time t, then $\lim_{\Delta t \to 0} (\sigma_{ij}/\Delta t)$ = objective stress rate of T_{ij}, $\lim_{\Delta t \to 0} (e_{ij}/\Delta t)$ = deformation rate d_{ij}, $\lim_{\Delta t \to 0} (\omega_{ij}/\Delta t)$ = rotation rate w_{ij},

$$d_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i}), \qquad w_{ij} = \tfrac{1}{2}(v_{i,j} - v_{j,i}) \qquad (21)$$

where v_i is the velocity of particle. The various time rates obtained from σ_{ij}^a, σ_{ij}^b, σ_{ij}^c, σ_{ij}^d represent the objective tensor rates (stress fluxes), i.e., tensors which are invariant at any observer transformation [35, 10, 29, 16]. The objective rates of the stress tensor T_{ij} obtained from the expressions (14*b*), (16*b*), (17*c*), and (20*a*), are as follows:

$$\text{for } \sigma_{ij}^a \ldots \hat{T}_{ij}^a = \dot{T}_{ij} - T_{kj}v_{i,k} - T_{ki}v_{j,k} + T_{ij}v_{k,k} \qquad (22a)$$

$$\text{for } \sigma_{ij}^b \ldots \hat{T}_{ij}^b = \dot{T}_{ij} - T_{kj}v_{i,k} - T_{ki}v_{j,k} - T_{ij}v_{k,k} + \tfrac{1}{2}(T_{jk}d_{ik} + T_{ik}d_{jk}) \qquad (22b)$$

$$\text{for } \sigma_{ij}^c \ldots \hat{T}_{ij}^c = \dot{T}_{ij} - T_{ki}w_{jk} - T_{kj}w_{ik} + T_{ij}v_{k,k} \qquad (22c)$$

$$\text{for } \sigma_{ij}^d \ldots \hat{T}_{ij}^d = \dot{T}_{ij} - T_{ki}w_{jk} - T_{kj}w_{ik} \qquad (22d)$$

where

$$\dot{T}_{ij} = \lim_{\Delta t \to 0} (\Delta T_{ij}/\Delta t)$$

Because ΔT_{ij} is the change of stress (of Cauchy) in a given particle, $\dot{T}_{ij}$ must represent the material rate which is expressed as $\dot{T}_{ij} = T_{ij,k}v_k + \partial T_{ij}/\partial t$ [29]. (If the material properties are defined by equation (18) for $\Delta t \to 0$, the material is called hypoelastic.)

It is readily recognized that expression (22*a*) is the Truesdell's stress rate [29, 10, 22] and expression (22*d*) is the corotational stress rate, due to Jaumann [29, 10, 35, 22]. The stress rates (22*b*) and (22*c*) probably have not been used so far.

Thus it may be concluded that the Truesdell's stress rate corresponds to the use of the classical Lagrangian strain tensor ϵ_{ij}^a in the theory of incremental deformations or stability, and to Trefftz's equations of neutral equilibrium. The correspondence between the Truesdell's stress rate and the stability theory leading to Trefftz's expression (2*a*) has already been shown by Masur [22].

The corotational (Jaumann's) stress rate corresponds to neutral equilibrium equation (2*d*) or (17*b*), due to Biot [5] and Neuber [25] but has no corresponding form of finite strain tensor unless the material is incompressible. The fact that for the material $\hat{T}_{ijkl}^d = C_{ijkl}^d d_{kl}$ the incremental potential exists for certain moduli such that $C_{ijkl}^d \neq C_{klij}^d$, and is nonexistent if $C_{ijkl}^d \neq C_{klij}^d$, is to be regarded as an inconvenience of the corotational (Jaumann's) stress rate (22*d*). This feature disappears for the stress rate (22*c*) which corresponds to the finite strain tensor (17*a*) (logarithmic strain) and the Biezeno and Hencky's neutral equilibrium equations given by (2*c*) and (17*c*). The difference between (22*c*) and (22*d*) can be important, of course, only for materials of high volume compressibility such as porous materials.

Another often used rate is the convected stress rate, due to Cotter and Rivlin [29, 10, 35]:

$$\hat{T}_{ij}^e = \dot{T}_{ij} + T_{ik}v_{k,j} + T_{jk}v_{k,i} \qquad (22e)$$

It seems that no corresponding equation of neutral equilibrium has been devised. It may be verified that if the strain tensor

$$\epsilon_{ij}^e = e_{ij} + \tfrac{1}{2}u_{k,i}u_{k,j} - \tfrac{3}{2}e_{ki}e_{kj} \qquad (23)$$

(which represents a second-order approximation to $e_{11}/(1 + e_{11})$) were used in equation (8), the corresponding stress rate would equal expression (22*e*) plus the term $T_{ij}v_{k,k}$. But no finite strain tensor could lead to equation (22*e*) itself. Hence, for strongly compressible materials the convected stress rate (22*e*) leads to the same kind of inconvenience as the corotational (Jaumann's) rate (22*d*).

Oldroyd's stress rate [10] equals Truesdell's rate (22*a*) if the term $T_{ij}u_{k,k}$ is omitted. Therefore, for an incompressible material this rate also corresponds to the classical Lagrangian tensor ϵ_{ij}^a. But for compressible materials the corresponding incremental moduli are again nonsymmetric if the incremental potential exists.

Stability Criterion

The diversity of admissible forms of equations considered hitherto is projected in other theorems of the theory of stability and incremental deformations. As an example consider the criterion of infinitesimal stability [35, 32, first attempt 7]. To avoid lengthy discussions let us restrict attention to conservative systems with dead loads. Then a given stressed state is infinitesimally stable if the work done at any further infinitesimal incremental deformation compatible with the boundary conditions of place is greater than or equal the work of given initial surface and volume loads. Noting that according to (4*a*) $W = T_{ij}^0\epsilon_{ij} + \tfrac{1}{2}C_{ijkl}e_{ij}e_{kl}$ (for small e_{ij}), this criterion may be expressed as follows:

$$\int_{(V)} (T_{ij}^0\epsilon_{ij} + \tfrac{1}{2}C_{ijkl}e_{ij}e_{kl})dV \geq \int_{(V)} \rho_0 f_i^0 u_i dV + \int_{(S)} p_i^0 u_i dS \qquad (24)$$

Subtracting equation (9*a*) written without the sign δ, the sufficient condition of stability under dead loads is obtained in form of the inequality:

$$\int_{(V)} [\tfrac{1}{2}C_{ijkl}e_{ij}e_{kl} + T_{ij}^0(\epsilon_{ij} - e_{ij})]dV \geq 0 \qquad (25)$$

which must be satisfied for any kinematically admissible incremental displacements u_i.[7] The special forms of the general criterion (25) are obtained by selection of a certain finite strain tensor ϵ_{ij}. If the classical Lagrangian form ϵ_{ij}^a is substituted and

[6] Biot's notation and terminology is: $\sigma_{ij}^d = s_{ij}$ = "incremental stresses," $\sigma_{ij}^b = t_{ij}$ = "alternative stresses," $C_{ijkl}^d = B_{ijkl}$, $C_{ijkl}^b = C_{ijkl}$.

[7] From criterion (25) it can be concluded that the body is at the limit of stability when the variation of (25) is zero. Using this condition, equation (7) for $p_i = f_i = 0$ can be deduced from criterion (25) [1].

the equality $C_{ijkl}e_{ij}e_{kl} = C_{ijkl}u_{i,j}u_{k,l}$ is considered, the stability criterion (25) takes on the form

$$\int_{(V)} (C^a_{ijkl} + \delta_{ik}T^0_{jl})u_{i,j}u_{k,l}dV \geq 0 \tag{26}$$

presented by Trefftz [34, equation (26)]. For the special case of an elastic body with displacements prescribed at the whole boundary, this criterion was obtained by Hadamard ([12], equations (VI.18, VI.20); [9]; cf. also [27]), [13, equation (15)], [35, equation (68b.2)], [29, equation (X.4.15)].[8] It is easy to verify that equation (26) is also equivalent to the criterion given by Pearson [27, p. 137, equation (13)], Hill [13, equation (22)], Prager [29, equations (4.18, 4.22), p. 219], Truesdell and Noll [35, equations (68b. 19)] and Goodier and Plass [14, equation (15)].

Substituting for C^a_{ijkl} from equations (19a) and (19b), other forms of criterion (25) may be obtained, e.g.,

$$\int_{(V)} [C^b_{ijkl} + \tfrac{1}{4}(3\delta_{ik}T^0_{jl} - \delta_{il}T^0_{jk} - \delta_{jl}T^0_{ik} - \delta_{jk}T^0_{il})] \times u_{i,j}u_{k,l}dV \geq 0 \tag{27a}$$

$$\int_{(V)} [C^c_{ijkl} + \tfrac{1}{2}(\delta_{ik}T^0_{jl} - \delta_{jl}T^0_{il} - \delta_{il}T^0_{jk} - \delta_{jl}T^0_{ik})] \times u_{i,j}u_{k,l} \geq 0 \tag{27b}$$

Elastic Materials in Finite Strain

In the preceding, only the incremental properties were assumed to be elastic, while the initial state could have been inelastic. Let us now examine the relationship between the definition of an elastic material and the formulation of incremental deformations. The elastic material may be defined by the condition that a certain material stress tensor be a tensorial function of a certain finite strain tensor. If the Piola-Kirchhoff stress tensor of the second kind and the classical Lagrangian strain tensor with respect to the unstressed state are used, the definition of the elastic material may be written as follows [35, 16]:

$$T_{ij} = F_{ir}f_{rs}(E^a_{kl})F_{js}/J_0 \tag{28}$$

where $F_{ir} = \partial x_i/\partial X_r$ = transformation matrix with respect to the stress-free state, X_i and x_i coordinates of a particle in the stress-free state and after deformation, respectively; J_0 = det (F_{ir}), E^a_{kl} = Cauchy-Green deformation tensor [35, 16] = $F_{kr}F_{rl} = \delta_{kr} + \epsilon^a_{kr}$ where ϵ^a_{kr} = classical Lagrangian finite strain tensor as referred to the stress-free state; f_{rs} = given tensorial function.

For incremental deformations, superposed on finite deformations,

$$\Delta T_{ij} = T_{ij} - T^0_{ij} \doteq [\partial T_{ij}/\partial F_{km}]_{F_{rs}}\Delta F_{km} \tag{29}$$

(Note that ΔT_{ij}, defined by equation (13a), is not an objective stress increment.) Substitution of equation (28), rearrangement and omission of higher-order terms [16, equations (21.7–21.8)], [35, equations (68.16–17)][9] leads then to the relationship

$$C^a_{ijkl}e_{kl} \doteq \Delta T_{ij} - T^0_{kj}u_{i,k} - T'_{kj}u_{j,k} + T^0_{ij}u_{k,k} \tag{30}$$

where

$$C^a_{ijkl} = F^0_{ir}F^0_{js}F^0_{kp}F^0_{lq}(\partial f_{rs}/\partial F_{pq})/J_0 \tag{31}$$

It is recognized that expression (30) represents the material stress increment σ^a_{ij} (equation (14b)), from which the Trefftz's equations of neutral equilibrium (equation (2a) or (14a)) may be obtained by application of equation (8).

For other forms of definition of an elastic material [18, 30, 37] which are not based on the Piola-Kirchhoff tensor of the second kind, the derivative $\partial T_{ij}/\partial F_{km}$ needed in equation (29) is given by complex expressions (even if only second-order terms are considered).

However, simple expressions for C_{ijkl} in all formulations are possible if the material is hyperelastic, i.e., is defined by potential energy per unit initial volume. Then, according to the discussion following equation (4),

$$C^a_{ijkl} = \partial^2U/\partial\epsilon^a_{ij}\partial\epsilon^a_{kl}, \quad C^b_{ijkl} = \partial^2U/\partial\epsilon^b_{ij}\partial\epsilon^b_{kl}, \quad C^c_{ijkl} = \partial^2U/\partial\epsilon^c_{ij}\partial\epsilon^c_{kl} \tag{31a}$$

Example of Buckling of an Orthotropic Half Space

To demonstrate the formal differences between various theories, let us consider the problem of surface buckling of an incompressible elastic orthotropic halfspace $x_2 \leq 0$. The discussion will be restricted to the plane-strain problem in the plane (x_1, x_2).

First, the orthotropic two-dimensional stress-strain relationships for infinitesimal incremental deformations must be introduced. In terms of moduli C^a_{ijkl} (equation (14b)), these relationships are

$$\begin{aligned} \sigma^a_{11} &= C^a_{1111}e_{11} + C^a_{1122}e_{22} \\ \sigma^a_{22} &= C^a_{2211}e_{11} + C^a_{2222}e_{22} \\ \sigma^a_{12} &= 2C^a_{1212}e_{12} \end{aligned} \tag{32}$$

Introducing the condition of incompressibility in plane strain, $e_{11} = -e_{22}$, the relationships (32) take the form

$$\sigma^a_{11} - \sigma^a = 2N^ae_{11}, \quad \sigma^a_{22} - \sigma^a = 2N^ae_{22}, \quad \sigma^a_{12} = 2Q^ae_{12} \tag{33}$$

in which $\sigma^a = (\sigma^a_{11} + \sigma^a_{22})/2$ and

$$N^a = \tfrac{1}{4}(C^a_{1111} + C^a_{2222} - C^a_{1122} - C^a_{2211}), \quad Q^a = C^a_{1212} \tag{34}$$

There are thus only two independent elastic constants, Q^a and N^a. For an isotropic material $N^a = Q^a$.

Proceeding in an analogous manner, one can obtain for σ^b_{ij} and σ^c_{ij} equations of the same form as (33) in which

$$N^b = \tfrac{1}{4}(C^b_{1111} + C^b_{2222} - C^b_{1122} - C^b_{2211}), \quad Q^b = C^b_{1212} \tag{34a}$$

$$N^c = \tfrac{1}{4}(C^c_{1111} + C^c_{2222} - C^c_{1122} - C^c_{2211}), \quad Q^c = C^c_{1212} \tag{34b}$$

Because equations (20a) for stress σ_{ij} with nonsymmetric moduli C^d_{ijkl} differ from equations (17c) only by the effect of volume compressibility,

$$N^d = N^c, \quad Q^d = Q^c \tag{35c}$$

Using the relationships (19a, b, c), it may be established that

$$N^d = N^b + \tfrac{1}{4}(T^0_{11} + T^0_{22}), \quad Q^d = Q^b + \tfrac{1}{4}(T^0_{11} + T^0_{22}) \tag{36a}$$

$$N^d = N^a + \tfrac{1}{2}(T^0_{11} + T^0_{22}), \quad Q^d = Q^a + \tfrac{1}{2}(T^0_{11} + T^0_{22}) \tag{36b}$$

terms of moduli N^d, Q^d, and formulation (2d), the exact solution of surface buckling of an orthotropic incompressible half space in plane strain was presented by Biot [5, pp. 204–212]. In his solution the axis of orthotropy is assumed as parallel to the surface of the half space $x_2 \leq 0$ and the initial stress field is considered as uniaxial compression $T^0_{11} = -P(T^0_{22} = T^0_{12} = 0)$.

[8] From (26) it was deduced by Hadamard [35, equation (68b.3)] that in each point of body the inequality

$$(C^a_{ijkl} + \delta_{ik}T_{jl})\lambda_i\lambda_k\mu_j\mu_l \geq 0 \tag{27c}$$

must be satisfied for any two vectors λ_i, μ_i. This is a necessary (but not sufficient) local condition of infinitesimal stability. However, if displacements are prescribed on the whole boundary and the initial stress is homogeneous, equation (27c) is also a sufficient condition, irrespective of the shape of the body [35, equation (68b. 18)]. Equation (27c) may be also given alternate forms if moduli C^b_{ijkl} or C^c_{ijkl} are used.

[9] The procedure leading from (29)–(30) is not given in detail because it may be found in a book by Jaunzemis [16, p. 492].

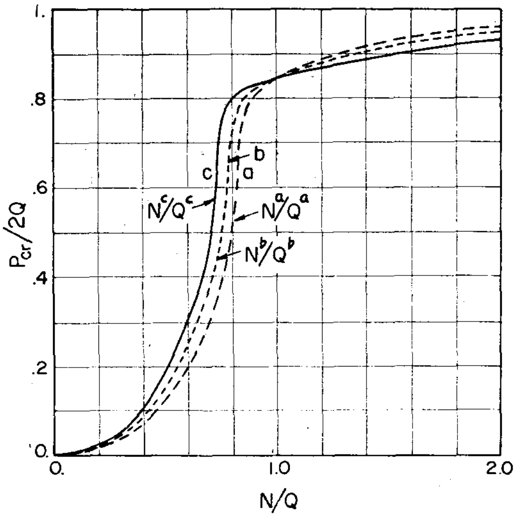

Fig. 1

The resulting dependence of the buckling stress P_{cr} on the ratio N^d/Q^d according to Biot's formula [5, equation (IV.4.41), p. 211] is plotted in Fig. 1 as curve c.

If other equilibrium equations and other incremental moduli are used, the whole analysis need not be performed again, because at each stage of analysis relationships (26a, b) or (19a, b, c) must be satisfied (although the equations may look quite different). To obtain the solution in terms of moduli N^a, Q^a or N^b, Q^b, the relationships (36a, b) may be substituted into the resulting formula for the diagram in Fig. 1 (curve c). This substitution takes on the form

$$N^b/Q^b = (N^d/Q^d + P_{cr}/2Q^d)/(1 + P_{cr}/2Q^d) \quad (37a)$$

$$N^a/Q^a = (2N^d/Q^d + P_{cr}/2Q^d)/(2 + P_{cr}/2Q^d) \quad (37b)$$

Using these relationships, the plots of buckling stress P_{cr} as a function of N^a/Q^a or N^b/Q^b may be obtained from curve c in Fig. 1. They are represented by curves a and b. It is seen that the differences between various formulations of stability theory, i.e., between the various definitions of the incremental moduli, can be substantial. It is found, for instance, that the values of P_{cr} for the case $N^a/Q^a = 0.75$ and the case $N^d/Q^d = 0.75$ are almost in the ratio 1:2. Differences disappear for $N^a = Q^a$, i.e., for an isotropic incompressible half space. It is especially noteworthy that the ratio between the values N^d/Q^d, N^b/Q^b and N^a/Q^a leading to the same P_{cr} approaches 1:1:1 as N^d/Q^d tends to zero (or $P_{cr} \rightarrow 0$), which is the case of a strongly orthotropic medium.

Example of a Column in Flexure and Shear (Timoshenko Beam)

The formal differences between various formulations of stability theory also disappear for thin bodies, such as shells, plates and bars, if the assumption is made that the cross sections (or normals) remain plane and perpendicular to the deflected middle line (or surface). However, differences are encountered if shear is considered, i.e., the cross sections do not remain perpendicular to the deflected middle line. (Shear must be taken into account, e.g., in orthotropic columns or built-up columns.)

As an example, consider now the buckling of a perfect pin-ended orthotropic column of rectangular cross section of area A and moment of inertia I. Let axis x be the longitudinal axis and consider that the column is initially under uniaxial stress T_{11}^0. The longitudinal displacements equal $u_1 = z\psi(x)$ where ψ is the rotation of cross section. The shear angle is $\gamma = \psi + w_{,x}$ where w is the deflection in the sense of $x_3 = z$; $u_3 = w(x)$ and $w_{,x} = dw/dx$. (Equations $u_1 = z\psi(x)$, $u_3 = w(x)$ may be viewed as assumed expressions for the first terms of Kantorovich's direct variational method.) As a starting point, let us use the variational principle (10). Obviously one can write $\int_{(A)} \sigma_{ij}\delta e_{ij} dA = M\delta\psi_{,x} + T\delta\gamma$ where $M = EI\psi_{,x}$ = bending moment; $T = GA\gamma$ = shearing force; A = area of cross section; $I = \int_{(A)} z^2 dA$; E = longitudinal Young's modulus; G = shear modulus for directions x and z (adjusted for the shear correction coefficient for the cross section considered); $\psi_{,x} = d\psi/dx$. All components of T_{ij}^0 are zero except T_{11}^0. Furthermore, $u_{1,1} = e_{11} = z\psi_{,x}$, $u_{3,1} = w_{,x}$, $e_{13} = e_{31} = \frac{1}{2}\gamma$, $u_{1,3} = \psi$, $u_{3,3} = e_{33} = 0$.

If the Green's strain tensor ϵ_{ij}^a given by (14) is used and the relation $\epsilon_{11}^a - e_{11} = \frac{1}{2}(z^2\psi_{,x}^2 + w_{,x}^2)$ is noted, equation (10) with $\tau_{ij}\delta u_{i,j} = \sigma_{ij}\delta e_{ij} + T_{ij}^0\delta(\epsilon_{ij}^a - e_{ij})$ now reads:

$$\int_0^L \left[\int_{(A)} T_{11}^0 \tfrac{1}{2}\delta(z^2\psi_{,x}^2 + w_{,x}^2) dA + G^a A(\psi + w_{,x})(\delta\psi + \delta w_{,x}) + E^a I\psi_{,x}\delta\psi_{,x} \right] dx = 0 \quad (38)$$

where $x = 0$ and $x = L$ are the ends of column. Integrating by parts with respect to $\delta\psi_{,x}$ and noting that $\psi = 0$ at the hinged ends of column, it follows, after rearrangement:

$$\int_0^L [T_{11}^0 A w_{,x}\delta w_{,x} - (T_{11}^0 I\psi_{,x})_{,x}\delta\psi + G^a A(\psi + w_{,x})(\delta\psi + \delta w_{,x}) - (E^a I\psi_{,x})_{,x}\delta\psi] dx = 0 \quad (39)$$

This equation is satisfied for any $\delta\psi$ and $\delta w_{,x}$ if, and only if,

$$-(T_{11}^0 I\psi_{,x} - E^a I\psi_{,x})_{,x} + G^a A(\psi + w_{,x}) = 0,$$
$$T_{11}^0 A w_{,x} + G^a A(\psi + w_{,x}) = 0 \quad (39a)$$

If the second equation is subtracted from the first one and is integrated from 0 to x (noting that $\psi_{,x} = w = 0$ for $x = 0$), equations (39a) are brought to the form

$$E^a I \frac{d\psi}{dx} - Pw + T_{11}^0 I \frac{d\psi}{dx} = 0,$$
$$G^a \left(\psi + \frac{dw}{dx}\right) + T_{11}^0 \frac{dw}{dx} = 0 \quad (40)$$

Solution of these equations satisfying the boundary conditions $w = \psi_{,x} = 0$ at $x = 0$ and $x = L$ may be sought in the form $\psi = A \cos \pi x/L$, $w = B \sin \pi x/L$, with A, B as arbitrary constants. Substitution in (40) yields the characteristic equation

$$T_{11}^{0^2} - T_{11}^0(E^a + G^a + E^a G^a/T_E^a) + E^a G^a = 0 \quad (41)$$

and the smallest critical value of initial stress

$$T_{11}^0 = \tfrac{1}{2} c - \sqrt{\tfrac{1}{4}c^2 - E^a G^a} \quad \text{where } c = E^a + G^a + E^a G^a/T_E^a \quad (42)$$

Here $T_E^a = E^a I\pi^2/(L^2 A)$ = Euler's critical stress for modulus E^a. It is easy to verify that for a very slender column, i.e., $T_E^a/E^a \rightarrow 0$, $T_E^a/G^a \rightarrow 0$, expression (42) has the limit T_E^a.

Solutions for the other formulations of incremental deformation may be obtained by substituting in equations (40) or (41) the relations:

$$E^a = E^b - T_{11}^0, \qquad G^a = G^b - \tfrac{1}{4}T_{11}^0 \quad (43a)$$

$$E^a = E^c - 2T_{11}^0, \qquad G^a = G^c - \tfrac{1}{2}T_{11}^0 \quad (43b)$$

which represent a special case of (19a, b) for uniaxial stress. It can be verified that a direct derivation, similar as above but based on strain tensors ϵ_{ij}^b or ϵ_{ij}^c, yields the same results.

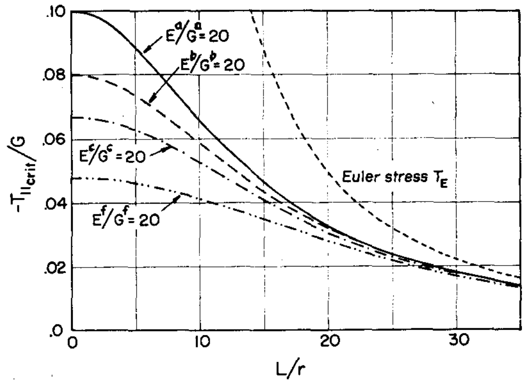

Fig. 2

Thus, for instance, the characteristic equation (41) can take on considerably different forms

$$T_{11}^{0^2}(10 + E^b/T_E^b) - T_{11}^0(5E^b + 8G^b + 4E^bG^b/T_E^b) + 4E^bG^b = 0 \quad (44a)$$

$$T_{11}^{0^2}(9 + E^c/T_E^c) - T_{11}^0(3E^c + 6G^c + 2E^cG^c/T_E^c) + 2E^cG^c = 0 \quad (44b)$$

where

$$T_E^b = E^bI\pi^2/(L^2A), \qquad T_E^c = E^cI\pi^2/(L^2A).$$

It can be verified that for slender columns ($T_{11}^0 \ll E^a$) equations (40) and (42) are identical with the Engesser's equations for buckling with shear [33, p. 133]. These equations were originally derived by consideration of equilibrium from the assumption that the shear force equals $-Pw_{,x}$, which represents the component of initial axial force $P = -T_{11}^0A$ along the cross section which is perpendicular to the deflected middle line.

Alternatively, the problem has also been solved assuming [33, p. 134] that the shear force equals $P\psi$, which represents the component of P along the cross section that was perpendicular to the beam axis in the initial state. From the equilibrium condition the equation analogous to the second of equations (40) is then obtained (putting $G = G^f$) in the form

$$G^f(\psi + w_{,x}) - T_{11}^0\psi \quad (45)$$

It is interesting to note that this formulation can be found to correspond to still another form of finite strain tensor, namely, $\epsilon^f = \epsilon_{ij}^a - 2e_{ki}e_{kj}$, and to the incremental moduli $E^f = E^a + 4T_{11}^0$, $G^f = G^a + T_{11}^0$.

In engineering literature the term $T_{11}^0Id\psi/dx$ in the first of equations (40) has been, as a rule, neglected. This is an admissible approximation for slender columns in which $T_{11}^0 < E^a$ but is exact only for Biot's incremental moduli E^b and G^b.

To illustrate the importance of distinguishing properly between E^a, E^b, etc., consider four different (incrementally orthotropic) hypoelastic materials, for which $E^a/G^a = 20$ or $E^b/G^b = 20$ or $E^c/G^c = 20$ or $E^f/G^f = 20$ at any T_{11}^0. The plots of the smallest critical stress T_{11}^0 versus column slenderness $L/r = L\sqrt{A/I}$ have been computed from equations (42), (44a), (44b), and are shown in Fig. 2.

Conclusions

1 Expression (6) for the incremental mixed (Piola-Kirchhoff) stress tensor (of the first kind) is the basic relationship which provides a unified formulation of the incremental equilibrium equations and enables to determine the relationships between various special forms. In different formulations the incremental material stress tensors are not identical, as is exemplified by

Table 1

Form	*a*	*b*	*c*	*d*	*e*	*f*
Finite strain tensor	ϵ_{ij}^a $\epsilon_{11}^a = e_{11} + \frac{1}{2}e_{11}^2$ equation (14) (Green's, classical Lagrangian)	$\epsilon_{ij}^b = \epsilon_{ij}^a - \frac{1}{2}e_{ki}e_{kj}$ $\epsilon_{11}^b = e_{11}$ equation (15) (pure deformation part of displacement gradient)	$\epsilon_{ij}^c = \epsilon_{ij}^a - e_{ki}e_{kj}$ $\epsilon_{11}^c = e_{11} - \frac{1}{2}e_{11}^2 \approx \ln(1 + \epsilon_{11})$ equation (18a) (logarithmic strain)	$\epsilon_{ij}^d = \epsilon_{ij}^c$ if the material is incompressible, otherwise nonexistent	$\epsilon_{ij}^e = \epsilon_{ij}^a - \frac{3}{2}e_{ki}e_{kj}$ $\epsilon_{11}^e = e_{11} - e_{11}^2$ equation 23	$\epsilon_{ij}^f = \epsilon_{ij}^a - 2e_{ki}e_{kj}$ $\epsilon_{11}^f = e_{11} - \frac{3}{2}e_{11}^2$
Incremental equilibrium equation	equation (2*a*) or (14*a*) (Trefftz)	equation (2*b*) or (16*b*) (Biot)	equation (2*c*) or (17*b*) (Biezeno and Hencky)	equations (2*d*) and (2*e*) (Biot, Neuber, Southwell)		
Objective material stress tensor	equation (14*b*) (Piola-Kirchhoff tensor of the second kind)	equation (16*b*) (Biot's alternative stress)	equation (17*c*)	equation (20*a*) (Biot's incremental stress)		
Incremental moduli if a potential exists	C_{ijkl}^a symmetric	C_{ijkl}^b symmetric	C_{ijkl}^c symmetric	C_{ijkl}^d $C_{ijkl}^d \neq C_{klij}^d$	C_{ijk}^e $C_{ijk}^e \neq C_{klij}^e$	C_{ijk}^f
Objective stress rate	$\hat{T}_{ij}^a$ equation (22*a*) (Truesdell's rate; if incompressible, also Oldroyd's rate)	$\hat{T}_{ij}^b$ equation (22*b*)	$\hat{T}_{ij}^c$ equation (22*c*)	$\hat{T}_{ij}^d$ equation (22*d*) (Jaumann 's corotational rate)	$\hat{T}_{ij}^e$ equation (22*e*) (convected rate of Cotter and Rivlin)	
Stability criterion	equation (26) (Hadamard, Pearson, Hill)	equation (27*a*)	equation (27*b*)			
Definition of elastic material in finite strain	equation (28)					
Buckling of column with shear	equations (40) and (41) (Engesser)	equation (44*a*)	equation (44*b*)			equation (45) (Timoshenko)

equations (14*b*), (16*b*), (17*c*) and (20*a*), and the values of incremental moduli are not the same. Their mutual relationships are given by equations (19*a*, *b*, *c*) or (36*a*, *b*).

2 The arbitrariness of choice between the (infinitely) many possible forms of equilibrium equations for incremental deformations corresponds to the arbitrariness in the choice of (*a*) the finite strain tensor, ((*b*) the material strain tensor, (*c*) the objective stress rate, (*d*) the form of stability criterion, and (*e*) the definition of an elastic material in finite strain. The correspondence between various special forms of these quantities or equations, established in this paper, is summarized in Table 1.

3 The solution procedures of a certain problem according to different formulations must be completely analogous and satisfy at each stage relationships (19*a*, *b*, *c*) (although the equations may look quite different). Using equations (19*a*, *b*, *c*), it is possible to obtain from the solution according to one formulation a solution for any other formulation.

4 All of the formulations of incremental deformations and elastic stability discussed in this paper, are correct and mutually equivalent.[10] But formulations in which τ_{ij} cannot be expressed in form of equation (6) are inadmissible, excepting the case in which the deviation from expression (8) is proportional to $T_{ij}^0 u_{k,k}$. In this case, however, the elastic moduli are unsymmetric if the incremental potential exists and the material is compressible. This inconvenient feature is characteristic for the formulations in columns *d* and *e* in Table 1.

5 Although the choice of a certain formulation is ultimately a matter of convenience, the formulation in column *a* in Table 1, which is associated with the classical Lagrangian (Cauchy-Green) strain tensor and directly follows from the condition of frame indifference, is basically preferable. The other formulations have the disadvantage that the associated finite strain tensors do not possess exact closed expressions, which is impractical in dealing with any material in finite strain. For an elastic material, only the formulation in column *a* in Table 1 admits a simple direct expression of the incremental moduli from the tensorial function f_{rs} defining the material, equation (28).

On the other hand, in certain practical problems [5, 18, 30, 37] other formulations might be more convenient. Biot's equation (2*b*) allows a simple geometrical interpretation of stresses σ_{ij}^b and seems to be practical when dealing with incompressible materials under zero initial shears [5, 37]. This is due to the fact that with the finite strain tensor (15), representing the pure deformation part of the displacement gradient, the nonlinear material effects are separated from the nonlinear geometrical effects. For instance, only for this tensor the work done at any infinitesimal strain increments e_{ij}, less the work expressed as a quadratic form in e_{ij}, equals the work of T_{ij}^0 expressed as if T_{ij}^0 represented forces which are held constant during the incremental deformation.

6 When the achievable initial stresses are not small with respect to the incremental moduli, the predicted buckling stresses for the various well-known formulations can differ substantially if the proper distinction between the various definitions of the incremental moduli is not made; e.g., in the problem of surface buckling, Fig. 1, the differences in buckling stress almost reach the ratio 1:2. This is important for highly deformable rubber-like materials, or lightly strain-hardening metals in the plastic range. On the other hand, in thin structures the achievable initial stresses T_{ij}^0 are so small that the differences in predicted buckling loads are negligible. Undoubtedly, it is just this practical fact which has clouded the fine points of the subject.

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[10] Recent rejection of some of the older formulations of stability theory has not been justified as far as the final forms of relationships are concerned although the concepts from which some of the older theories have been derived might have been dubious.

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