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DIRICHLET SERIES CREEP FUNCTION FOR AGING CONCRETE

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INTRODUCTION

Some modern concrete structures, such as prestressed concrete reactor vessels, deep submergence shells, and large span bridges and shells, are rather sensitive to creep effects and the accuracy of their prediction is of great importance. Creep analysis of concrete structures in the working stress range is, fortunately, facilitated by the fact that concrete may be approximately regarded as a time-variable linearly visco-elastic material, provided that no large strain reversals, no cyclic strains, and no significant changes in water content and temperature occur. However, even under these restrictions, which will be supposed throughout this paper, the analysis is complicated by the variation of material properties with the progress of cement hydration, a phenomenon called aging or maturing. As is well known, the presence of aging admits only step-by-step time integration methods, unless severe distortions of the measured creep function are committed [as in the rate of creep method, effective modulus method, Arutyunian-Maslov's creep function, etc. (1,3,19)]. In the usual step-by-step procedure, based on approximation of the hereditary integrals with finite sums, the stress values from all previous time steps must be stored, in addition to the current values. Considering that about 100 time steps are needed in a typical problem, the storage requirements thus become about 100-fold of those in the corresponding elastic problem. When the number of degrees-of-freedom is large, as is typical in finite element models, storage of the previous history on tapes is thus unavoidable. Then, however, the computing time becomes unacceptably long because lengthy sums over all previous stress values stored on tapes must be evaluated in each time step. Such difficulties have been experienced, e.g., by Cederberg and David (10) and Rashid (23).

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It is thus imperative to find a way of characterizing the complete stress history with only a few parameters. This can be achieved by the rate-type formulation of the creep law (also called the equation of state) which can be based either on the Kelvin (or Kelvin-Voigt) chain model or the Maxwell chain model. In this paper chiefly the former one, which is more closely related to creep test data, will be considered, while the latter one is left to a subsequent paper. In application to concrete, the properties of the model must be, of course, regarded as age-dependent, which makes classical viscoelasticity inapplicable. The general form of the stress-strain relations taking age-dependency into account was given in Ref. 2.

One problem in using the rate-type creep formulation stems from the great breadth of the retardation spectrum of concrete, which is indicated by the fact that the slope of the creep curve in the logarithmic time scale is significant for elapsed times ranging from seconds to decades of years. This property requires that the time step by gradually increased if the long-time (e.g., 30-yr) response under permanent load should be reached with an acceptable number of steps. However, the usual step-by-step algorithms, such as the Euler method and the Runge-Kutta or predictor-corrector methods, do not allow the size of the time step to exceed the shortest retardation time of the material without causing numerical instability. A numerically stable algorithm of creep analysis which permits arbitrary increase of the time step has recently been presented, for the case of a Kelvin chain, in Ref. 8. Another algorithm applicable directly to the expansion of creep function into a series of exponentials has been developed in Ref. 6. (see also Appendix I).

Another problem in using the rate-type creep formulation consists in identification of the material parameters from the available test data. This problem has recently been studied from a very theoretical point of view as a nonlinear optimization problem in control theory (12) (for an excellent survey of this approach see Ref. 21). In the writers' opinion, however, this type of approach, though mathematically elegant, is unnecessarily sophisticated in view of the great scatter of test data, which calls for the simplest possible approach to allow good insight into the problem.

To date, no identification of a rate-type creep law from the extensive data on concrete creep has yet been carried out, to the best of the writers' knowledge. Solution of this problem is chosen as the main objective of this paper. A general, but simple and straightforward, method will be presented for this purpose. Numerical results for the best creep data available will also be given.

LINEAR CREEP LAW OF CONCRETE

The dependence of strain on the histories of water content and temperature (8) is so complex that it must be left aside in this paper. Attention will thus be restricted to concrete at nearly constant water content and temperature, in which case the strain is a functional of stress history alone. As long as no abrupt changes in the microstructure occur, this functional must be continuous. Then the functional admits a generalized Taylor series expansion, whose linear term, if present, must be a sufficiently good approximation of the material behavior for sufficiently small stresses and sufficiently short time histories. Practically, the linear creep law has been found acceptable in the working stress range provided the strain (not stress) reversals and cyclic strains are excluded. For this range, in fact, no better creep law than a linear one is known today. The linearity implies validity of the principle of superposition, whose application yields the uniaxial creep law in the form

$$\epsilon(t) - \epsilon^0(t) = \int_0^t J(t, t')d\sigma(t') \text{ (Stieltjes integral)} \quad \ldots\ldots\ldots\ldots (1)$$

in which t = time from casting of concrete; σ = stress; ϵ = strain; $J(t, t')$ = creep function (or creep compliance, creep kernel) = strain in time t caused by a constant unit stress applied at time t'; and ϵ^0 = stress-independent inelastic strain comprising shrinkage and thermal dilatation.

It must be admitted that in the case of strain reversals caused by unloading, the stress-strain laws of the well-known rate-of-creep method due to Glanville (called Dischinger's law in some countries) and, especially, the rate-of-flow method of England and Illston (19) are somewhat less inaccurate than Eq. 1. However, since these creep laws are also linear, they obey the principle of superposition as well. But in these laws the superposition is not based on the actually measured creep function but on a certain distorted creep function, so that prediction of creep effects for other types of stress histories cannot be satisfactory. Correctly, any deviation from the principle of superposition must be treated in terms of a nonlinear stress-strain law.

In the case of multiaxial stress, the isotropic linear stress-strain law consists of two independent relations of the same form as Eq. 1 for the volumetric and the deviatoric components of the stress and strain tensor. The Poisson's ratio, ν, at constant stress is, in general, also a function of t and t' but test results are inconclusive at present. Anyhow, ν varies only little (at maximum between 0.15 and 0.25) and may approximately be assumed as constant, $\nu = 0.18$. Then, in analogy with the elastic relations $G^{-1} = 2(1 + \nu)E^{-1}$ and $K^{-1} = 6(1/2 - \nu)E^{-1}$ for shear modulus G and bulk modulus K, the deviatoric and volumetric creep functions may be expressed, because of isotropy of concrete, as

$$J^D(t, t') = 2(1 + \nu)J(t, t'), \quad J^V(t, t') = 6\left(\frac{1}{2} - \nu\right)J(t, t') \quad \ldots\ldots (2)$$

In view of these simple relations, only the uniaxial creep function $J(t, t')$ will be considered in the sequel.

DIRICHLET SERIES CREEP FUNCTION AND ITS IDENTIFICATION FROM DATA

As will be seen, one way of converting creep law given by Eq. 1 into a rate-type form consists in approximation of the creep function, $J(t, t')$, by a series of real exponentials

$$J(t, t') = \frac{1}{E(t')} + \sum_{\mu=1}^{n} \frac{1}{\hat{E}_\mu(t')}\left(1 - e^{-(t-t')/\tau_\mu}\right) \quad \ldots\ldots\ldots\ldots (3)$$

in which τ_μ are constants which will be called retardation times; $\hat{E}_\mu$ are coefficients which depend only on t' and have the dimension of elastic moduli; and $E(t') = 1/J(t', t')$ = instantaneous elastic modulus. Eq. 3 represents the

Dirichlet series (16,25). (Its standard form is $\sum_\mu C_\mu e^{-x/\tau_\mu}$, where x = variable; C_μ = constants, but this is obviously equivalent to Eq. 3. When $\tau_\mu - \tau_{\mu-1}$ = constant, the series is also called Prony series.)

Note that Eq. 3 is a special case of the so-called degenerate kernel of integral equation, Eq. 1, i.e., a kernel of the type $J(t, t') = \sum_\mu A_\mu(t)B_\mu(t')$, in which A_μ and B_μ are functions of one variable.

For fitting of given creep data by Eq. 3 it is most convenient to select first a number of t'-values, i.e., t'_α, $\alpha = 1, 2, 3, \ldots$, and fit $J(t, t')$ as a function of t alone for each fixed t' separately. To achieve good accuracy, the t' values considered should be distributed uniformly in log t'-scale, about one value in each in each decade (order of magnitude).

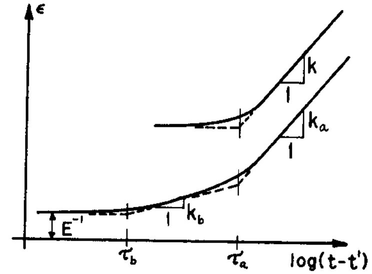

FIG. 1.—APPROXIMATION OF CREEP CURVES BY ONE OR TWO STRAIGHT-LINE SEGMENTS IN LOG-TIME

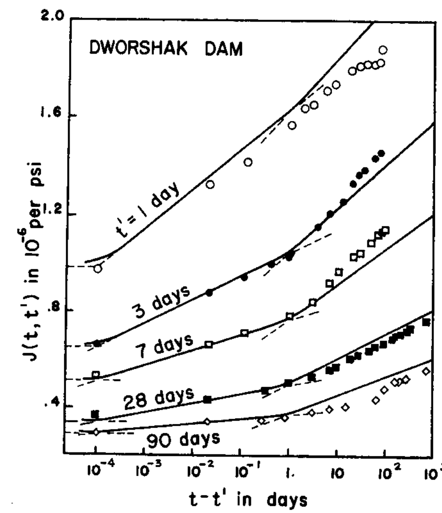

FIG. 2.—FIT OF CREEP DATA OF PIRTZ, DWORSHAK DAM, AS GIVEN IN TABLE 1

Fitting of curves by Dirichlet series is not as simple as for their complex counterpart, the Fourier series. It has been found (17) that the solution is unique and not oversensitive to the inaccuracy of data (i.e., unstable function of J) only if the τ_μ-values are specified and the choice of τ_μ is such that the difference between any two τ_μ-values is not too small. On the other hand, for good accuracy these differences may not be too large and with close enough distribution of the τ_μ-values any creep curve can be fitted with any desired accuracy (excepting the curves whose slope in log $(t - t')$-scale is locally too large, which probably cannot occur for a real material, and the case of stationary creep rate which requires addition of a linear term). Practically, the values, $\tau_1, \ldots, \tau_m$, are best chosen as evenly distributed in log τ_μ-scale. The accuracy of fit is sufficient when the τ_μ-values are distributed by decades, i.e.:

$$\tau_\mu = 10^{\mu-1}\,\tau_1 \;(\mu = 1, \ldots, n) \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (4)$$

Here τ_1 should roughly coincide with the point where the creep curve plotted in the log $(t - t')$-scale begins to rise, and τ_n with the longest elapsed time $t - t'$ to be considered in the analysis, or with the point where the creep curve in log $(t - t')$-scale levels off (if at all), whichever is smaller. (In comparison with the usual inaccuracy of data, it is even sufficient to use $\tau_\mu = 60^{\mu-1}\,\tau_1$, which leads to about 1.5 times fewer τ_μ - s for the same time range to be covered.)

The fit of the creep curve by Eq. 3 can be best obtained by the method of least squares. It is usually more convenient to impose the least square condition on point values rather than continuous curves. A series of values $(t - t')_\beta$, $\beta = 1, 2, 3, \ldots$, uniformly distributed in log $(t - t')$-scale, is chosen. Four values per decade, $(t - t')_\beta = 10^{1/4}\,(t - t')_{\beta-1}$, are a suitable choice. Usually $J(t, t')$ is given by measured values which are not spaced in this manner, and so interpolation must be used to determine the values in the desired times $(t - t')$. Denoting by $\tilde{J}$ the given data to be fitted and by J the expression in Eq. 3, and putting $t_\beta = t' + (t - t')_\beta$, the values $\hat{E}_1^{-1}, \hat{E}_2^{-1}, \ldots$ for one chosen t' have to be determined by minimizing the expression

$$\Phi = \sum_\beta\,[J(t_\beta, t') - \tilde{J}(t_\beta, t')]^2 + \sum_\mu \Big[w_1(\hat{E}_{\mu+1}^{-1} - \hat{E}_\mu^{-1})^2$$

TABLE 1.—EXPRESSIONS FITTING DATA IN FIGS. 2-9 (VALID ONLY WITHIN RANGE OF DATA POINTS IN FIGURES)

Figure number (1)	τ_a, in days (2)	Expression for $J(t - t', t')$ in 10^{-6} per psi, with notation $F_\mu = 1 - \exp[-(t - t')/(10^\mu\, 5.63\, \tau_a)]$ (3)	Reference number (4)
2,9	1	$0.208 + 0.93t'^{-0.5} - 0.16t'^{-0.46} + \sum_{\mu=-3}^{-1}\left(0.002 + 0.16t'^{-0.46}\right)F_\mu + \sum_{\mu=0}^{\infty}\left(0.003 + 0.23t'^{-0.25}\right)F_\mu$	Pirtz—Dworshak Dam (20)
3	3	$0.14 + 0.69t'^{-0.50} + \sum_{\mu=0}^{\infty}\left(0.025 + 0.175t'^{-0.25}\right)F_\mu$	Ross Dam (14,15)
4	1	$0.16 + 0.35t'^{-0.33} + \sum_{\mu=0}^{\infty}\left(0.005 + 0.175t'^{-0.33}\right)F_\mu$	Canyon Ferry Dam (14,15,18)
5	1	$0.19 + 1.63t'^{-0.75} + \sum_{\mu=0}^{\infty}\left(0.0025 + 0.25t'^{-0.33}\right)F_\mu$	Shasta Dam (14,15)
6	1	$0.20 + 0.45t'^{-0.50} + \sum_{\mu=0}^{\infty}\left(0.0075 + 0.233t'^{-0.33}\right)F_\mu$	Browne—Wylfa Vessel (9)
7	1	$0.232 + 0.68t'^{-0.75} - 0.066t'^{-0.33} + \sum_{\mu=-4}^{\infty}\left(0.004 + 0.066t'^{-0.33}\right)F_\mu$	Gamble and Thomass (13)
8	10	$0.515 + \sum_{\mu=0}^{\infty} 0.126\,F_\mu$ for $t' = 28$ days	Troxell, Raphael, and Davis (28)

TABLE 2.—INFORMATION ON

Figure number (1)	Temperature, in degrees Fahrenheit (2)	Humidity (3)	28 day cylinder strength, in pounds per square inch (4)	Stress-strength ratio (5)	Water-cement ratio (6)
2	70	sealed	2,080	0.21-0.385	0.56
3	~70	sealed	4,970	0.33	0.56
4	~70	sealed	2,920	0.33	0.50
5	~70	sealed	3,230	<0.33	0.58
6[a]	68	sealed	6,600	0.36	0.42
7	75	94 %	4,900	0.42-0.46	0.70?
8	70	in fog, in water	2,500	0.33	0.59

[a] Data points in Fig. 6 were obtained by averaging the creep curves presented in

$$+ w_2(\hat{E}_{\mu+2}^{-1} - 2\hat{E}_{\mu+1}^{-1} + \hat{E}_\mu^{-1})^2 + w_3(\hat{E}_{\mu+3}^{-1} - 3\hat{E}_{\mu+2}^{-1} + 3\hat{E}_{\mu+1}^{-1} - \hat{E}_\mu^{-1})^2\Big] \quad (5)$$

The second sum represents a penalty term which is added as a smoothing device. It contains squares of the first, second, and third differences between $\hat{E}_1^{-1}, \hat{E}_2^{-1}, \ldots$ and causes the resulting $\hat{E}_\mu^{-1}$ to vary smoothly with μ and also reduces the sensitivity of $\hat{E}_\mu^{-1}$ values to the scatter of data. Suitable values for the weights, w_1, w_2, w_3, can be assumed only by computing experience. The lowest values for which sufficient smoothness is achieved should be used. (If the J-data have been smoothed in advance or the scatter of measurements was unusually low, the smoothing term can sometimes be left out, i.e., $w_1 = w_2 = w_3 = 0$.) The minimization conditions, $\partial\Phi/\partial(\hat{E}_\mu^{-1}) = 0$ $(\mu = 1, 2, \ldots, n)$ and $\partial\Phi/\partial(E^{-1}) = 0$, yield a system of $(n + 1)$ linear algebraic equations for E^{-1} and $\hat{E}_\mu^{-1}$.

The procedure as just described has been programmed in FORTRAN IV and has worked satisfactorily. (The present method of fitting of curves by Dirichlet series has been found more efficient than the alternative methods proposed in Refs. 11, 24, and 29).

The least square fitting procedure is applicable for any shape of creep curves. However, for most data on creep of sealed concrete the number of independent parameters can be further reduced noting that most creep curves can be fitted quite well by two straight line segments in log $(t - t')$-scale with a relatively short transition curve between them. To take advantage of this fact the Dirichlet series approximation to the straight-line creep curve (Fig. 1) is needed first, thus

$$\epsilon = E^{-1} + k\,[\log\,(t - t') - \log\,\tau_a] \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (6)$$

It can be verified that for $t - t' \geq 0.22\,\tau_0$ the foregoing straight-line is approximated, with an error of less than $\pm$ 0.018 k, by the Dirichlet series

$$E^{-1} + \sum_{\mu=0}^{\infty} k\left[1 - e^{-(t-t')/\tau_\mu}\right] \quad \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \quad (7)$$

DATA USED IN ANALYSIS

Aggregate cement ratio (7)	Maximum aggregate size, in inches (8)	Type of cement (9)	Type of aggregate (10)	Size of cylinder, in inches (11)
6.8	1.5	II	granite-gneiss	6 × 18
—	1.5	II	—	6 × 16
—	1.5	II	—	6 × 16
—	1.5	IV	—	6 × 26
4.4	1.5	II	foraminiferal limestone	6 × 12
5.1	.19	I	crushed greywacke	4 × 10 and other
5.67	1.5	I	granite	4 × 14

Ref. 9.

in which $\tau_\mu = 10^\mu(5.63\ \tau_a)$. It is noteworthy that the coefficients of all exponential terms in this series are equal to k. To the left of point 0.22 τ_0, the foregoing series rapidly approaches the horizontal line $\epsilon = E^{-1}$ (see Fig. 1), i.e., the elastic compliance. If the series in Eq. 7 is truncated at $\mu = n$, the fit of the straight line, given by Eq. 6, is good only up to time τ_{n-1}. But it can be made good up to time 1.5 τ_n, by increasing the coefficient of the last term ($\mu = n$) as

$$E^{-1} + \sum_{\mu=0}^{n-1} k\left[1 - e^{-(t-t')/\tau_\mu}\right] + 1.2\ k\left[1 - e^{-(t-t')/\tau_n}\right] \ldots\ldots. \quad (8)$$

Superimposing two expressions of form of Eq. 7, a creep curve given by two straight line segments of slopes, k_a, k_b (see Fig. 1), can be found. It has a short smooth transition in the vicinity of the intersection point, τ_a, and it can be closely approximated, for $t - t' \geq 0.22\ \tau_{-m}$ ($\mu = -m$), by the Dirichlet series

$$E^{-1} + \sum_{\mu=-m}^{-1} k_a\left[1 - e^{-(t-t')/\tau_\mu}\right] + \sum_{\mu=0}^{\infty} k_b\left[1 - e^{-(t-t')/\tau_\mu}\right] \ \ldots \quad (9)$$

in which $\tau_\mu = 10^\mu(5.63\ \tau_a)$ and $E^{-1} = E_a^{-1} - mk_a$ = instantaneous elastic compliance; m = number of retardation times to the left of point τ_a; and E_a^{-1} = ordinate ϵ of the intersection point at τ_a. To the left of point 0.22 τ_m, this expression rapidly approaches the horizontal line, $\epsilon = E^{-1}$ (Fig. 1). The slopes, k_a, k_b, as well as E^{-1} and the intersection point, τ_a, can be obtained from creep data quite easily, even in a graphical manner. (τ_a is defined in Fig. 1.)

The latter procedure just described has been used to fit various creep data available in the literature (9,13,14,15,18,20,28). Most of the data had to be replotted in the logarithmic time scale. This was done from the numerical table of data whenever possible (14,20). (Unfortunately, most authors in the past

have been showing fits in the actual time scale. Such fits succeed easily and look satisfactory for almost any theory because both the short and the long

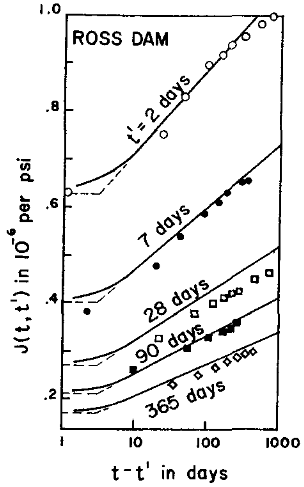

FIG. 3.—FIT OF CREEP DATA, ROSS DAM, AS GIVEN IN TABLE 1

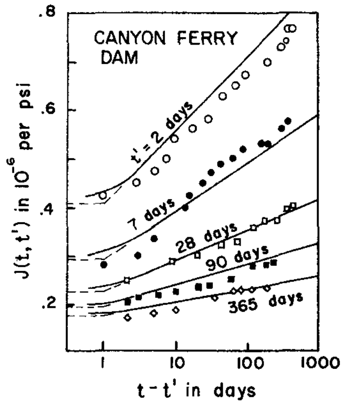

FIG. 4.—FIT OF CREEP DATA, CANYON FERRY DAM, AS GIVEN IN TABLE 1

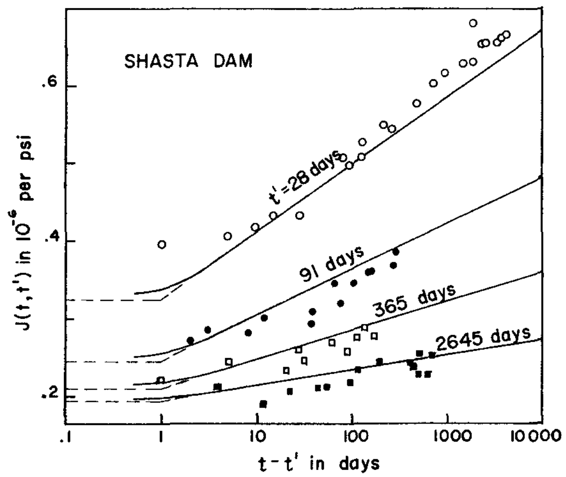

FIG. 5.—FIT OF CREEP DATA, SHASTA DAM, AS GIVEN IN TABLE 1

elapsed times are obscured.) Only the tests at constant (room) temperature and constant (or almost constant) water content have been considered. All of the tests of creep of drying specimens have been left out because of the nonuniform state of stress in such specimens, which requires a much more sophisticated theory (whose development is presently in progress at Northwestern University). Furthermore, only the data which include widely different ages at loading t' could have been used. To assure that the problem of optimum fit has a unique solution, there should be at least one t' for each order of magnitude over the whole time range considered. But unfortunately, even in the best data available the information on the effect of t' is incomplete in this regard.

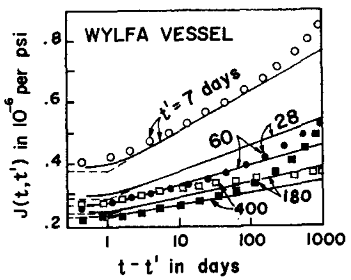

FIG. 6.—FIT OF CREEP DATA, WYLFA VESSEL, AS GIVEN IN TABLE 1

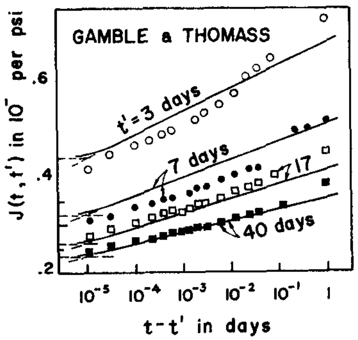

FIG. 7.—FIT OF CREEP DATA, GAMBLE AND THOMASS, AS GIVEN IN TABLE 1

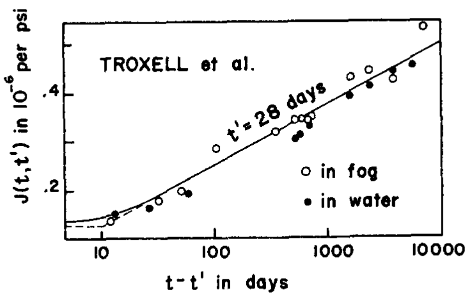

FIG. 8.—FIT OF CREEP DATA, TROXELL, RAPHAEL, AND DAVIS, AS GIVEN IN TABLE 1

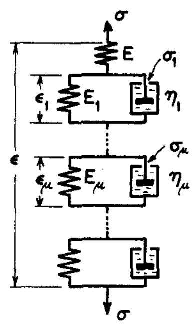

FIG. 9.—KELVIN CHAIN MODEL WITH AGE-DEPENDENT PROPERTIES

For this reason, and also because the dependence of the parameters, k_a, k_b, E^{-1}, upon t' showed enough scatter that called for further smoothing, it has been assumed that k_a, k_b, and E^{-1} all depend on t' as

$$a + bt'^{-n} \quad \text{(10)}$$

in which a, b, and n are constants. Several other expressions were also tried

but Eq. 10 gave the best fits among the simple ones. The value of τ_a was considered as independent of t'.

The fits achieved are shown in Figs. 2-8 by solid lines and the fitting expressions are listed in Table 1. Further information on the data analyzed is compiled in Table 2. Note that the individual creep curves could have been fitted more closely in Fig. 2, e.g., if no smoothing with respect to t' has been done. Also, in a few instances the fit could have been somewhat closer with a negative value of a in Eq. 10. But this was not admitted because it would imply a negative value of the corresponding parameter at $t' \to \infty$ which is impossible. The fit in Fig. 6 is seen to be poor but examination of the data reveals that this is due to unusually large scatter with respect to t'. The set of the data analyzed in this study pertains to a very wide range of elapsed times $t - t'$, from 10^{-5} day (~ 1 sec, Fig. 7) to 23 yr. The 23-yr data (Fig. 8), though available only for a single age, t', is shown here to emphasize that the creep curve appears straight in log $(t - t')$-scale even for this duration. The data in Fig. 7 are shown only up to the 1-day time lag because for longer times the mildly drying environment (94 %) used in these tests probably distorted the creep curves. The data points in Fig. 6 represent the averages of several measurements (quite scattered).

It is also of interest that the expressions for $J(t, t')$ listed in Table 1 are not of the type which is presently recommended by ACI (American Concrete Institute) Committee 209 (22) as well as CEB (19), namely $\phi_u(t')f(t - t')$ where ϕ_u and f are functions of one variable. Using this simple form of dependence upon t' the fits of data would have been poorer, especially in Fig. 2.

If relaxation data for various initial ages are given instead of creep data, the methods presented can also be applied since relaxation data can be converted to creep data by a well-known numerical procedure (7).

Formulation of creep function in terms of Dirichlet series greatly facilitates the numerical step-by-step analysis of large structures, as is demonstrated in Appendix I.

RELATION BETWEEN DIRICHLET SERIES CREEP FUNCTION AND KELVIN CHAIN

Assuming the Kelvin chain model (Fig. 9) with age-dependent properties, the relation between stress σ and strain ϵ (in the uniaxial case) has the form

$$\dot{\epsilon} - \dot{\epsilon}^0 = \frac{\dot{\sigma}}{E} + \sum_{\mu} \dot{\epsilon}_\mu \quad \text{(11)}$$

$$\dot{\epsilon}_\mu = \frac{\sigma_\mu}{\eta_\mu} \quad \text{(12a)}$$

$$\sigma - \sigma_\mu = E_\mu \epsilon_\mu \quad (\mu = 1, \ldots, n) \quad \text{(12b)}$$

in which E = instantaneous elastic modulus and E_μ, η_μ = elastic modulus and viscosity of the μth Kelvin unit, which all depend on time t; $\epsilon^0 = \epsilon^0(t)$ = prescribed stress independent inelastic strain, such as thermal dilatation and shrinkage; ϵ_μ = strain in the μth Kelvin unit, called hidden strain (or partial strain); and σ_μ = stress in the μth dashpot. [It is important to realize that the relation for the μth spring cannot be replaced by the relation, $\sigma - \sigma_\mu = E_\mu \epsilon_\mu$, if E_μ increases with age, as has been pointed out in Ref. 2 and justified in Ref. 5 by the fact that hydration is adding solid matter to the existing solid microstructure in an unstressed (rather than stressed) state.]

Expressing σ_μ from Eq. 12*a* and substituting it into Eq. 12*b*, a single equation for ϵ_μ is obtained which reads

$$\ddot{\epsilon}_\mu + \frac{\dot{\epsilon}_\mu}{\tau_\mu} = \frac{\dot{\sigma}}{\eta_\mu} \quad (\mu = 1, 2, \ldots, n) \quad (13)$$

in which $$\frac{1}{\tau_\mu} = \frac{E_\mu + \dot{\eta}_\mu}{\eta_\mu} \quad (14)$$

Assume now that all τ_μ - s are constants and integrate the equation for ϵ_μ taking $\sigma(t)$ as a step function, i.e., $\sigma = 0$ for $t < t'$ and $\sigma = 1$ for $t > t'$. The initial conditions at $t = t'$ are $\epsilon_\mu = 0$ and $\dot{\epsilon}_\mu = 1/\eta_\mu(t')$. It can be verified by back substitution that the integrals of Eq. 13 are

$$\dot{\epsilon}_\mu(t) = \eta_\mu^{-1}(t')\, e^{-(t-t')/\tau_\mu}; \quad \epsilon_\mu(t) = \tau_\mu \eta_\mu^{-1}(t') \left[1 - e^{-(t-t')/\tau_\mu}\right] \quad (15)$$

The corresponding creep function, $J(t, t')$, is $E^{-1} + \sum_\mu \epsilon_\mu$ which is obviously identical with the Dirichlet series approximation (Eq. 3) if, and only if, $\tau_\mu/\eta_\mu(t') = 1/\hat{E}_\mu(t')$. Then, according to Eq. 14

$$\eta_\mu(t) = \tau_\mu \hat{E}_\mu(t); \quad E_\mu(t) = \hat{E}_\mu(t) - \tau_\mu \frac{d\hat{E}_\mu(t)}{dt} \quad (\mu = 1, 2, \ldots, n) \quad (16)$$

This leads to the conclusion that any creep function in the form of Dirichlet series can be represented by a Kelvin chain with age-dependent properties. Thus, the assumption of constancy of expression in Eq. 14, the key idea in the foregoing considerations, is not overly restrictive. Furthermore, any creep function can be represented by a Kelvin chain with any desired accuracy because this has been found true for Dirichlet series creep function. [This conclusion and Eq. 16 has already been reported in 1970 in unpublished class notes. For the Arutyunian-Maslov's creep function (1), which is a special case of Eq. 3 for $n = 1$, this conclusion has been reached already in 1966 (2).]

Substituting for $\hat{E}_\mu(t)$ in Eqs. 16 the coefficients of the expressions listed in Table 1, the Kelvin chain model fitting any of the data in Figs. 1-8 may be readily obtained. The numerical step-by-step algorithm of structural analysis for this model has been presented in Ref. 8 (Eqs. 70-79).

Knowledge of the rate-type approximation of the creep law, such as Eqs. 13 and 14, is useful not only for the numerical structural analysis but also for further refinement and generalization of the creep law on the basis of a model of the physical processes in the microstructure of the material (8). These processes consist of various kinds of diffusions or thermally activated rate processes, each of which is mathematically formulated by a certain first-order differential equation in time for some kind of microstrains and microstresses. Each of these equations may be associated with one of the first-order differential equations in the stress-strain law, such as Eqs. 13. A correlation of this sort would be of considerable interest because it would allow generalization to cases of variable temperature and water content.

There are some plausible restrictions on E_μ and η_μ when a physical interpretation is considered. For instance, none of the elastic moduli, E_μ, or viscosities, η_μ, may then be negative. This condition has been checked by evaluating E_μ and η_μ from Eq. 13 for the expressions in Table 1 and, unfortunately, it was found that the E_μ indeed become negative for small t and large τ_μ, such as $\tau_\mu >$ about 1,000 days. [For the special case of Eq. 3 with $n = 1$, i.e., Arutyunian-Maslov's function, this observation was already made earlier (2).] Although this fact is of no consequence for the use of stress-strain relations Eqs. 11 and 12 for step-by-step creep analysis, it precludes physical interpretation of the hidden strains, ϵ_μ, and stresses σ_μ as microstrains and microstresses. Thus, for this sort of interpretation other alternatives must be sought.

OTHER REPRESENTATIONS OF CONCRETE CREEP BY KELVIN CHAIN

In view of the conclusion just reached, it is of interest to consider whether some other Kelvin chain models are possible which would avoid negativeness of E_μ, η_μ. For example, Eqs. 11 and 12 for the Kelvin chain may be also easily integrated if it is assumed that

$$\frac{E_\mu}{\eta_\mu} = \text{constant} = \frac{1}{\tau'_\mu} \quad (17)$$

Elimination of σ_μ from Eq. 12 then yields

$$\ddot{\epsilon}_\mu + \left(\frac{1}{\tau'_\mu} + \frac{\dot{\eta}_\mu}{\eta_\mu}\right) \dot{\epsilon}_\mu = \frac{\dot{\sigma}}{\eta_\mu} \quad (18)$$

Considering σ as a step function of time t, i.e., $\sigma = 0$ for $t < t'$ and $\sigma = 1$ for $t > t'$, the initial condition is $\dot{\epsilon}_\mu(t') = 1/\eta_\mu(t)$ and the integral of Eq. 18 is $\dot{\epsilon}_\mu(t) = [1/\eta_\mu(t)]\, e^{-(t-t')/\tau'_\mu}$ as can be verified by back substitution in Eq. 18. Thus

$$\dot{\epsilon}(t) = \sum_\mu \frac{1}{\eta_\mu(t)}\, e^{-(t-t')/\tau'_\mu} \quad (19)$$

Further integration, however, would not yield a closed expression for $J(t, t')$. It is thus better to work with the creep rate, $\dot{\epsilon}$. It is seen that Eq. 19 for $\dot{\epsilon}$ is again a Dirichlet series. But its coefficients are now functions of t rather than t' and so the curves which need to be expanded in Dirichlet series are the curves of $\dot{\epsilon}$ versus $t - t'$ at constant t rather than constant t'. Such curves represent the history preceding fixed t rather than the history which follows fixed t'. They have been constructed from the creep data in Figs. 2-6. To this end the $J(t, t')$ curve for every fixed t' was smoothed graphically (by hand) and then characterized by the J-values at the points of slope change. From these the slope in log $(t - t')$ was calculated, from which $\dot{\epsilon} = [dJ/d \log (t - t')][(t - t') \ln 10]^{-1}$ was obtained. For determination of $\dot{\epsilon}$ for t'-values other than those in the given data, linear interpolation in log t' was used. (All this has been done with a computer, of course.) In this manner the creep data in Fig. 2 have been converted to data on $\dot{\epsilon}$, which are graphically represented in

Fig. 10 in the coordinates log $(t - t')$ and log $\dot{\epsilon}$.

Having the plot shown in Fig. 10 (or its equivalent in terms of a table of values), the expansion (Eq. 19) in Dirichlet series may be obtained by the least square method already described. There is, however, one important modification to be mentioned. The creep rate, $\dot{\epsilon}$, is decreasing with $t - t'$ about exponentially and if a uniform weight were applied on the square of the error, the error in $\dot{\epsilon}$ for high $t - t'$ would surpass the $\dot{\epsilon}$ value. Requiring that the error should appear about uniform in the graph of log $\dot{\epsilon}$ in Fig. 9, the error in $\dot{\epsilon}$ in the least square condition must be multiplied by a weight equal to $(t - t')$. With these weights the program which has been set up for the expansion in Dirichlet series worked satisfactorily.

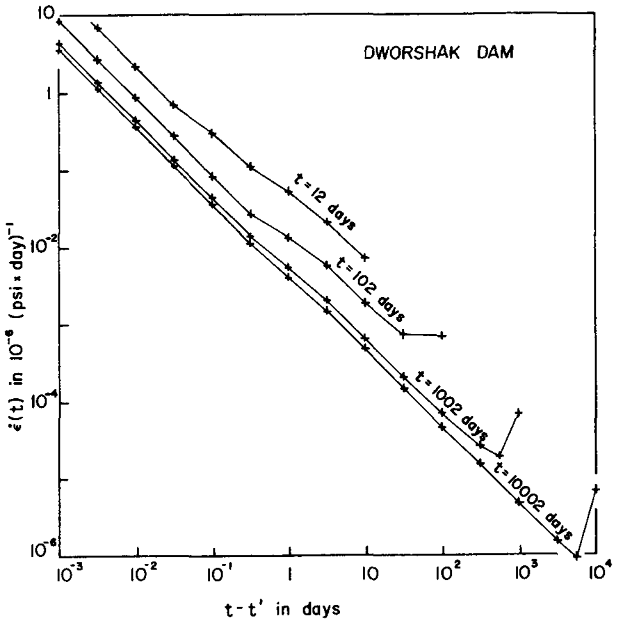

FIG. 10.—TYPICAL CREEP DATA PLOTTED IN TERMS OF CREEP RATE VERSUS LOGARITHM OF ELAPSED TIME AT FIXED CURRENT TIME

It has been found, however, that some of the values of η_μ and $E_\mu = \eta_\mu/\tau_\mu$ are negative for some periods of time. Although this fact is irrelevant for the numerical step-by-step analysis using this model, it precludes again giving any physical interpretation to the hidden strains, ϵ_μ, and stresses σ_μ, as with the model considered previously. The negativeness is, roughly speaking, caused by the fact that in Fig. 10 at least some $\dot{\epsilon}$-curves rise near the end, as has been found for the test data studied.

It has been examined how large an error is committed when the rising end portion of the $\dot{\epsilon}$-curves in Fig. 10 is neglected to make all of the viscosities η_μ positive. But no satisfactory fit of creep curves was obtained in this case.

Other Kelvin chain models may be obtained by choosing some other algebraic relation between E_μ, η_μ, and $\dot{\eta}_\mu$. One tempting choice is $\dot{\eta}_\mu = 0$ or η_μ = constant but it does not yield any closed expressions for ϵ or $\dot{\epsilon}$, which would make data fitting rather complicated. Another simple choice would be E_μ = constant. Although this does not allow closed expressions, it is possible to construct easily the creep curves corresponding to the hypothetical case that the progress of hydration (aging) is stopped from the instant, t', of loading. Because E_μ-s are constant, the slowing down or cessation of hydration changes only η_μ which causes the strain response for loading at a certain t' to shift horizontally to the left in the diagrams of $\dot{\epsilon}$ versus log $(t - t')$. The distance of shift can be determined if selected points on the actual creep curves are labeled by the corresponding ages [obtained as $t' + (t - t')$] and then interpolation between the labeled ages is used to pass creep the curves for constant age. However, applying this procedure to the actual data it has been found that the slope of the hypothetical $\dot{\epsilon}$-curves for stopped aging increases in log $(t - t')$-scale without bounds, which makes an expansion in Dirichlet series impossible. The assumption that E_μ = constant is thus unacceptable.

Still another Kelvin model representation is derived directly from Eq. 1 at the end of Appendix I (Eqs. 33, 34 with Eq. 11). Its spring moduli are $E_\mu = \hat{E}_\mu(t)$ and are thus always positive. However, the relations for dashpots are $\sigma_\mu = \eta_\mu \ddot{\epsilon}_\mu$ (Eq. 34) which is different from the usual form, Eq. 12, and is physically inadmissible. Namely, considering constant ϵ_μ, Eq. 12 gives $\sigma_\mu = \dot{\eta}_\mu \epsilon_\mu$ while Eq. 34 yields $\sigma_\mu = 0$. But in reality $\sigma_\mu > 0$ because the solid matter and hindered adsorbed layers (8) being added by hydration to the existing solid framework must increase the resistance to a given deformation rate.

From the foregoing considerations it is clear that there exist Kelvin chain models of different properties which can approximate a given creep function $J(t, t')$ with any desired accuracy. Thus the solution to the identification problem of an age-dependent material is not unique (not counting the nonuniqueness due to various possible choices of τ_μ, as mentioned before). This contrasts with classical viscoelasticity where there is only one Kelvin chain (one retardation spectrum) representing the material. (All possible Kelvin chains are, of course, equivalent, provided no generalization to variable temperature and water content, of the type outlined in Ref. 8, is attempted.)

NOTES ON MAXWELL CHAIN REPRESENTATION

From the preceding considerations it seems to be almost certain that no Kelvin chain model exists which would admit correlation with physical processes in the microstructure. Subsequent studies revealed, however, that the Maxwell chain model (with dashpots governed by the first of Eqs. 12) does allow such a correlation because none of its elastic moduli E_μ and viscosities η_μ ever become negative. But this advantage is gained at the cost of losing a conceptually simple relationship between the model and the creep test data, as is exhibited by the Kelvin chain.

There are again different possible forms of Maxwell chain representation and the most convenient one is characterized by age-independent relaxation times, $\tau_\mu = \eta_\mu/E_\mu$ = constant. In this case, instead of the creep curves, the stress relaxation curves for unit strain imposed at various ages t' [i.e., relaxation function $E_R(t, t')$] are obtained in the form of a Dirichlet series: $\sum_\mu E_\mu(t') \exp[-(t - t')/\tau_\mu]$. The data may be fitted in a similar procedure, provided that all creep data are first converted to relaxation data, which may be accomplished quite easily with a program presented in Ref. 7. A detailed discussion of this approach and fits of numerous data will be given in a future publication along with an unconditionally stable algorithm for step-by-step creep structural analysis which is analogous to the algorithm in Appendix I.

The possibility of correlation with physical processes in the microstructure is utilized in another future publication where the concept of activation energy is revoked to fit with Maxwell chains very closely all of the best known data on creep of aging concrete at variable temperature.

BASIC CONCLUSIONS

1. The creep function of concrete can be approximated with any desired accuracy by the Dirichlet series (Eq. 3) in the elapsed time, $t - t'$, with coefficients depending on the age at loading, t'. Expansion in Dirichlet series is easily accomplished with a computer by the method described.
2. Close and smoothed fits of test data can be obtained, as Figs. 2-8 demonstrate.
3. The Dirichlet series creep function corresponds to the Kelvin chain model (Fig. 9) with age-dependent properties (see Eqs. 16).
4. By contrast with nonaging materials, representation of a given creep function by the Kelvin chain model is not unique. Alternate forms may be based on Eqs. 17 and 19 with the Dirichlet series expansions of the plots of creep rate $\dot{\epsilon}$ versus elapsed time $t - t'$ for various times t Fig. 9, or on Eqs. 33 and 34.
5. If the creep function is expanded in Dirichlet series or characterized directly by a Kelvin chain, the step-by-step time integration of a creep problem can be formulated in a new algorithm which allows arbitrary increase of the time step and uses only a few hidden state variables for characterization of the whole history of the material (see Appendix I). The inherent savings in machine time and storage make possible creep analysis of large finite element systems.
6. In none of the Kelvin chain models examined was it possible to avoid either negativeness of some spring moduli for some periods of time or an unrealistic form of the equations for dashpots (Eq. 34). This is of no consequence for the creep analysis of structures but precludes any microstructural physical interpretation of the hidden state variables, which would be desirable for generalizations to variable temperature and water content. This drawback may be avoided by the Maxwell chain model.

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APPENDIX I—NUMERICALLY STABLE ALGORITHM WITH HIDDEN STRAINS FOR DIRICHLET SERIES CREEP FUNCTION

As has been mentioned in the Introduction, expansion of the creep function in Dirichlet series (Eq. 3) allows the step-by-step integration of creep problems

in time to be formulated in an algorithm in which the past history is characterized by a few hidden state variables and the time step can be arbitrarily increased without causing numerical instability. Because this algorithm was published in conference preprints (6) of limited circulation, its abbreviated description is given here for the reader's convenience.

Inserting Eq. 3 into Eq. 1, Eq. 1 can be written in the form

$$\epsilon(t) = \int_0^t \left(E^{-1} + \sum_{\mu} \hat{E}_{\mu}^{-1}\right)_{t'} d\sigma(t') - \sum_{\mu=1} \epsilon_{\mu}^{*}(t) + \epsilon^{0}(t) \quad \ldots\ldots (20)$$

in which the quantities

$$\epsilon_{\mu}^{*}(t) = e^{-t/\tau_{\mu}} \int_0^t e^{t'/\tau_{\mu}} \hat{E}_{\mu}^{-1}(t')\, d\sigma(t') \quad (\mu = 1, 2, \ldots, n) \quad \ldots (21)$$

may be viewed as hidden material variables which characterize the past history. Let time t be subdivided by discrete times $t_0, t_1, \ldots, t_N$ ($t_0 = t'$) into steps $\Delta t_r = t_r - t_{r-1}$. [For structures under constant load, the steps, Δt_r, are suitably chosen in a geometric progression (7), appearing constant in log $(t - t')$-scale.] Because the term, $e^{t'/\tau_{\mu}}$ in Eq. 21, can become extremely large when $t' >> \tau_{\mu}$, a good numerical approximation of the integral in Eq. 21 with regard to the exponential term is imperative. Assuming that $d\sigma(t)/dt$ and $E_{\mu}(t)$ have constant values within each time step (t_{r-1}, t) and change discontinuously in discrete times $t_0, t_1, t_2, \ldots$, Eq. 21 may be brought to the form

$$\epsilon_{\mu_r}^{*} = e^{-t_r/\tau_{\mu}} \sum_{s=1}^{r} \left(\hat{E}_{\mu}^{-1} \frac{d\sigma}{dt}\right)_{s-1/2} \int_{t_{s-1}}^{t_s} e^{t'/\tau_{\mu}}\, dt' \quad \ldots\ldots (22)$$

which can be integrated (exactly) as

$$\epsilon_{\mu_r}^{*} = e^{-t_r/\tau_{\mu}} \sum_{s=1}^{r} e^{t_s/\tau_{\mu}} \frac{\lambda_{\mu_s} \Delta\sigma_s}{\hat{E}_{\mu_{s-1/2}}} \quad \ldots\ldots (23)$$

$$\text{in which } \lambda_{\mu_s} = \left(1 - e^{-\Delta t_s/\tau_{\mu}}\right) \frac{\tau_{\mu}}{\Delta t_s} \quad \ldots\ldots (24)$$

Subscript r stands for time t_r, e.g., $\epsilon_{\mu_r}^{*} = \epsilon_{\mu}^{*}(t_r)$, or the increment during the rth step, e.g., $\Delta\sigma_s = \sigma_s - \sigma_{s-1} = \sigma(t_s) - \sigma(t_{s-1})$. Subscript $s - 1/2$ refers to the average value in the sth step, Δt, e.g., $\hat{E}_{\mu_{s-1/2}} = (1/2)\left(\hat{E}_{\mu_{s-1}} + E_{\mu_s}\right)$.

Subtracting from Eq. 23 the analogous expression for $\epsilon_{\mu_{r-1}}^{*}$, the following recurrent formula may be acquired:

$$\epsilon_{\mu_r}^{*} = \frac{\lambda_{\mu_r} \Delta\sigma_r}{E_{\mu_{r-1/2}}} + \epsilon_{\mu_{r-1}}^{*} e^{-\Delta t_r/\tau_{\mu}}; \quad \mu = 1, \ldots, n \quad \ldots\ldots (25)$$

Thus, the values of $\epsilon^*_{\mu_s}$ for $s < r - 1$ need not be stored.

Eq. 20 may be approximated by the following finite difference equation:

$$\Delta\epsilon_r = \left(E^{-1} + \sum_n \hat{E}_\mu^{-1}\right)_{r-1/2} \Delta\sigma_r - \sum_{\mu=1}^{n} \Delta\epsilon^*_{\mu_r} + \Delta\epsilon^0_r \quad \ldots\ldots\ldots (26)$$

Expressing the increment, $\Delta\epsilon^*_{\mu_r} = \epsilon^*_{\mu_r} - \epsilon^*_{\mu_{r-1}}$, with the help of Eqs. 25, Eq. 26 takes the form:

$$\Delta\epsilon_r = \frac{\Delta\sigma_r}{E''_r} + \Delta\epsilon''_r \quad \ldots\ldots\ldots (27)$$

in which

$$\frac{1}{E''_r} = \frac{1}{E_{r-1/2}} + \sum_{\mu=1}^{n} \frac{1-\lambda_{\mu_r}}{\hat{E}_{\mu_{r-1/2}}} \quad \ldots\ldots\ldots (28)$$

$$\Delta\epsilon''_r = \sum_{\mu=1}^{n} \left(1 - e^{-\Delta t_r/\tau_\mu}\right) \epsilon^*_{\mu_{r-1}} + \Delta\epsilon^0_r \quad \ldots\ldots\ldots (29)$$

Supposing that in a given creep problem, the stresses have already been calculated up to time t_{r-1}, values E''_r and $\Delta\epsilon''_r$ may be determined from Eqs. 28 and 29. Eq. 27 may then be regarded as a fictitious linear elastic stress-strain law, in which $\Delta\epsilon''_r$ = pseudo-inelastic (or pseudo-initial) strain increment and E''_r = pseudo-instantaneous elastic modulus. The problem of solving $\Delta\sigma_r$ and $\Delta\epsilon_r$ in all points of the structure is thus formally reduced to a linear elasticity problem with inelastic strains. After this problem is solved (e.g., by finite element method), the new values of hidden variables ϵ^*_μ at time t_r are computed from Eq. 29 and the analysis of the next time step, Δt_{r+1}, may be begun. The creep problem is thus converted to a series of linear elasticity problems.

Although the foregoing expressions have been given only for the case of uniaxial stress, the generalization to the case of multiaxial stress is obvious (for detail see Ref. 6). Eqs. 27-29 and Eq. 25 are then transformed by replacing E''_r with the pseudo-instantaneous shear and bulk moduli G''_r and K''_r; σ, ϵ, and ϵ_μ with the deviatoric stresses and strains, σ^D_{ij}, ϵ^D_{ij}, $\epsilon^D_{ij_\mu}$; and the volumetric stress and strain, σ^V and ϵ^V, ϵ^V_μ, and $\hat{E}_\mu$, with the analogous coefficients in the expansion of the deviatoric and bulk creep functions.

To check whether or not numerical instability may arise, it is possible to restrict attention to the solution of strains corresponding to a prescribed stress history. Eq. 25 may then be regarded as a linear difference equation for the discrete variable, $\epsilon^*_{\mu_r}$. The equation is nonhomogeneous because of the presence of a term with prescribed values of $\Delta\sigma_r$. The solution of the corresponding homogeneous equation may be sought in the form $\epsilon^*_{\mu_r} = a^r$ whose substitution in Eq. 25 yields the characteristic equation $a = e^{-\Delta t_r/\tau_\mu}$. Obviously $0 < a < 1$. Thus, the solutions of the homogeneous part of Eq. 25 are

of an exponentially decaying character, i.e. stable. For comparison note that if the usual forward difference approximation of Eqs. 11 and 12 is used, solutions of the type, a^r, with $a < -1$ are found when $\Delta t > 2\tau_1$ = double the shortest retardation time. Therefore, such methods are numerically unstable and do not allow arbitrary increase of the time step. A simple numerical example demonstrating the numerical stability and efficiency of the aforementioned method is given in Ref. 6.

Further it is of interest to note that $\lambda_{\mu_r} \to 1$ for $\Delta t_r/\tau_\mu \to 0$, $\lambda_{\mu_r} \to 0$ for $\Delta t_r/\tau_\mu \to \infty$, and always $0 < \lambda_{\mu_r} < 1$. Thus, according to Eq. 28, $1/E''_r$ is bounded, as it must be. Denoting by p the value of μ for which $\tau_p < \Delta t_r < \tau_{p+1}$, $\lambda_{\mu_r} \doteq 0$ may be put for $\mu = 1, 2, \ldots, (p-1)$, provided that $\tau_\mu = \tau_0\, 10^{\mu-1}$, and according to Eqs. 28 and 30:

$$\frac{1}{E''_r} = \frac{1}{E_{r-1/2}} + \sum_{\mu=1}^{p-1} \frac{1}{\hat{E}_{\mu_{r-1/2}}} + \sum_{\mu=p}^{n} \frac{1-\lambda_{\mu_r}}{\hat{E}_{\mu_{r-1/2}}} \quad \ldots\ldots\ldots (30)$$

$$\Delta\epsilon''_r = \sum_{\mu=p}^{n} \left(1 - e^{-\Delta t_r/\tau_\mu}\right) \epsilon^*_{\mu_{r-1}} + \Delta\epsilon^0_r \quad \ldots\ldots\ldots (31)$$

As a crude approximation $e^{-\Delta t_r/\tau_\mu} \approx 0$ and $\lambda_{\mu_r} \approx 1$ may be put for $\mu > p + 1$. Thus, the deformation due to all exponential components of creep function (Eq. 3) whose retardation times τ_μ are substantially less than Δt_r may be taken as instantaneous. (As a crude approximation, for τ_μ substantially larger than Δt_r the deformation contribution may be neglected.) This agrees with what may be expected on the basis of intuitive judgment.

A similar algorithm, applicable directly to the Kelvin chain model, Eqs. 11 and 12, may be found in Ref. 8 (Eqs. 70-79). A different type of algorithm which also eliminates storage of stress history but does not allow an arbitrary augmentation of the time step was given by Selna (26) and Bažant (4). Note that the preceding algorithm, as well as that of Ref. 8, bears some marks of analogy with the algorithm of Taylor, Pister, and Goudreau (27) and the algorithm of Zienkiewicz, Watson, and King (30), which both apply only for a special class of time-variable materials with constant spring moduli (polymers), characterized either by a Maxwell chain (27), or by a Kelvin chain with time-variable dashpot viscosities (30). (The restriction to constant moduli has not been spelled out in Ref. 30 but follows from the fact that relations of the form, $\sigma_\mu = E_\mu\,\epsilon_\mu$, have been used for springs, which is thermodynamically inadmissible when E_μ grows.)

With regard to the rate-type creep formulation it is interesting to note that hidden variables ϵ^*_μ given by Eq. 21 satisfy the first order differential equations

$$\dot{\epsilon}^*_\mu + \frac{\epsilon^*_\mu}{\tau_\mu} = \frac{\dot{\sigma}}{\hat{E}_\mu} \quad \ldots\ldots\ldots (32)$$

as can be checked by substitution of Eq. 12 and its derivative. Furthermore, integration of Eq. 32 may be shown to yield Eq. 21, so that these two equations are equivalent. The derivative of Eq. 20 can be written in form of Eq. 11 in which

$$\dot{\epsilon}_\mu = \frac{\dot{\sigma} - \dot{\sigma}_\mu}{\hat{E}_\mu(t)} \quad (33)$$

and $\dot{\sigma}_\mu = \hat{E}_\mu(t)\,\dot{\epsilon}^*_\mu$. Then, subtracting Eq. 33 from Eq. 32, it is found that $\epsilon_\mu = \dot{\epsilon}^*_\mu / \tau_\mu$ and

$$\dot{\sigma}_\mu = \eta_\mu(t)\ \bar{\epsilon}_\mu; \ (\mu = 1, \ldots, n) \quad (34)$$

in which $\eta_\mu(t) = \hat{E}_\mu(t)\tau_\mu$ or $\eta_\mu(t)/\hat{E}_\mu(t) = \tau_\mu$ = constant. Eqs. 34, 33, and 11 are seen to correspond to the Kelvin chain model with nonstandard equations for dashpots (Eq. 34).

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APPENDIX III.—NOTATION

The following symbols are used in this paper:

$E(t) = 1/J(t, t)$ = instantaneous Young's modulus of concrete;
E'' = pseudo-instantaneous Young's modulus (Eq. 28);
E_μ = spring modulus of μth Kelvin unit in chain (Eq. 12, Fig. 8);
$\hat{E}_\mu$ = coefficients of Eq. 3;
$J(t, t')$ = creep function (compliance) = strain in time t caused by constant unit stress applied in time t' (Eq. 1);
$\tilde{J}(t, t')$ = measured data on J;
k = slope in Eq. 6 and Fig. 1;
t, t' = time from casting of concrete, in days;

t', t_0 = time of application of constant stress;
ϵ, σ = strain and stress;
ϵ_μ, σ_μ = hidden strains and hidden stresses in Eq. 12 = strain and stress in the μth dashpot in Fig. 8;
ϵ^0, ϵ'' = prescribed stress-independent inelastic strain (Eq. 1) and the pseudo-inelastic strain in Eq. 29;
ϵ^*_μ = hidden state variables defined by Eq. 21;
η_μ = viscosity of the μth dashpot in Kelvin chain (Fig. 8, Eq. 12);
λ_μ = parameter given by Eq. 24;
τ_a = point of change of slope in Fig. 1 and Eq. 6;
τ_μ = retardation times in Eq. 3;
τ'_μ = retardation times given by Eq. 17; and
Dots = time derivatives, e.g., $\ddot{\epsilon} = d^2\epsilon/dt^2$.

Subscripts

r, s = for discrete times t_r, t_s in step-by-step analysis;
α, β = for selected values of t' and $(t - t')$ used in least square condition (Eq. 5); and
μ = for μth exponential in Eq. 3 or μth Kelvin unit in Fig. 8.

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KEY WORDS: **Aging tests (materials); Concrete; Creep; Engineering mechanics;** Numerical analysis; Viscoelasticity

ABSTRACT: A simple method is presented by which the linear creep function of concrete can be approximated, with any desired accuracy, by Dirichlet series with varibale coefficients. Smooth fits of the best known data on creep at constant temperature and water content are demonstated. It is shown that the approximation is equivalent to the Kelvin chain model with age-dependent properties. Other approximations leading to the Kelvin chain are also presented. It is found, however, that no Kelvin chain approximation can avoid negativeness of some spring moduli for some periods of time, which precludes physical interpretation of hidden strains. But representations with Maxwell chain are free from this deficiency. The Dirichlet series approximation allows formulation of an efficient algorithm of step-by-step time integration of creep problems, for which arbitrary increase of the time step is possible and storage of the stress history can be dispensed with.

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