THREE-DIMENSIONAL HARMONIC FUNCTIONS NEAR TERMINATION OR INTERSECTION OF GRADIENT SINGULARITY LINES: A GENERAL NUMERICAL METHOD

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Abstract—The harmonic function $u$ near point 0 from which a single singularity ray emanates is assumed to be dominated by the term $r^pU$ where $r=$distance from point 0, $p=$known constant and $p=$chosen function of angular spherical coordinates $\theta, \phi$, for which a partial differential equation with boundary conditions, especially those at the singularity rays, and a variational principle, are derived. Because grad $U$ is nonsingular, a numerical solution is possible, using, e.g. the finite difference or finite element methods. This reduces the problem to finding $\lambda$ of the smallest real part satisfying the equation $\det (A_{ij}) = 0$ where $A_{ij}$ is a large matrix whose coefficients depend linearly on $\mu = \lambda(\lambda + 1)$. In general $\lambda$ and $A_{ij}$ are complex. Solutions can be obtained either by reduction to a standard matrix eigenvalue problem for $\mu$, or by successive conversions to nonhomogeneous linear equation systems. Computer studies have confirmed the feasibility of the method and have shown that highly accurate results can be obtained. Solutions for cracks and notches ending at a plane or conical surface, and for cracks ending obliquely at a halfspace surface, are presented. In these cases, $\lambda$ is real and the singularity is always weaker ($\lambda > p$) than on the singularity line and may even disappear ($\lambda > 1$). Furthermore, elastic stresses under a wedge-shaped rigid sliding stamp or at a corner of a crack edge, and also harmonic functions at three-sided pyramidal notches, have been analyzed. Here $\lambda < p$ was found to occur. A simple analytical solution for one class of special cases has also been found and used to check some of the numerical results.

INTRODUCTION

In three-dimensional potential theory there exists a great number of problems in which one or several lines of singularity of the potential gradient terminate or intersect. The singularity at the point of termination or intersection can naturally be expected to differ from the singularity on the line or lines entering this point. Solutions to these problems are of considerable interest, e.g. for scattering of waves at a corner of a screen, for distribution of electric charge, for problems of heat conduction, potential flow, seepage etc., and perhaps most importantly, for fracture mechanics of elastic bodies. At present, however, most of these problems remain unsolved.

One problem of this type is the distribution of pressure below the tip of a rigid frictionless stamp of wedge-shaped contact, pressed upon an elastic halfspace. The problem can be reduced to a potential theory problem[4]. It was discussed in 1947 by Galin[5, 4] and an approximate solution restricted to non-oscillating singularities and based on mapping of a hemisphere upon a circle was presented in 1957 by Rvachev[11]. He applied the Galerkin method in the mapped domain and, using up to six terms, made rough estimates of the real values of the singularity exponent $\lambda$. He concluded that the pressure near the tip behaves as $r^{\lambda-1}$ where $\lambda$ monotonously increases from 0 to 1 as the wedge angle $2\alpha$ grows from 0 to $2\pi$ ($r=$distance from the tip). Recently Aleksandrov and Babeshko[1] used Green's function approach to study analytically the limiting case of the wedge angle $2\alpha$ tending to zero. They concluded that near the tip the pressure exhibits an oscillating singularity of the type $r^{-1.5} \cos (k \ln r)$ ($k =$ parameter), which obviously prevails over the non-oscillating singularity $r^{-1}$ found for $\alpha \to 0$ by Rvachev[11].
But it seems that no complete solutions of the near singularity fields, including the singularity on the edge of the stamp near the tip, have been presented so far.

A mathematically equivalent problem for the Helmholtz reduced wave equation, consisting in wave scattering near the tip of a right-angled corner of a screen, was studied analytically by Radlow [9]. He found that near the tip the harmonic wave function is of the type $r^{-0.25}$, without checking for a possible imaginary part of the exponent, and without deducing the near-tip field itself. Diffraction of elastic waves by a right-angled wedge was also studied analytically by Kraut [16].

The intent of the present study is to propose a general method for problems of this type. The method will consist of reduction to a two-dimensional eigenvalue problem for a function without gradient singularity, which can presumably be solved numerically with any desired accuracy.

**TYPE OF PROBLEMS TO BE DISCUSSED**

Using spherical coordinates $\theta$, $\phi$, $r$, the Laplace equation for the potential $u(\theta, \phi, r)$ reads

$$\nabla^2 u = \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} = 0. \tag{1}$$

The problem to be discussed herein consists in finding the limiting form of the solution of this equation as $r \to 0$, assuming that the domain of the solution is an infinite space bounded by a certain surface $B(\theta, \phi) = 0$ formed by rays emanating from point 0. The boundary conditions are $u = 0$ on part of the surface and $\partial u / \partial n = 0$ on the remaining part, $n$ being a normal to the surface. (The case of the boundary conditions $u = \text{constant}$ on part of the surface is easily reduced to the present case.) No boundary conditions will be prescribed at any point on radial rays that do not lie entirely on $B = 0$, so that an infinite number of solutions can be expected. Assuming these solutions to comprise a complete system of linearly independent functions, a certain linear combination of these solutions should satisfy the additional boundary conditions on the radial rays occurring in real problems. As will be seen, a class of admissible solutions is of the form $r^*F(\theta, \phi)$ and consequently, regardless of the type of the afore-mentioned linear combination, the solution for any boundary conditions on the radial rays will be dominated in a sufficiently small neighborhood of point 0 by that solution of the present problem for which $Re(\lambda)$ is smallest. When $Re(\lambda) < 1$, grad $u$ near point 0 becomes unbounded, i.e. the solution exhibits a singularity at point 0. However, for the given boundary conditions on the radial rays the singularity need not necessarily occur because the solutions with $Re(\lambda) < 1$ need not be present in the afore-mentioned linear combination (as in the case of a homogeneous normal stress field parallel to a crack). The sufficient conditions of singular behavior, as well as the uniqueness aspects, cannot be included in the present study.

If $u$ is required to be finite, only $Re(\lambda) > 0$ is admissible. (However, in some physically meaningful problems, such as the case of a line load on an elastic half-space, $u$ is not finite at point 0.)

The surface formed by radial rays and the corresponding boundary conditions are assumed to be of such a form that the surface contains a finite number of rays on which grad $u$ is singular (unbounded). Further, it is assumed that at any finite distance $L$ from point 0, the field within a neighborhood of the singularity ray which is sufficiently smal-
ler than $L$ is known, and is of the same type at any point of the ray except at point 0. This field may be determined by solving the plane problem in a plane normal to the ray. (The latter property is verified in some special cases by the analytical solutions reported in the sequel.}

Consider the plane section normal to a singularity ray passing through point 0' at a finite distance $L$ from point 0. It is assumed that within a neighborhood of point 0' which is sufficiently smaller than $L$, the normal plane section of the body is bounded by two straight rays $O'A, O'B$ from point 0', forming angle $\alpha$ (Fig. 1). The boundary condition on the rays may be either $u = 0$ or $\partial u / \partial \phi = 0$ where $\phi =$ angular coordinate in Fig. 1. The solution near point 0' in the normal plane section may be sought in the form $u = r^2 f_1(\phi)$ where $r$ is the distance from point 0', and $p$ and $f_1(\phi)$ are assumed to be the same at any finite distance $L$ from point 0.

To give an example of the determination of $p$ and $f_1(\phi)$, consider that the boundary conditions are $\partial u / \partial \phi = 0$ on ray $O'A$, and $u = 0$ on ray $O'B$. Substitution of $u = r^2 f_1(\phi)$ into the Laplace equation yields a homogeneous ordinary differential equation for $f_1(\phi)$, whose general solution is $A \cos p \phi + B \sin p \phi$. The boundary conditions reduce to $df_1(0)/d \phi = 0, \ f_1(\pi - \alpha) = 0$, which yield

$$u = r^2 f_1(\phi) \quad \text{where} \quad f_1(\phi) = \cos p \phi, \ p = \frac{1}{2} \pi / (\pi - \alpha).$$

The same result is obtained when rays $O'A, O'B$ form angle $2\alpha$ and $u = 0$ on both rays.

This technique of the determination of the near-singularity fields has been used in plane elasticity (where it is not as trivial as here) by Knein[8] (who thanked T. von Kármán for suggesting the basic approach), and later independently by Williams[14] and Karp and Karal[7]. In analogy with this technique, the variables may be separated in the three-dimensional problem as follows:

$$u(\theta, \phi, r) = r^nf(\theta, \phi)$$

where $n$ is, in general, complex and $f$ is a function of $\theta$ and $\phi$ only. It may be imagined as a function defined in a certain domain $\Omega$ on a unit sphere about point 0. Substitution of (3) into the Laplace equation (1) allows $r$ to be completely eliminated from the equation and results in a differential equation for $f(\theta, \phi)$.

It should be noted that in the case of the Helmholtz reduced wave equation, $\nabla^2 u + k^2 u = 0$ (arising in wave scattering problems), or the Poisson equation, $\nabla^2 u = k$, the variable $r$ is not eliminated since the additional term $k^2 r^2 f$ or $kr^{3-n}$, respectively, appears in the equation for $f$. However, in a sufficiently small neighborhood of point 0, this term becomes negligible, as compared with other terms in the equation (provided that $f$ is
bounded and, in the case of the Poisson equation, \( n < 2 \). Thus, the solutions of the reduced wave equation and the Poisson equation exhibit the same type of singularity as the harmonic functions.

The gradient singularity at point 0 is only partly interpreted by the term \( r^n \) in equation (3) since along certain radial rays there is an additional singularity which must be embedded in function \( f \). Consequently, function \( f \) is not suitable for determination by approximate numerical methods.

REMOVING SINGULARITY ON RADIAL SURFACE RAYS

It is desired to separate the variables in such a manner that, in any plane normal to the singularity ray, function \( u \) near the ray be of the same form as equation (3) in which \( f \), has no gradient singularity. Considering, for the sake of brevity, the case of a single singularity ray, this may be achieved by writing

\[
u(\theta, \phi, r) = r^n r_2^n U(\theta, \phi)
\]

where \( r_1 = \) chosen continuous smooth function of \( \theta, \phi, \) and \( r \), which is identical to the distance from the singularity ray in its vicinity and is nonzero everywhere except on the ray. Obviously, if \( p \) is the correct exponent for the gradient singularity on the ray, as determined from (2), \( U(\theta, \phi) \) must be a continuous smooth function of \( \theta \) and \( \phi \) which has (a) no gradient singularity on the singularity ray, and (b) in its vicinity is proportional to the function \( f \) from equation (2).

Function \( r_1 \) may always be introduced in the form

\[
r_1 = \rho_1(\theta, \phi)
\]

where \( \rho_1 = \) chosen continuous smooth function of \( \theta, \phi \) which is nonzero everywhere except on the singularity ray in whose vicinity it is identical with the distance from the ray measured on the unit sphere. Thus

\[
u(\theta, \phi, r) = r^n F(\theta, \phi) = r^{\lambda} \rho^n U(\theta, \phi)
\]

where

\[
\lambda = n + p.
\]

If the singularity ray is located at the pole, \( \theta = 0 \), a very simple choice of \( \rho \) is possible:

\[
\rho = \theta \quad \text{or} \quad \rho = \sin \theta.
\]

With the latter choice, \( r_1 \) represents the exact distance from the ray not only in its vicinity but everywhere in the domain. It must be warned, however, that \( \sin \theta \) cannot be used when \( \theta = \pi \) is part of the domain and no singularity ray exists at \( \theta = \pi \). The function \( \rho = \sin k\theta \) may be suitable when symmetric boundary conditions at \( \theta = \pi/2k \) are desired.

If the singularity ray is located at \((\theta_1, \phi_1)\), it is possible to choose, e.g.
Three-dimensional harmonic functions

\[ p = \{((\theta - \theta_1)^2 + ((\phi - \phi_1) \sin \theta_1)^2\}^{1/2}. \]  

Generalization to the case of several singularity rays contained within the domain on the unit sphere is obvious. For three rays, e.g.

\[ u(\theta, \phi, r) = r^\lambda F(\theta, \phi) = r^{\rho_1 + \rho_2 + \rho_3} U(\theta, \phi) \]  

where

\[ \lambda = n + p_1 + p_2 + p_3 \]  

\( p_1, p_2, p_3 \) being known exponents for the three rays and \( \rho_1, \rho_2, \rho_3 \) functions of the same type as \( \rho (\theta, \phi) \) was before.

Now consider a singularity ray which is placed into the pole of a spherical coordinate system (e.g. ray 00' in Fig. 2). In the plane \((\theta, \phi)\) the pole appears as a straight line segment \(\theta = 0\) with end points given by the angles \(\phi_1, \phi_2\) forming the crack or notch that creates the singularity (Fig. 3). This line segment is a part of the boundary of domain \(\Omega\) in the plane \((\theta, \phi)\) on which function \(U(\theta, \phi)\) has to be solved. It is obvious that a certain boundary condition must be imposed at this line segment (00' in Fig. 3). The boundary condition may be deduced from the fact that \(U\) must be proportional to the function \(f,\) from equation (2) in the vicinity of \(\theta = 0.\) In equation (2) \(f,\) depends only on \(\phi,\) i.e. is independent of \(\theta.\) Hence, the condition

\[ \frac{\partial U}{\partial \theta} = 0 \quad \text{at} \quad \theta = 0 \]  

must be satisfied.

Fig. 2. Various cases of a notch or crack ending into a conical or planar surface. (The sphere is not the body surface and serves only for visualizing the situation in spherical coordinates.)
DIFFERENTIAL EQUATION OF THE PROBLEM AND VARIATIONAL FORMULATION

Substitution of equation (6) into equation (1) yields, after rearrangement and simplification, the partial differential equation:

$$
p^2 \frac{\partial^2 u}{\partial \theta^2} + \frac{\rho^2}{\sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} + \left(2 p \rho \frac{\partial \rho}{\partial \theta} + \rho^2 \cot \theta \right) \frac{\partial u}{\partial \theta} + \left\{ \lambda (\lambda + 1) \rho^2 + p \left[ \rho \frac{\partial \rho}{\partial \theta} \cot \theta + (p - 1) \left( \frac{\partial \rho}{\partial \theta} \right)^2 + \rho \frac{\partial^2 \rho}{\partial \theta^2} \right] \right\} u = 0. \quad (13)$$

In the special case where $\rho = \sin \theta$, equation (13) becomes

$$
\sin^2 \theta \frac{\partial^2 U}{\partial \theta^2} + \frac{\partial^2 U}{\partial \phi^2} + \left( p + \frac{1}{2} \right) \sin 2 \theta \frac{\partial U}{\partial \theta} + \left\{ \lambda (\lambda + 1) - p(p + 1) \right\} \sin^2 \theta + p^2 \right\} U = 0. \quad (14)
$$

The problem consists in finding a nonzero solution of equation (13) or (14) which satisfies the boundary conditions at a surface formed by the radial rays and gives the smallest positive real part of $\lambda$.

It should be noted that for $\theta \to 0$, equation (14) becomes $\partial^2 U / \partial \phi^2 + p^2 U = 0$, which gives $U = \cos p \phi$, as is required by equation (2).

For numerical solution, the variational formulation is of particular interest. The functional associated with the Laplace equation is known to be

$$W = \int_V \frac{1}{2} (\nabla u)^2 \, dV + \mathcal{W}, \quad \mathcal{W} = \frac{1}{2} \int_V \left[ \left( \frac{\partial u}{\partial r} \right)^2 + \left( \frac{1}{r} \frac{\partial u}{\partial \theta} \right)^2 + \left( \frac{1}{r \sin \theta} \frac{\partial u}{\partial \phi} \right)^2 \right] \, dV + \mathcal{W}, \quad (15)$$

where $V = \text{volume}$, $\mathcal{W}$ = certain integral over the boundary surface. The functional which is associated with equation (13) or (14) may be obtained by reducing functional $W$ to a two-dimensional one. In conformity with the separation of variables by substitution (6), the minimizing condition for $W$ may be considered to be of the form $\delta(\delta, W) = 0$ where $\delta$ is the variation with regard to functions of $r$, and $\delta$ is the variation with regard to functions of $\theta$ and $\phi$. Thus, according to (15) and (7),

Fig. 3. Domain $O'ACO'$ from Fig. 2 visualized in $(\theta, \varphi)$ plane, with a nodal net used for numerical solution.
where $R = r^*$, $R' = dR/dr$. Integration by parts with respect to $(\delta R)'$ provides

$$\int_r^s \left[ \delta \int_{\Omega} \left[ -\lambda (\lambda + 1) F^2 \frac{1}{\sin^2 \theta} \left( \frac{\partial F}{\partial \phi} \right)^2 \right] \sin \theta d\theta d\phi \right] \delta R \, dr + \delta (\delta W_i) = 0$$

where $\Omega$ is the given domain on the unit sphere. Since this equation must be satisfied for any variation $\delta R$ as a function of $r$, the minimizing condition reduces to $\delta W = 0$, where $W$ is the following functional

$$W = \int \int_{\Omega} \frac{1}{2} \left[ \left( \frac{\partial F}{\partial \theta} \right)^2 + \frac{1}{\sin^2 \theta} \left( \frac{\partial F}{\partial \phi} \right)^2 - \lambda (\lambda + 1) F^2 \right] \sin \theta d\theta d\phi + W_i$$

in which $F = \rho^n U$; $W_i$ is a certain integral arising from $W_i$. Because the surface of the body in three dimensions consists of radial rays, $W_i$ represents a certain contour integral along the boundary of the domain $\Omega$. The foregoing simple derivation is, of course, not rigorous, but it is easy to verify using standard procedures that equation (13) is indeed associated with the variational principle $\delta W = 0$.

SOME CASES ADMITTING SIMPLE ANALYTICAL SOLUTIONS

Consider solutions of equation (14) which have the form

$$U(\theta, \phi) = Y(\theta) \sin \phi.$$

This obviously satisfies the boundary conditions

$$U = 0 \text{ at } \phi = 0 \quad \text{and} \quad U = 2(\pi - \alpha) = \pi/p$$

or

$$U = 0 \text{ at } \phi = 0 \quad \text{and} \quad \partial U/\partial \phi = 0 \text{ at } \phi = \pi - \alpha = \pi/2p$$

that is, $U = 0$ at the planes $00'A$ and $00'B$ in Fig. 2 where $\alpha$ and $p$ are related as given by equation (2). Thus, if domain $\Omega$ on the unit sphere is rectangular in $\theta$ and $\phi$ (Fig. 3), i.e.

$$0 \leq \phi \leq \pi/p, \quad 0 \leq \theta \leq \beta$$

function $Y(\theta)$ in (19) needs to satisfy only the boundary conditions on $\theta = 0$ and $\theta = \beta$. Considering that there is a free surface at $\theta = \beta$ (a cone, and for $\beta = \pi/2$ a plane; Fig. 2), and taking equation (12) into account, one may write

$$\partial U/\partial \theta = 0 \text{ at } \theta = 0, \quad U = 0 \text{ at } \theta = \beta.$$

Thus, it is necessary and sufficient that $Y(\theta)$ fulfill the boundary conditions:

$$\partial Y/\partial \theta = 0 \text{ at } \theta = 0, \quad Y = 0 \text{ at } \theta = \beta.$$
For $\alpha < \pi/2$ there is a single singularity ray in this problem, the ray \(00'\) in Fig. 2, and for $\alpha \geq \pi/2$ there is none. (On rays \(0A, 0B\) there is no singularity.) Substituting equation (19) into (14), equation (14) is found to be satisfied if

$$\frac{d^2Y}{d\theta^2} + (2p + 1) \cot \theta \frac{dY}{d\theta} + [\lambda(\lambda + 1) - p(p + 1)]Y = 0. \quad (24)$$

Upon introduction of a new independent variable $x = \sin^2 \theta$, equation (24) may be transformed to the form

$$x(1-x) \frac{d^2Y}{dx^2} + \left[ p + 1 - \left( p + \frac{3}{2} \right) x \right] \frac{dY}{dx} + \frac{\lambda(\lambda + 1) - p(p + 1)}{4} Y = 0 \quad (25)$$

which is seen to be a special form of the hypergeometric (Gauss) differential equation. The solution of this equation [12] leads to hypergeometric functions and it can be shown that it may be expressed in terms of elementary functions only if $\beta = \pi/2$, that is if the conical surface degenerates into a plane (Fig. 2b). Then the root $\lambda$ having the smallest real part is obtained as

$$\lambda = p + 1 \quad (26)$$

and the corresponding solution is

$$u(\theta, \phi) = Cr^{p+1} (\sin \theta)^p \cos \theta \sin p\phi \quad (27)$$

where $C$ is an arbitrary constant. Equation (27) can be proved by substitution into (14). Note that, for any angle $\alpha$, $\lambda$ is greater than 1.

An interesting result is thus obtained: the gradient singularity at the ray (edge of a crack or notch, e.g.) disappears at the surface of the body (cases in Fig. 2b, c, d).

One important case is the half-space $-\pi \leq \phi \leq 0$ whose boundary conditions at the surface $\phi = 0$ are $u = 0$ on the infinite wedge $0 \leq \theta \leq 2\beta$ and $\partial u/\partial n = \partial u/\partial \phi = 0$ on the rest of the surface (Fig. 11a). Simple analytical solutions exist [11] for two limiting cases:

$$u = C \ln \left( \tan \frac{\theta}{2} \right) \quad \text{for } \beta \to 0 \quad (\lambda = 0) \quad (28)$$

$$u = Cr \sin \theta \sin \phi \quad \text{for } \beta \to \pi \quad (\lambda = 1) \quad (29)$$

and, of course, for the case $\alpha = \pi/2$ in which the problem reduces to a plane one and has a well known solution yielding $\lambda = \frac{1}{2}$.

REDUCTION TO A NONLINEAR EIGENVALUE PROBLEM FOR A MATRIX

Because function $U(\theta, \phi)$ has no gradient singularities, standard numerical methods may be applied to its determination. For complicated geometries of the domain $\Omega$ on the unit sphere, the most versatile approach will certainly be the finite element method. However, development of finite elements on the basis of the functional (18) exceeds the scope of this paper and will be treated separately. For domains of simpler geometries,
the use of the finite difference method appears to be easier and has been adopted for all computer studies reported herein. The differential equation (14) or (13) is replaced by its finite difference approximation in terms of nodal values $U_i (i = 1, 2, \ldots)$ and is written for all nodes of the chosen network inside the domain $\Omega$. Some of the equations involve nodal points outside the boundary for which further equations expressing the boundary conditions must be added.

Application of the finite difference method, as well as the finite element method, yields a large system of $N$ homogeneous linear algebraic equations for values $U_i$ of function $U$ in the nodes $j = 1, 2, \ldots, N$,

$$\sum_{j=1}^{N} A_{ij}(\lambda) U_j = 0 \quad (i = 1, 2, \ldots, N).$$

(30)

The determinant of matrix $A_{ij}$, in general complex, must vanish for a non-zero solution to exist. The problem is thus reduced to finding a root $\lambda$ having the smallest positive real part. The solution is complicated by the large size of matrix $[A_{ij}]$ needed for accurate results, and also by the fact that $A_{ij}$ are nonlinear functions of $\lambda$ (polynomials). Various methods of solution are possible.

**Method A. Reduction to a matrix eigenvalue problem**

Noticing that equation (13) or (14) involves $\lambda$ solely through the term $\lambda(\lambda + 1) = \mu$, matrix $A_{ij}$ is a linear function of $\mu$. Furthermore, if the finite difference method is used, $\mu$ appears only in the diagonal coefficients of $A_{ij}$, and if the equations expressing the boundary conditions are eliminated, $\mu$ appears in all diagonal coefficients. Mere division of each equation by the coefficient at $\mu$ reduces the matrix $A_{ij}$ to the form:

$$A_{ij}(\lambda) = K_{ij} - \mu \delta_{ij}$$

(31)

where $K_{ij}$ is a matrix which is independent of $\lambda$, and $\delta_{ij} = 1$ for $i = j$, $\delta_{ij} = 0$ for $i \neq j$. Thus, a standard eigenvalue problem in $\mu$ for a real, general (nonsymmetrical) matrix is obtained. There are two roots $\lambda$ which correspond to each root $\mu$,

$$\lambda = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + \mu}.$$ 

(32)

If the roots $\lambda$ are all real, the smallest positive $\lambda$ evidently corresponds to the smallest $\mu$ ($\pm$ being replaced by $+\mu$). But this is not true in the case of complex roots and all roots $\mu$ must be checked to determine the smallest $Re(\lambda)$.

For the solution of the eigenvalue problem obtained, standard subroutines are available [3, 6, 10, 15]. Nevertheless, even with the contemporary electronic computers it is difficult and costly to exceed 150 equations, approximately, which usually suffices for accurate but not highly accurate results. The cost is also substantially increased by the fact that for the finite difference method, matrix $A_{ij}$ is non-symmetric.

The finite element method, which may be based on minimizing the functional (18), yields symmetric matrices $A_{ij}$ and $K_{ij}$. However, in contrast with the finite difference method, $\mu$ no longer appears exclusively in the diagonal terms of $A_{ij}$ and inversion of a matrix of the size $N \times N$ is necessary to obtain a standard eigenvalue problem in $\mu$.

Taking advantage of the sparsity of matrix $A_{ij}$, a much larger matrix can be stored in
rapid access memory and the eigenvalue could be obtained by iteration\cite{3,6,10}. However, this would be economical only for the eigenvalue of largest absolute value. Attempts have been made to recast the problem in terms of the highest eigenvalue, e.g. replacing $\lambda$ with $1/\gamma$ and linearizing with respect to $\gamma$. It appears, however, that this approach is impossible because there are always some spurious negative roots whose absolute value is larger than the largest positive $\gamma$.

**Method B. Conversion to nonhomogeneous equation systems**

(a) **The case of real root.** In this method, which has been used with success\cite{2} to solve buckling problems for very large elastic structural systems leading to equations of type (30), a certain value $\lambda_0$ is first chosen and all coefficients $A_{ik}(\lambda_0)$ are evaluated. One equation, whose number will be referred to as $k$, is replaced by the equation $U_i = 1$, which makes the system nonhomogeneous. Normally the matrix of the new system is nonsingular and all $U_i$ ($i = 1, 2, \ldots, N$) may be solved using the standard subroutines for systems of linear equations. The obtained values of $U_i$ are then substituted into the original $k^{th}$ equation, previously replaced, and the corresponding right-hand side $\Sigma A_{ik} U_i = Q$ is computed. In general $Q$ will be nonzero and the aim is to find the smallest value of $\lambda$ for which $Q = 0$. This may be done by the regula falsi method. Several values of $\lambda_0$ are selected and the plot of $Q$ versus $\lambda_0$ is imagined as shown in Fig. 6. Subsequent choices of $\lambda_0$ are made by linear interpolation. Good accuracy and convergence is achieved only when the $k^{th}$ equation is chosen in such a manner that no $|\Sigma A_{ik}|$ is much larger than $U_i$.

From computer studies it has been found that the radius of convergence of this method is sometimes quite small. The interval of interest, such as $0 \leq \lambda \leq 2$, must be scanned in small steps of $\lambda$, about 0.05, to obtain a good initial guess of $\lambda$. Only then may the automatic improvements of $\lambda$ by the regula falsi method be started. This is obvious from the sharply varying slope of the plot in Fig. 6 (although there exists cases of slowly varying slope, as is shown by curve $c$ in Fig. 6). Much care is needed to avoid missing the smallest root. Nevertheless, if a great number of problems with the given data varying in small steps are being solved, as in the studies reported herein, an approximate value of $\lambda$ is usually known in advance and scanning of the large interval may be skipped. It is because of this fact that this method is usually much more economical than the general programs for eigenvalues. Moreover, in contrast with method A, this method allows handling up to one or two thousand equations without great difficulties (using either various elimination methods for banded matrices with tape storage or the sparse matrix storage in rapid access memory combined with an iterative solution).

If no estimate of $\lambda$ is available, computationally the most efficient procedure is a combination of methods A and B. First, method A with a low number of nodes is used to obtain an approximate value of $\lambda$ and the solution is then refined to a high accuracy using method B with a large number of nodes.

As an alternative to method B, it is also possible to apply the regula falsi method to the plot of the determinant of $A_{ij}$ versus $\lambda$. But the slope of this plot usually varies even more rapidly and the convergence properties were found to be poorer than in method B.

(b) **The case of complex root.** When root $\lambda$ of the smallest real part happens to be complex, a similar method may be used. In this case $U_i$, as well as matrix $A_{ij}$ depending on $\lambda$, must be considered as complex. The objective is to find complex $\lambda_0$ for which
Q = 0. First, an arbitrary value of \( \text{Im}(\lambda_0) = X \) is chosen and, keeping it fixed, the plot of \( \text{Re}(Q) \) versus \( \text{Re}(\lambda_0) \) is examined, choosing various values of \( \text{Re}(\lambda_0) \). The point at which \( \text{Re}(Q) = 0 \) may be found by the regula falsi method and the corresponding \( \text{Im}(Q) \) denoted as \( Y \). The foregoing procedure is then repeated for various \( \text{Im}(\lambda_0) = X \) and, considering the plot of \( Y \) versus \( X \), the point at which \( Y = 0 \) (and \( Q = 0 \)) is again found by the regula falsi method.

The fact that the regula falsi method must be applied in two dimensions makes the computations much more expensive than in the case of a real \( \lambda \). Moreover, because the matrix is complex, the running time of each equation system is longer. Nevertheless, using tape storage or compact storage of the sparse matrix, much larger systems can be handled than with method A.

**Method C. Successive linearizations with respect to \( \lambda \)**

If the technique presented here is applied in elasticity, \( A_{ij} \) will usually consist of general polynomials in \( \lambda \), complex or real. Method B will still be applicable, but instead of Method A successive solutions of eigenvalue problems will be required. Since \( A_{ij} \) are, in general, analytic functions of a complex variable \( \lambda \), expansion in a Taylor series about a chosen complex \( \lambda_0 \) is possible. The series may be truncated after the second term to achieve linearization. Equations (30) are then reduced to the form

\[
\sum_{i}^{N} (A_{ij}^0 - \lambda' A_{ij}^0)U_i = 0 \quad (i = 1,2, \ldots N)
\] 

where \( \lambda' = \lambda_0 - \lambda \), \( A_{ij}^0 = A_{ij}(\lambda_0) \), \( A_{ij}' = -\partial A_{ij}(\lambda_0)/\partial \lambda \). Multiplication of matrix equation (33) by the inverse of \( A_{ij}' \) then produces a standard eigenvalue problem, in general with a complex matrix. The objective is to find \( \lambda_0 \) for which \( \lambda' = 0 \). This can be achieved by application of the regula falsi method, in the same procedure as the condition \( Q = 0 \) is achieved in method B.

This method has so far been tested only in the problems with real \( \lambda \) as reported in the sequel. The convergence was rapid and the radius of convergence appeared to be quite large, that is the method automatically converged even if the initial guess of \( \lambda_0 \) was very poor (e.g. 0.2 in the case of plot (a) in Fig. 6). This is due to the slow change of the slope in this plot.

**APPLICATION AND RESULTS OF NUMERICAL STUDIES**

**Cracks and notches ending at a plane or conical surface**

To examine the numerical methods described above, the cases whose exact solution is known were solved first. These are the cases shown in Fig. 2 and characterized by boundary conditions (20a) and (22). There is only one singularity ray 00' in these problems. The domain \( \Omega \) is given by equation (21) and is rectangular in terms of \( \theta \) and \( \phi \), as shown in Fig. 3. The domain was subdivided by a rectangular network with steps \( \Delta \phi = (2\pi - \alpha)/2n \), \( \Delta \theta = \frac{1}{2}\pi/(n + \frac{1}{2}) \). It is noteworthy that no interior nodes of the network may be placed on the boundary line \( \theta = 0 \) because the finite difference equations for the nodes on this line do not involve any node with \( \theta \neq 0 \). However, these nodes may be considered as exterior nodes of the network used in formulating the boundary condition (as has been done in computing Table 2).

The solutions were carried out for various sizes of the step. The results for \( \alpha = 0 \), given in Fig. 7, indicate an excellent convergence of \( \lambda \) to the exact value 1.5 resulting...
from equations (26) and (2). For 72 equations the error is 0.4 per cent and for 1152 equations it is 0.04 per cent. The convergence is monotonous and from the slope of the line in Fig. 7 it may be inferred that the error \( \approx (\text{step size})^{1.9} \), that is, the convergence is almost quadratic. A similar picture is obtained for other values of \( \alpha \).

The solutions were also computed for non-zero values of \( \beta \), that is, the cases of cracks or notches ending at the apex of a conical surface, which do not admit simple analytical solutions. The results are plotted in Fig. 4 and some of the results are tabulated in Table 1 (where three decimals are believed to be correct).

The results for \( \beta \neq 0 \) were compared and agreed with one-dimensional finite difference solutions of the ordinary differential equation (24). (For \( \alpha = 0, \beta = \pi/2 \), step \( \Delta \theta = \beta/50 \) gave \( \lambda = 1.4998 \), which is only 0.013 per cent in error.) For several cases, the eigenstates, as characterized by function \( Y(\theta) \) in equation (19), are plotted in Fig. 5. As a check, some solutions have also been run with \( \rho^o = \sqrt{\theta} \) and the results were identical.

It is interesting to note that at a certain value \( \beta > \pi/2 \), depending on \( \alpha \), \( \lambda \) becomes less than 1.0 so that \( \text{grad} \ u \) is singular at point 0. But the singularity is always weaker than at the edge 00' (see Fig. 4).

**Cracks ending obliquely at a halfspace surface**

The general method presented herein was further applied to the case shown in Fig. 8b, which is equivalent to the case in Fig. 8a where the boundary conditions are also given. There is only one singularity ray 00' in this problem, with \( p = \frac{1}{2} \) from equation

![Fig. 4. Singularity strength for cases shown in Fig. 2.](image-url)
Three-dimensional harmonic functions

Table 1. Values of \( \lambda \) for various \( \alpha \) and \( \beta \) in the problem shown in Fig. 2

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \pi/4 )</th>
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(2). The pole of the spherical coordinates is conveniently placed into this ray. Then, using \( \rho = \sin \theta \), the problem is again governed by the differential equation (14). \( \Omega \) is the domain \( 0'ABCD0' \) (or \( 0'AECD0' \)) in Fig. 8 as is shown in plane \((\theta, \varphi)\) in Fig. 10. The boundary conditions are \( \partial u / \partial \varphi = 0 \) on side \( DC0' \) (plane of symmetry), \( \partial u / \partial \theta = 0 \) at the pole \( 0' \) (side \( 0'\varnothing' \) in Fig. 10), \( u = 0 \) on side \( 0'A \) (crack surface) and \( u = 0 \) on side \( ABC \) (or \( AEC \)). The latter side appears in the plane \((\theta, \varphi)\) as the curve defined as follows:

\[
\theta = \arctan (\tan \beta / \cos \varphi); \quad \text{if } \theta < 0 \text{ then } \theta \leftarrow \theta + \pi.
\]  

(34)

Formulation of the finite difference equivalents of the boundary condition \( u = 0 \) in terms of the nodal values near boundary \( ABC \) requires locating point \((\theta_c, \varphi_c)\) which is symmetrical with respect to plane \( ABC \) to a given outside node \((\theta_o, \varphi_o)\). To this end, the value of \( \theta \) which corresponds to \( \varphi = \varphi_o \) according to (34) is first calculated and, using spherical trigonometry, the following expressions are successively evaluated

\[
\begin{align*}
\cos \psi &= 1 - 2(\cos \beta \sin \varphi_o)^2 \\
\theta_c &= \arccos \left[ \cos \theta_o \cos (\theta_o - \theta) + \sin \theta_o \sin (\theta_o - \theta) \cos \psi \right] \\
\varphi_c &= \varphi_o + \arcsin \left[ \sin (\theta_o - \theta) \sin \psi / \sin \theta_o \right].
\end{align*}
\]  

(35)

Fig. 5. Plot of function \( Y(\theta) \) from equation (19) for some cases shown in Fig. 2.
Fig. 6. Examples of dependence of $\lambda$ and parameter $Q$ upon $\lambda_0$ in methods $A$ and $B$ for solving equations (30). (Curves $a$, $b$ correspond to Fig. 2c, curve $c$ to Fig. 12a.)

Fig. 7. Convergence of $\lambda$ to the exact value 1.5 with diminishing step size of the net, for the case in Fig. 2c.

The $U$-value at point $(\theta_c, \varphi_c)$ is then related by linear interpolation to the $U$-values at the nearest three interior nodes $p$, $q$, $r$ using area coordinates in the triangle $pqr$, i.e.

$$U_c = (a_{pq}U_p + a_{qr}U_q + a_{rp}U_r)/a_{pqr}$$

where $a_{pq}$ etc. are the areas of the triangles indicated by subscripts. The boundary condition $U = 0$ is expressed in the form $U_s = -U_c(\sin \theta_c/\sin \theta_0)^n$.

Introducing finite difference grids with 80–100 nodes, the problem has been solved for various values of $\beta$. The values of $\lambda$ of the smallest real part are all real and the computer results are shown as points on curve $a$ in Fig. 9. The limiting values $\lambda = 1$ and 2 for $\beta \to 0$ and $\beta \to \pi$ result from equation (2) for the plane problem. The value $\lambda = 1.5$
Three-dimensional harmonic functions

Fig. 8. (a) A quarterspace with boundary conditions giving rise to singularity on line 00'. (b) A halfspace with a crack whose edge is not perpendicular to the surface.

Fig. 9. Smallest real root $\lambda$ for cases shown in Fig. 11 (curve b), Fig. 8 (curve a) and Fig. 13 (curve c).

for $2\beta = \pi$ agrees with the result from Fig. 4. It is noteworthy that the singularity on the ray 00' disappears at the surface of the halfspace (Fig. 8b) for any angle $\beta$. The eigenstates $U(\theta, \varphi)$ for two special cases are drawn in Fig. 10. For the case $\beta = \pi/4$ (Fig. 12a) it has been verified by method A that no complex roots exist. This case has also been analyzed by method B with grid $\Delta \varphi = \pi/24, \Delta \theta = \pi/26$ (359 nodes), which yielded $\lambda = 1.131$, while the coarser grid ($\Delta \varphi = \pi/12, \Delta \theta = \pi/13$) resulted in $\lambda = 1.128$ (the second and third smallest $\lambda$ was 2.25 and 2.95). Assuming a similar convergence pattern as in Fig. 7, the correct value can be estimated as $\lambda = 1.131 + 0.001$, or $\lambda = 1.132$.

Elastic stresses under wedge-shaped stamps or at crack edge corners. Wave scattering by corners.

Furthermore, the case of a rigid sliding stamp with wedge-shaped contact of an
Table 2. Nodal values of function $U$; (A)—Case in Fig. 10a ($p = 0.5$); (B)—Fig. 12a ($p = 0.5$); (C)—Fig. 12b ($p = 0.5$); (D)—Fig. 14 ($p = 0.5$); (E)—Fig. 11 for $\beta = \pi/8$ ($p = 2/3$). Nodal coordinates are $\theta_k = K \Delta \theta$, $\varphi_k = J \Delta \varphi$ in cases (A), (B), (D) and (E), $\varphi_k = (J-1) \Delta \varphi$ in case (C); grid $\Delta \varphi = \pi/24$, $\Delta \theta = \pi/26$.

In Table (C) the left and right of Fig. 14 are interchanged. The top row of each case was obtained by extrapolation.

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In Table (C) the left and right of Fig. 14 are interchanged. The top row of each case was obtained by extrapolation.
arbitrary angle, $2\beta$, pressed upon the surface of an isotropic elastic half-space (Fig. 11a), has been studied. As is well known\[4\], this problem of elasticity may be reduced, with the help of Papkovich–Neuber potentials, to a potential theory problem, $\nabla^2 u = 0$. Introducing cartesian coordinates $x$, $y$, $z$ and considering the half-space $z \geq 0$ or $0 \leq \varphi \leq \pi$, the boundary conditions at the halfspace surface are

$$u = 0 \text{ on } \mathcal{F} \quad \text{and} \quad \frac{\partial u}{\partial z} = 0 \text{ outside } \mathcal{F}$$

where $\mathcal{F}$ is the wedge-shaped domain below the stamp ($00' \ A0''$ in Fig. 11a). Also, at infinity, $\text{grad } u$ is required to vanish. After solving for $u$, the displacements $u_x$, $u_y$, $u_z$ in cartesian coordinates $x$, $y$, $z$ at any point of the half-space $z \geq 0$ are given as\[4\]:

$$u_x = (1-2\nu) \int_z^{\infty} \frac{\partial u}{\partial x} \, dz - z \frac{\partial u}{\partial x}, \quad u_y = (1-2\nu) \int_z^{\infty} \frac{\partial u}{\partial y} \, dz - z \frac{\partial u}{\partial y}, \quad u_z = 2(1-\nu) u - z \frac{\partial u}{\partial z},$$

where $\nu$ is Poisson's ratio of the material. The normal stress below the stamp is

$$\sigma_z = (\partial u / \partial z) E / (1 + \nu) \quad \text{where } E = \text{Young's modulus}.$$

This problem is equivalent to the case of an infinite elastic body containing a sharp crack whose edge forms a sharp corner of an arbitrary angle $2\beta$ (Fig. 11b). The body is subjected at infinity to uniformly distributed normal stresses perpendicular to the crack.
Fig. 11. (a) Wedge stamp contact (unshaded) upon elastic halfspace or, alternatively, a screen, scattering waves. (b) Crack corner in an infinite elastic space. (The sphere is not the body surface and serves only for visualizing the situation in spherical coordinates.)

plane. Furthermore, the near field in this problem happens to be the same as in the scattering of waves by a sharp corner of a screen[9].

Although in Fig. 11 there are two singularity rays, 00' and 00", each with $p = \frac{1}{2}$ from equation (2), the problem may be reduced to one with a single singularity ray, taking advantage of the fact that the solution $u$ or $F$ must be symmetrical with regard to plane $OABC$ in Fig. 11b. The formulation of the problem then differs from the previous case (Fig. 8) solely by the boundary condition on side $ABC$ (or $AEC$) which reads:

$$\partial \left[(\sin \theta)^p U\right]/\partial n = 0$$

(38)

where $p = \frac{1}{2}$ and $n$ is the normal to the side $AEC$ (normal on the sphere, not in the plane $(\theta, \varphi)$). The finite difference formulation of the boundary condition is the same as in the previous case (Fig. 8, equations 34, 35), except that $U_a = U_c (\sin \theta_0/\sin \theta_0)\varphi$.

Computer results for grids with 80–100 nodes are shown by the points on curve $b$ in Fig. 9. Note that the points adjacent to $\lambda = 0.5$ fit a smooth curve passed through the well-known value of $\lambda = 0.5$ for $2\beta = \pi$. The point $\lambda = 0$ for $\beta \to 0$ follows from equation (28), and $\lambda = 1$ for $\beta \to \pi$ results from equation (29). The eigenstates $U(\theta, \varphi)$ are shown for two special cases by lines of equal $U$-value in Fig. 12. It is worth noting that the computed variation of $U$ along the top side $0^\prime 6^\prime$ of Fig. 12a or $b$ is proportional to $\sin (\theta/2)$, as is required by equation (2).

The results prove that the singularity at the sharp concave ($2\beta < \pi$) corner $0$ of a crack edge (Fig. 11b) in an elastic body is more severe ($\lambda < p$) than at a straight crack edge ($p = \frac{1}{2}$). This fact is of interest for fracture mechanics of brittle elastic materials. It shows that at a concave corner of the crack edge of a planar crack the crack will propagate faster or at a lower stress than at a straight edge. Obviously, this would tend to straighten the crack edge. A plane crack with a straight edge is thus more stable than a plane crack with zig-zag edge.

All roots have been computed by method $B$, restricting consideration to real $\lambda$. However, the case $2\beta = \pi/2$ has also been computed by method $A$; the result was identical and it was found that no complex roots $\lambda$ exist in this case (similar to cases $2\beta = \pi$ and $\beta = \pi$).
Three-dimensional harmonic functions

Fig. 12. Function $U$ computed for the case shown in Fig. 11 ($\rho^p = \sqrt{\sin \theta}$).

All points on curve $b$ in Fig. 9 have first been computed with grids $\Delta \varphi = \pi/12, \Delta \theta = \pi/13$ (80–100 nodes). For the case of a right-angled corner ($2\beta = \pi/2$; Fig. 12a) this grid (having 86 nodes, not counting those outside the domain) yielded $\lambda = 0.2905$. This case has then also been computed with a grid $\Delta \varphi = \pi/24, \Delta \theta = \pi/26$ (having 359 nodes), which has resulted in $\lambda = 0.2947$ (using method B). This differs by 0.0043 (i.e. 1.46 per cent) from the coarser grid. Nearly the same difference, namely 0.0044, is found between the grids with the same spacing in Fig. 7. Because the result for a grid with 359 nodes would be in Fig. 7 about 0.0015 below the exact value, it can be inferred that the correct value probably is $\lambda = 0.2947 + 0.0015$, i.e.

$$\lambda = 0.296 \quad (\text{for } 2\beta = \pi/2).$$

(However, Radlow [9] deduced for this case analytically the value $Re(\lambda) = 0.25$, whose difference from the present result is obviously much greater than the probable value of numerical error, and so reexamination of the analysis in [9] seems to be in order.) The second and third smallest $\lambda$, as found by method $A$ with 86 nodes for $p = \frac{1}{3}$, are 1.43 and 2.03. It was also checked that higher $p$ does not give smaller $\lambda$.

The case $2\beta = 3\pi/2$ (Fig. 12b) has also been computed with the finer grid, which has furnished $\lambda = 0.8148$; assuming similar convergence behavior as before, the correct
value is \( \lambda = 0.815 + 0.001 = 0.816 \). For \( 2\beta = \pi/4 \), the finer grid yielded \( \lambda = 0.2057 + 0.0015 = 0.207 \).

Rvachev's results for real roots\[1\], obtained by a mapping technique and the Galerkin method, differ considerably (by as much as 0.2) from curve b in Fig. 9. This is probably due to numerical error because Rvachev used only six unknown parameters.

For \( \beta \to 0 \), Aleksandrov and Babeshko\[1\] found \( \lambda \) to be complex, with \( \text{Re}(\lambda) = -0.5 \), which overrides the value \( \lambda = 0 \) indicated by equation (28) and shown in Fig. 9. However, no cases with very small \( \beta \) have yet been run to verify this result. This will be postponed until the finite element formulation is completed (because refinement of the nodal network near side OA in Fig. 10 will probably be desirable for high accuracy).

Singularity at the apex of pyramidal notches with three equal sides

The cases of pyramidal notches shown in Fig. 13 have 3 singularity rays and possess one plane of symmetry through each of the three edges, so that only the domain \( O'ACD'0' \) needs to be considered. This domain includes only one singularity ray and the formulation of the problem is the same as for the cases shown in Fig. 11, discussed previously, except for two differences. First, angle \( 2\alpha \) of the planes forming each radial edge depends on the apex angle \( 2\beta \) of each side, namely \( \sin \alpha = 1/(2 \cos \beta) \), and the value of \( p \) is then given by equation (2). Second, the location of the right boundary in the \((\theta,\varphi)\) plane is different, as is shown in Fig. 14 for the case of the right-angled pyramidal notch \((2\beta = \pi/2)\).

This case was analyzed only by method B, assuming \( \lambda \) to be real. The smallest roots computed with networks of about 80 nodes are given by the points on curve c in Fig. 9 and the eigenstate \( U \) for one case is shown in Fig. 14. Note that the cases \( 2\beta > 2\pi/3 \) represent a convex solid pyramid rather than a concave notch. For \( 2\beta = 2\pi/3 \) the pyramid degenerates into a planar surface \((\alpha = 0)\), for which \( \lambda = 1 \). The case \( \beta \to 0 \) should give the same value as for curve b. (According to the result in[1], the values in Fig. 9 will possibly be overridden by some complex \( \lambda \) at \( \beta \to 0 \).) The right-angled pyramidal notch (Figs. 14 and 13a) has also been computed with a finer grid \( \Delta \varphi = \pi/24, \Delta \theta = \pi/26 \), which has yielded \( \lambda = 0.4533 \). Assuming similar convergence behavior as before, the correct value is \( \lambda = 0.4533 + 0.0015 = 0.455 \). For \( 2\beta = \pi/4 \), the finer grid yielded \( \lambda = 0.2288 + 0.0015 = 0.230 \) (\( \alpha = 32.765^\circ, p = 0.61127 \)).

![Fig. 13. Pyramidal notches with three sides of equal apex angle. (The sphere is not the body surface and serves only for visualizing the situation in spherical coordinates.)](image-url)
Development of singular finite elements and extensions to elasticity

It should be noted that knowledge of the dominant near-singularity field is all that is needed for the development of finite elements comprising the singularity, following the path shown, e.g. by Walsh[13]. Table 2 gives sufficient information for formulation of such elements for the most important cases. This brings within reach the numerical solutions by the finite element method for three-dimensional domains with arbitrary boundaries.

The general method presented can also be applied to problems of three-dimensional elasticity which are not reducible to potential theory problems. Here expressions of the form (6) or (10) augmented by rigid-body displacement terms for the neighborhood of the ray must be introduced for each of the three displacement components. The analysis may be based either on Navier's differential equations of equilibrium in spherical coordinates or the Papkovich–Neuber potentials. In either case the problem is reduced to three simultaneous second order partial differential equations in $\theta, \phi$ for three functions with nonsingular gradients. Applying a numerical method, the problem is again reduced to the system of algebraic equations (30). But its matrix is about three times larger than for the same nodal network in a potential theory problem, has a wider band, and is less sparse. This will make computations more costly.

Application of the numerical method to singularities in plane elasticity

There is a number of unanalyzed singular problems in plane elasticity, especially in combinations of cracks, notches and bonded dissimilar materials. Analytical solutions are complicated and a great effort is being made to solve various cases one by one, as can be seen in the recent literature. The present method of numerical solution can be useful in all these problems, too. The technique of Klein[8] and Williams[13] reduces the plane problem to an ordinary rather than a partial differential equation. The system of equations (30) arising in numerical solution is therefore of a much smaller size, and usually also narrowly banded, which substantially simplifies computations according to methods $A$, $B$ or $C$ outlined previously.
CONCLUSION

Three-dimensional harmonic functions near termination or intersection of gradient singularity lines can be determined by the general numerical method presented herein. Knowledge of the near field will enable the development of finite elements comprising the singularity.

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so dass \( u \sim (pr)^r \) nahe dem Strahl für jedes endliche \( r \). \( U \) ist eine unbekannte Funktion von \( \theta, \varphi \), für die eine partielle Differentialgleichung mit Grenzbedingungen, besonders die an den Singularitätsstrahlen, und ein Variationsprinzip abgeleitet werden. Weil Gradient \( U \) nicht-singular ist, ist eine numerische Lösung möglich, verwendend, zum Beispiel, die Netzverfahren oder die Methode der endlichen Elementen. Dies reduziert das Problem auf das Finden von \( \lambda \) des kleinsten reellen Teiles, der die Gleichung \( \text{Det} (A_{ii}) = 0 \) befriedigt, wo \( A_{ii} \) eine große Matrix ist, deren Koeffizienten linear von \( \mu = \lambda(\lambda + 1) \) abhängen. Im Allgemeinen sind \( \lambda \) und \( A_{ii} \) komplex.

Lösungen können entweder durch Reduktion auf ein gewöhnliches Matrizeigenwertproblem für \( \mu \) erhalten werden, oder durch aufeinanderfolgende Umwandlungen auf nicht-homogene lineare Gleichungssysteme. Untersuchungen an Rechenanlagen bestätigten die Durchführbarkeit der Methode und zeigten, dass sehr genaue Resultate erhalten werden können. Es werden Lösungen für Risse und Kerben vorgelegt, die an einer Ebene oder Kegeloberfläche enden, und für Risse, die quer an einer Halbraumoberfläche enden. In diesen Fällen ist \( \lambda \) reell und die Singularität ist immer schwächer (\( \lambda > p \)) als an der die Singularitätslinie und kann sogar verschwinden (\( \lambda > 1 \)). Weiter wurden elastische Spannungen unter einem keilförmigen starren gleitenden Stempel oder an der Ecke einer Risskante analysiert, wie auch harmonische Funktionen an dreiseitigen pyramidalen Kerben. Es wurde gefunden, dass \( \lambda > p \) hier vorkommt. Es wurde auch eine einfache analytische Lösung für eine Klasse von Spezialfällen gefunden und wurde verwendet, um einige der numerischen Resultate zu prüfen.

Sommario—Si assume che la funzione armonica \( u \), vicino al punto 0 da cui emana un raggio con singularità singola, sia dominata dal termine \( r^pU \) dove \( r = \) la distanza dal punto 0, \( p \) è una costante nota e \( \rho \) è una funzione scelta delle coordinate sferiche angolari \( \theta, \varphi \) tale che \( u \sim (pr)^r \) vicino al raggio per ogni valore finito di \( r \). \( U \) è una funzione non nota di \( \theta, \varphi \) per la quale vengono derivati un'equazione a differenziali parziali con condizioni allo strato limite, specialmente quelle sui raggi con singularità, ed un principio variazionale. Poiché \( \nabla U \) non è singolare, una soluzione numerica è possibile, usando per esempio il metodo delle differenze finite o quello degli elementi finiti. Le soluzioni possono venir ottenute per riduzione al problema del valore speciale di una matrice normale per \( \mu \), 0 per conversioni successive a sistemi di equazioni lineari non omogenei. Degli studi effettuati su calcolatori hanno confermato la praticabilità di questo metodo e hanno dimostrato che si possono ottenere risultati molto accurati. Vengono presentate soluzioni per fenditure e intagli terminanti su una superficie piana o conica, e per fenditure terminanti obliquamente sulla superficie di un semispazio. In questi casi \( \lambda \) è reale e la singularità è sempre più debole (\( \lambda > p \)) che sulla linea di singularità e può perfino scomparire (\( \lambda > 1 \)). Sono state analizzate inoltre le sollecitazioni elastiche sotto uno stampo scorrevole rigido a forma di cuneo o su un angolo del bordo della fenditura, come pure le funzioni armoniche su intagli piramidali a tre facce. È stata inoltre trovata una semplice soluzione analitica per una classe di casi speciali, e la soluzione è stata usata per controllare alcuni dei risultati numerici.

Абстракт—Принято, что гармоническая функция \( u \) вблизи точки 0, от которого излучается один особенность-луч, подвержен к этому члена \( r^pU \), где \( r \) = расстояние от точки 0, \( p \) = известное постоянное, \( \rho \) = выбор функции от углов сферических координат \( \theta, \varphi \), так что \( u \sim (pr)^r \) вблизи луча для любого конечного значения \( r \). При этом \( U \) есть неизвестная функция от \( \theta, \varphi \), для которого дан вывод частного дифференциального уравнения с граничными условиями, в частности, для тех на особых качествах, а также выводится вариационный принцип. Так как \( grad U \) есть неособенность, возможно получить численное решение, применяя например методы конечных разностей или адаптированных методы. Поэтому проблема сводится к нахождению \( \lambda \) для наименьшей реальной части, удовлетворяющей уравнению \( \text{Det} (A_{ij}) = 0 \), где \( A_{ij} \) — большая матрица, коэффициенты которой зависят линейно от \( \mu = \lambda(\lambda + 1) \). Все это \( A_{ij} \) — комплексными. Решения можно получить или приведение проблема собственного значения стандартной матрицы относительно \( \mu \), или исследуя преобразованный систему нелинейных уравнений. Из исследований компьютером подтверждена практическая осуществимость метода, а также возможность получения высокоточных результатов. Представлены решения для трещин и надрезов, ограничивающихся плоизкая или конической поверхности, и для трещин, ограничивающихся косое у полупространственной поверхности. Во всех случаях \( \lambda \) есть реальное, а особенность всегда более слабое (\( \lambda > p \), чем значение ее на особенности линии, возможно и исчезновение ее (\( \lambda = 1 \)). Кроме того, проанализированы напряжения под кин-образным, жестким, скользящим штампом, или в угле края трещины, и гармонические функции трехграниных пирамидальных надрезов. Здесь установлено, что \( \lambda < p \). Получено и простое аналитическое решение для одного класса частных случаев, примененное для проверки некоторых численных результатов.