MECH. RES. COMM. Vol.4(5), 353-366, 1977. Pergamon Press. Printed in USA.

STABILITY CONDITIONS FOR PROPAGATION OF A SYSTEM OF CRACKS IN A BRITTLE SOLID

Zdeněk P. Bažant and Hideomi Ohtsubo*
Department of Civil Engineering, Northwestern University,
Evanston, Illinois 60201

(Received 11 February 1977; accepted for print 26 August 1977.)

Introduction

In one proposed geothermal energy scheme [1,2], a large vertical main crack is produced in a hot dry rock mass by hydraulic fracture. To be able to remove heat from rock mass which is remote from the crack face, it is necessary to induce by cooling a secondary crack system normal to the wall of the main crack. Significant heat removal is possible only if the opening of secondary cracks is sufficient to allow rapid water circulation in them. The crack opening is wider, the larger is the spacing of cracks. The rate $\dot{H}$ of heat removal from secondary cracks by non-turbulent water circulation is roughly proportional to w^3/h, where w = width of cracks and h = their spacing; w is, in turn, proportional to h, and so $\dot{H} \sim h^2$. Likewise, crack spacing is of importance when dealing with shrinkage cracks in reinforced concrete, for the opening of such cracks has a decisive effect on the rate of corrosion of the embedded steel reinforcement and on the shear transfer capability of aggregate interlock on rough crack surfaces. Other problems in which crack spacing is of interest include the vertical cracking of lava beds extruded and solidified at ocean floor [3], as well as cracking of mud flats and permafrost soils caused by drying [4,3].

Cooling of a homogeneous brittle elastic halfspace may be expected to produce a system of equally long parallel equidistant cracks normal to the surface. However, crack spacing is not unique according to the Griffith criterion, and also other equilibrium solutions in which the length alternates from one crack to another are possible. This suggests investigation of uniqueness and stability. It seems that stability questions have so far been considered only with regard to the propagation of a single crack and its direction of propagation [5-7] (Sih's criterion of maximum strain energy density, criterion of maximum energy release rate). This paper attempts to lay down foundations of stability analysis of a system of cracks for each of which the propagation direction is known. This problem is much less difficult than the problem of crack direction.

---

*On leave from Department of Naval Architecture, University of Tokyo, Bunkyo-Ku, Tokyo, Japan

Scientific Communication

Conditions of Stability of a General Crack Configuration

Consider first the general case of a brittle elastic body which contains a number of cracks of arbitrary shape (Fig. 1a). For the sake of simplicity, assume that the body is in a state of either plane strain or plane stress, and that propagation of the cracks is governed only by Mode I (opening mode) stress intensity factors, K_i [6], where subscript i refers to the i^{th} crack tip, i = 1, 2, ...N. Also, assume that the cracks do not branch and that they propagate in given directions along straight or curved trajectories. Let a_i denote the length of crack up to its tip (Fig. 1a).

It is well known that the condition of stability of equilibrium of a single critical crack of length a_i is

$$\partial K_i/\partial a_i < 0 \qquad (i = 1, 2,\ldots N) \tag{1}$$

This holds for a general elastic body, and because a body with many cracks of which only one extends is a special case of a general elastic body with one extending crack, this condition also represents a necessary condition of stability of a crack system. It is not at all clear, however, whether Eq. (1) represents a sufficient condition, i.e., whether there are other conditions that have to be satisfied to assure stability.

To investigate equilibrium and stability, it is necessary to consider the work, W (more precisely, Helmholtz free energy), that would have to be supplied to the body in order to extend the cracks;

$$W = U(a_1,a_2,\ldots a_N;D) + \sum_{i=1}^{N} \int_0^{a_i} 2\gamma_i da_i' \tag{2}$$

Here U = elastic strain energy of the body, $2\gamma_i$ = specific energy of extension of the i^{th} crack; and D = loading parameter. In particular, D will represent here the penetration depth of cooling. If yielding and microcracking near the advancing crack tip were absent and the crack surfaces were not rough but perfectly plane, γ_i would equal the surface energy of the material. But these effects are always present and often they dissipate much energy; then γ_i is a constant which is much higher than the surface energy.

Consider now that the crack tips number i = 1,...m extend ($\delta a_i > 0$), the cracks numbered m+1,...n close and shorten ($\delta a_i < 0$), and the crack tips numbered n+1,...N remain stationary ($\delta a_i = 0$); $0 \le m \le n \le N$. This includes the case m = n when no crack closes, and the case n = N when no crack remains immobile. The work, ΔW, that would have to be supplied in order to change the crack lengths by δa_i (at constant loading parameter D and for applied loads doing no work) is a function of δa_i. This function must admit Taylor series expansion, i.e.,

$$\Delta W = \delta W + \delta^2 W + \ldots ; \quad \delta W = \sum_{i=1}^{m} \left(\frac{\partial U}{\partial a_i} + 2\gamma_i \right) \delta a_i + \sum_{j=m+1}^{n} \frac{\partial U}{\partial a_j} \delta a_j \tag{3a}$$

$$\delta^2 W = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 U}{\partial a_i \partial a_j} \delta a_i \, \delta a_j + \sum_{i=1}^{m} \frac{\partial \gamma_i}{\partial a_i} (\delta a_i)^2 = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} W_{ij} \delta a_i \delta a_j \tag{3b}$$

in which δW and $\delta^2 W$ are the first and second variations; and

$$W_{ij} = W_{ji} = \frac{\partial^2 U}{\partial a_i \partial a_j} + 2 \frac{\partial \gamma_i}{\partial a_i} \delta_{ij} H(\delta a_i) \quad (i,j = 1,\ldots n) \tag{4}$$

where δ_{ij} = Kronecker delta and H = Heaviside step function, i.e., $H(\delta a_i) = 1$ where $\delta a_i > 0$ and $H(\delta a_i) = 0$ when $\delta a_i < 0$. Usually the fracture properties of the body can be considered homogeneous, and then $\partial \gamma_i / \partial a_i = 0$.

For the cracks to change their length in an equilibrium manner, δW must vanish for any δa_i. It is necessary to distinguish whether a crack extends ($\delta a_i > 0$) or closes ($\delta a_i < 0$). According to Eq. (2), $\delta W = 0$ occurs if, and only if

$$\text{for } \delta a_i > 0: \; -\frac{\partial U}{\partial a_i} = 2\gamma_i ; \quad \text{for } \delta a_i < 0: \; \frac{\partial U}{\partial a_i} = 0 . \tag{5}$$

Eq. (5) includes the well-known Griffith fracture criterion. Note that the strain energy release rate is $-\partial U / \partial a_i$.

An equivalent form of Eq. (5) can be given in terms of the stress intensity factor, $K_i = \lim (\sigma\sqrt{2\pi r})$ for $r \to 0$ where r = distance from the crack tip and σ = transverse normal stress straight ahead of the crack. It is well known [6] that for plane strain $\partial U / \partial a_i = -K_i^2 / E'$ with $E' = E/(1-\nu^2)$ where E = Young's modulus, ν = Poisson ratio. Thus, Eq. (5) is equivalent to

$$\text{for } \delta a_i > 0: \; K_i = K_{c_i} ; \quad \text{for } \delta a_i < 0: \; K_i = 0 \tag{6}$$

where $K_{c_i} = (2\gamma_i E')^{\frac{1}{2}}$ = critical value of the stress intensity factor = fracture toughness of the material. Using $\partial U / \partial a_i = -K_i^2 / E'$, one may write

$$\frac{E'}{2} \frac{\partial^2 U}{\partial a_i \partial a_j} = -K_i \frac{\partial K_i}{\partial a_j} = -K_j \frac{\partial K_j}{\partial a_i} \tag{7}$$

Having stated the conditions of equilibrium, it is natural to ask whether the equilibrium configuration is stable. The crack system is said to be

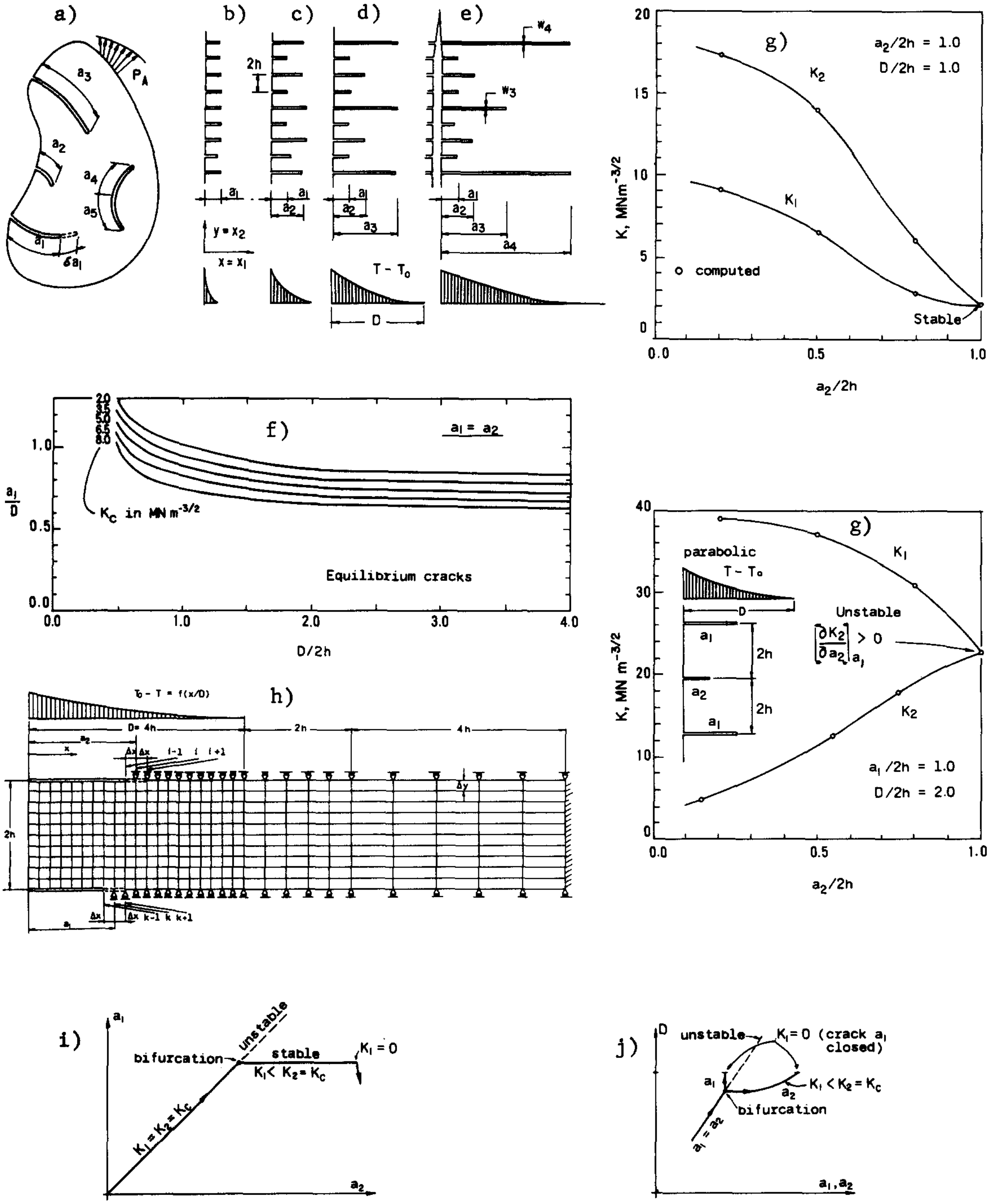

FIG. 1 General and Special Crack Systems Investigated (a-e), Some Numerical Results (f,g) with Grid Used (h; 2h = 1m), and Bifurcation of Equilibrium Path (i,j)

stable if and only if no a_i can change without changing the loads (or D). Thus, stability is ensured if and only if the work ΔW that is done at any admissible crack length increments δa_i is positive, for if this work is not done δa_i cannot occur. On the other hand, if $\Delta W < 0$ for some δa_i, energy is released, and when a release of energy is possible, changes δa_i will occur spontaneously, ΔW being transformed into kinetic energy and ultimately dissipated as heat (which follows from the second law of the thermodynamics).

One well-known unstable situation arises when $K_i > K_{c_i}$ (or $-\partial U/\partial a_i > 2\gamma_i$) for $\delta a_i > 0$. Indeed, $\delta W < 0$ for $\delta a_i > 0$, and so $K_i > K_{c_i}$ is impossible. Similarly, the case $K_i < 0$ (or $-\partial U/\partial a_i < 0$) for $\delta a_i < 0$ is also unstable. Therefore, with regard to non-negativeness of δW, it is necessary that $0 \leq -\partial U/\partial a_i \leq 2\gamma_i$ or $0 \leq K_i \leq K_{c_i}$ at all times. Combining the foregoing conditions and Eq. (6), it follows that with regard to the first variation, δW, only the following variations δa_i are admissible:

$$\text{for } K_i = K_{c_i}\text{: } \delta a_i \geq 0\text{; for } K_i = 0\text{: } \delta a_i \leq 0 \tag{8a}$$

$$\text{for } 0 < K_i < K_{c_i}\text{: } \delta a_i = 0 \tag{8b}$$

If $\delta W = 0$, stability will be ensured if

$$2\delta^2 W = \sum_{i=1}^{n} \sum_{j=1}^{n} W_{ij}\, \delta a_i\, \delta a_j > 0 \text{ for any admissible } \delta a_i. \tag{9}$$

Conversely, instability occurs if $\delta^2 W < 0$ for some admissible choice of δa_i. The admissible increments δa_i are given by Eq. (8). If matrix W_{ij} is positive definite, stability is assured. However, if W_{ij} is not positive definite, the crack configuration may or may not be unstable. It is unstable if $\delta^2 W < 0$ at $K_i = K_{c_i}$ or $K_i = 0$ for some admissible δa_i. Critical state occurs when $\delta^2 W = 0$ at $K_i = K_{c_i}$ or $K_i = 0$ for some admissible δa_i.

Array of Parallel Cooling Cracks Penetrating a Halfspace

Consider now a homogeneous isotropic elastic halfspace which is initially (at time $t = 0$) at constant temperature, $T = T_o$, and is then cooled at the surface $x = 0$ to temperature T_s. This produces an array of straight parallel equidistant cracks normal to the surface (Fig. 1 b-d). The temperature field is assumed to have the form $T - T_o = f(x/D)\,(T_s - T_o)$ where $D = D(t)$ = penetration depth of cooling. If all heat is transferred by conduction through the solid, one has $f(\xi) = \text{erfc}\ \xi = 2\int_{\xi}^{\infty} \exp(-\eta^2)\, d\eta/\sqrt{\pi}$, $\xi = \sqrt{3}\, x/D$, $D = \sqrt{12\, ct}$, c = heat diffusivity. Because T is constant along lines parallel to the surface, it is logical to assume a periodic pattern of crack lengths. Accordingly, consider that every other crack has one length, a_2, and the cracks inbetween have another length, a_1; $a_2 \geq a_1$ (Fig. 1c). Cracks of equal lengths ($a_2 = a_1$) represent one possible equilibrium state. These states (not all necessarily stable) are plotted in Fig.1f on the basis of finite element calculations for $T_o - T_s = 100^oC$ (with error function T-profile), $\alpha = 8 \times 10^{-6}$ per ^oC (linear thermal expansion coefficient), $E = 37600\ MN/m^2$, and $\nu = 0.305$ (2h=1m).

Various interesting properties of the parallel crack system can be analyzed even without numerical solutions of K_1 and W_{ij}. Since K_c is a constant, a sufficient condition of stability is the positive definiteness of matrix $W_{ij} = \partial^2 U/\partial a_i \partial a_j$, which requires that

$$W_{22} = W_{11} > 0 \text{ and } \begin{vmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{vmatrix} > 0 \qquad (10)$$

A critical state, corresponding to a bifurcation point on the basic equilibrium path $a_1 = a_2$, would arise if $\delta^2 W = \frac{1}{2} \Sigma_i (\Sigma_j W_{ij} \delta a_j) \delta a_i = 0$ for some admissible δa_i. This condition is satisfied if $\Sigma_j W_{ij}\ \delta a_j = 0$ or

$$\begin{aligned} W_{11}\ \delta a_1 + W_{12}\ \delta a_2 &= 0\ , \\ W_{21}\ \delta a_1 + W_{22}\ \delta a_2 &= 0 \end{aligned} \qquad (11)$$

in which $W_{11} = W_{22}$ and $W_{12} = W_{21}$. Setting the determinant zero, one has $W_{11} W_{22} - W_{12}^2 = 0$, and using Eq. (7) this becomes $(-K_2\ \partial K_2/\partial a_2)^2 - (-K_2\ \partial K_2/\partial a_1)^2 = 0$. Noting that $W_{11} = W_{22}$ and $K_2 = K_1$ for $a_2 = a_1$, one has $W_{22}^2 - W_{12}^2 = 0$, yielding $(\partial K_2/\partial a_2)^2 - (\partial K_2/\partial a_1)^2 = 0$ as a condition of possible critical state. Admissibility of the corresponding eigenvector (δa_1, δa_2) must be checked, though. Since $W_{11} = W_{22}$, $W_{12} = W_{21}$, and at the same time $W_{22} = \pm W_{12}$, Eq. 11 suggests

$$\delta a_1/\delta a_2 = \pm 1 \qquad (12)$$

as possible critical states.

The plus sign in Eq. (12) yields $\delta a_1 = \delta a_2$. In this case Eq. (11) reduces to $(W_{11} + W_{12})\ \delta a = 0$ where $\delta a = \delta a_1 = \delta a_2$. Noting that, by virtue of the chain rule of differentiation, $W_{11} + W_{12} = (\partial W_1/\partial a_1)\partial a_1/\partial a + (\partial W_1/\partial a_2)\partial a_2$

$/\partial a = dW_1/da$ with $W_1 = \partial W/\partial a_1 = -K_1^2/E'$, one concludes that Eq. (11) degenerates into the condition $(dK_1^2/da)\ \delta a = 0$ or $dK_1/da = 0$, which is a condition of instability of the basic equilibrium path, $a_1 = a_2$. The condition of stability of this path is $dK_1/da < 0$ or $dW_1/da > 0$, which is analogous to the well-known stability condition for a single crack (Eq. 1).

Bifurcation of the equilibrium path would be obtained if Eq. (12) admitted the minus sign, i.e., $\delta a_1 = -\delta a_2$. For Eq. (11) to allow this, W_{11} and W_{12}, and thus also $\partial K_2/\partial a_2$ and $\partial K_2/\partial a_1$, would have to be of the same sign. Of these, $\partial K_2/\partial a_2$ must be negative, or else a critical state of another type, associated with the first condition in Eq. (10) would precede this bifurcation. As far as $\partial K_2/\partial a_1$ is concerned, the finite element calculations described in the sequel indicated that for the present crack system with $a_1 = a_2$ it is always negative, which also agrees with some intuitive considerations.

Since in the present case both $\partial K_2/\partial a_2$ and $\partial K_2/\partial a_1$ are negative, it appears that Eq. (11) would indeed admit the minus sign. However, this means that either δa_1 or δa_2 must be negative, and this violates condition (8) because $K_1 = K_2 = K_c$. Hence, a bifurcation of the type given by Eqs. (11) and (12) is seen to be impossible.

The remaining possible critical state according to Eq. (10) is given by the condition $W_{22} = W_{11} = 0$ or

$$[\partial K_2/\partial a_2]_{a_1 = \text{const.}} = 0 \ . \tag{13}$$

The associated second variation is $\delta^2 W = \frac{1}{2} W_{22}\ (\delta a_2)^2$ with $\delta a_1 = 0$, and the bifurcation ("instability") mode is

$$\delta a_2 > 0, \ \delta a_1 = 0 \ . \tag{14}$$

This mode also represents a bifurcation point on the basic equilibrium path $a_1 = a_2$ (Fig. 1i). According to Eq. (8b), K_1 does not have to remain equal to K_c but it may decrease, i.e., the tip of the crack a_1 may be unloading. In fact K_1 ought to decrease after bifurcation since extension of crack a_2 should have a non zero effect on K_1. It might be also of interest to note that if $\partial K_2/\partial a_2 > 0$ (or $W_{11} = W_{22} < 0$), then $W_{11} W_{22} - W_{12}^2$ or det (W_{ij}) is always negative.

Further light may be shed on the problem if the path of equilibrium states is regarded as a function of parameter D (cooling penetration depth);

see (Fig. 1j). Denoting $W_i = \partial W/\partial a_i$ where W is given by Eq. (2), the equilibrium path is distinguished by the conditions $W_i = 0 (i = 1,2)$. Derivatives W_i at equilibrium states are functions of a_1 and a_2. However, by contrast to W in Eq. (2), a_i are generally not independent of D because along the equilibrium path the crack lengths a_1 and a_2 depend on D. Thus, W_i along the equilibrium path are implicit functions of D; i.e.,

$$[\partial W/\partial a_i]_{D=\text{const.}} = W_i\ [a_1(D),\ a_2\ (D)] = 0 \quad (i = 1,2) \tag{15}$$

Assume that on the basic equilibrium path $a_2 = a_1$ there is a critical point (bifurcation point) correspdnding to $D = D_0$ (Fig. 1j). Functions W_i ought to admit Taylor series expansions at $D = D_0$. The cracks should also be in equilibrium at adjacent states with D sufficiently close to D_0. If both δa_1 and δa_2 are assumed to be positive, then W_i would have to be constant for all such D-values. Consequently, $dW_i/dD = 0$, $d^2W_i/dD^2 = 0$, etc., must be true at $D = D_0$. This yields

$$\sum_{j=1}^{2} \left[\frac{\partial W_i}{\partial a_j}\right]_{D_0} a'_j = 0 \quad (i = 1,2) \tag{16}$$

$$\sum_{j=1}^{2} \sum_{k=1}^{2} \left[\frac{\partial^2 W_i}{\partial a_j \partial a_k}\right]_{D_0} a'_j\, a'_k + \sum_{j=1}^{2} \left[\frac{\partial W_i}{\partial a_j}\right]_{D_0} a''_j = 0 \ (i = 1,2) \tag{17}$$

where $a'_j = da_j/dD$ along the equilibrium path at $D = D_0$. These equations represent conditions of continuing equilibrium, analogous to those which follow from the perturbation method of structural stability theory [8]. If they admit solution for D → const., then a critical state is reached. Setting $a'_i \sim \delta a_i$, the condition in Eq. (16) is obviously identical to Eq. (11), which is a bifurcation of a type that is inadmissible. However, there exists no reason why a higher-order bifurcation governed by Eq. (17) could not take place. Since Eq. (16) is not satisfied, such higher-order bifurcation would have to conform to $a'_1 = a'_2$, i.e., to the increments for the basic path ($\delta a_1 = \delta a_2$) and would have to take place at increasing D (i.e., at increasing cooling penetration depth). Therefore a'_1 and a'_2 in Eq. (17) must be equal, and a higher-order bifurcation, with the secondary path being tangent to the basic path ($a_1 = a_2$) at the bifurcation point, would occur if Eq. (17) admitted a solution with $a''_1 \neq a''_2$.

Eqs. (16) and (17) were written under the assumption that both δa_1 and δa_2 are positive (and $K_1 = K_2 = K_c$). Consider now that $\delta a_2 > 0$ and $\delta a_1 = 0$, i.e., one crack stops growing. In this case Eqs. (16) - (17) still represent a possible condition of critical state, but in view of Eq. (8b) it is also possible that only K_2 remains at its critical value K_c while K_1 decreases below K_c. In fact, since bifurcation given by Eq. (16) was shown to be inadmissible (according to Eq. 8), it is not possible that K_1 and K_2 remain equal K_c during bifurcation. Hence, it is necessary that

$$\partial K_1/\partial a_2 < 0 \qquad (18)$$

during bifurcation. Thus $\partial W_1/\partial D$ cannot be zero at $D = D_0$, and only the condition $\partial W_2/\partial D = 0$ applies as a condition of continuing equilibrium, yielding

$$\frac{\partial W_2}{\partial a_2} a_2' = 0 \qquad (19)$$

where $a_2' = da_2/dD$. Eq. (19) represents the condition of a criticial state, provided that it holds true in the limit for $D \rightarrow$ const. This case is identical with Eq. (13) derived from the condition $\delta^2 W > 0$.

It is seen that the present variational stability analysis yields the condition in Eq. 13. This condition is identical to the elementary condition in Eq. 1, which is immediately obvious even without the variational analysis. It remains to be seen whether, for the particular crack system at hand, $\partial K_2/\partial a_2$ can indeed change its sign. Therefore, some finite element computations have been carried out. The grid in Fig. 1f, composed of four-node quadrilateral elements, formed by condensing a block of four constant-strain triangles, was used. The derivatives of potential energy ($\partial W/\partial a_1$, $\partial^2 W/\partial a_1 \partial a_2$, etc.) were calculated from their finite difference approximations, using the potential energy (Eq. 2) in the whole grid for various crack length a_1 and a_2. The temperature profile was approximated as parabolic, and the Young's modulus $E = 37{,}600$ MN/m^2, the Poisson ratio $\nu = 0.305$ and the linear thermal expansion coefficient $\alpha = 8 \times 10^{-6}$ per $^{\circ}$C (all typical of granite) were used. Some of the results are shown in Fig. 1g, in which the intersections of curves K_1 and K_2 represent equilibrium states if $K_c = 22.8$ MNm$^{-3/2}$. In one of these intersections the slope of the curve of K_2 versus a_2 is positive, which violates Eq. 13 and indicates that the equilibrium is unstable. This proves that instability due to the violation of Eq. 13 is indeed possible. However, it is not at all clear from Fig. 1g that the instability governed by Eq. 13

(or Eq. 1) is the case which controls. Without a complete solution of the crack problem, the higher-order bifurcation with common tangent (Eq. 17) cannot be ruled out in a general case. In fact, on the basis of some crude finite element calculations of the equilibrium path it was initially thought that in the present problem this latter type of bifurcation (Eq. 17) occurred before the bifurcation given by Eq. 13 and that it, therefore, controlled. On the other hand, S. Nemat-Nasser intuitively expected the elementary condition in Eq. 1 (or Eq. 13) to control. Following the present finite element calculations which proved that the bifurcation due to $\partial K_2/\partial a_2$ is possible, L. M. Keer et al.[1] demonstrated by a complete analytical solution of the problem based on singular integral equations that the bifurcation due to $\partial K_2/\partial a_2$ turning zero is not merely a possibility but a phenomenon which does actually occur. Simultaneously, refined finite element calculations were being performed by the authors together with K. Aoh of University of Tokyo (to be reported separately) and these calculations led to the same conclusion. These calculations and the work of Keer et al.[1] also indicated that (for granite and for a temperature drop of 100°C) the bifurcation is reached when a_1 and a_2 are roughly equal 1.8 times crack spacing.

Since K_1 was shown to decrease after bifurcation, the equilibrium path must have a straight horizontal segment of finite length after the bifurcation point; see (Fig. 1i). Assuming that the trend remains unchanged, the segment would end by a state in which $K_1 = 0$, $K_2 = K_c$, and subsequently crack a_1 would begin to close, $\delta a_1 < 0$. The fact that the bifurcation for $\partial K_2/\partial a_2 = 0$ occurs at constant a_1 means that in the plot of D versus a_2 the equilibrium path must have a horizontal tangent at the bifurcation point; see (Fig. 1j). If the path continued as a straight horizontal line beyond the bifurcation point, there would be infinitely many equilibrium crack lengths a_2 corresponding to the same D and same a_1, and this would require the potential energy release rate to be independent of D. Obviously, this is impossible. Hence, the path of D versus a_2 after the birfurcation point (Fig. 1j) must curve either upward or downward. If it curved downward, it would mean that a longer crack a_2 corresponded to a smaller cooling penetration depth D (at constant a_1), i.e.,

---

[1] Manuscript "Growth and Stability of Thermally Induced Cracks in Brittle Solids", communicated to the authors by L. M. Keer, S. Nemat-Nasser and K. Parihar of Northwestern University in September 1976.

equilibrium extension of cracks a_2 would require withdrawal rather than supply of energy. Therefore, if adjacent equilibrium states exist after the bifurcation point, their path in Fig. 1j must curve upward, i.e., with increasing penetration depth of cooling the leading cracks must get longer, not shorter, as may naturally be expected. It must be emphasized, however, that the shape of the post-bifurcation paths in Figs. 1i and j has been deduced here only qualitatively. Prior to formulating this qualitative deduction, the post-bifurcation paths of the type shown in Figs. 1i and j were obtained quantitatively by Keer et al.[2] by means of a singular integral equation approach.

The possibility that every other crack (a_1) might close is suggested by empirical observations of drying shrinkage cracks, e.g., in mud flats or in concrete. This was also suggested by the behavior of cracks in an experiment at Los Alamos Scientific Laboratory [9] in which a concrete slab was cooled by liquid nitrogen and hexagonal crack patterns at the surface were made easily observable by formation of nitrogen bubbles on evaporation from open cracks. The possibility of crack closing was also evidenced at the outset of the finite element work by the fact that for a sufficiently large value of $a_2/2h$ the normal stress σ_y along the line of symmetry between two adjacent cracks (a_1 and a_2) became compressive up to a certain depth from the surface. This showed that on this line of symmetry it is possible to introduce a closed crack up to a certain depth without causing any change of the stress state in the entire elastic half-space. It follows that shorter closed cracks may exist between opened leading cracks and this suggests that every other crack (cracks a_1) may close after bifurcation. However, it does not follow theoretically that every other crack must close. Keer et al.[2] demonstrated by analytical solution of the problem that cracks a_1 must indeed close after bifurcation.[3] So, it is certain that the spacing of the opened (leading) cracks doubles whenever the ratio of the opened cracks to their depth reaches a certain fixed value (about 1.8). This type of behavior, which has been suggested before on the basis of empirical observations [3, 4], is favorable for the afore-mentioned scheme for extracting geothermal heat, because it would mean that the width of the opened cracks is proportional to the penetration depth

---

[2] Ibid.

[3] The fact of closing is distinguished from the fact that K_1 must decrease after bifurcation, which is here established by Eq. (18).

of cooling and that the flux of water through the cracks is proportional to the square of the crack depth. These crude projections may, however, be greatly modified when the effect of water circulation on the temperature profile and possible development of eddy currents in the cracks is taken into account.

Conclusions

1. A system of cracks is stable if and only if the work ΔW needed to produce any admissible crack length increments is positive definite. This is assured if the second variation $\delta^2 W$ of W is positive definite for all admissible crack length increments.

2. A system of identical parallel equidistant cooling cracks propagating into a halfspace can exhibit instability. The critical state is indicated by the vanishing of the derivative of the stress intensity factor of cracks a_2 with regard to a_2 at constant a_1, which is the same as the well-known stability condition for a single crack considered separately. At the critical state every other crack, of length a_1, ceases to grow ($\delta a_1 = 0$) while the intermediate cracks of length $a_2 = a_1$ continue to advance ($\delta a_2 > 0$) at constant temperature. The path of the equilibrium states plotted in the space (a_1, a_2) or in the space (a_i, D) then bifurcates (D = penetration depth of cooling). After bifurcation, cracks a_1 gradually close. The plot of a_1 versus D has a horizontal tangent at bifurcation point.

3. Without numerical results, it cannot be ruled out that a higher-order bifurcation, in which the bifurcating path and the main path $a_1 = a_2$ have a common tangent, might be also possible for a system of parallel cracks.

4. Vanishing of the determinant of the second derivatives of work W with respect to a_1 and a_2 does not cause bifurcation in a system of parallel cracks because associated eigenvector (δa_1, δa_2) indicates negative δa_1.

Remark. - Equilibrium path bifurcation is characteristic of a perfect crack system. An imperfect crack system, e.g., a system of cracks which are almost but not exactly equidistant, would probably not exhibit bifurcation of equilibrium path, just like an imperfect column does not. However, such a case would be much more difficult to solve.

Acknowledgment

Support of the U. S. National Science Foundation under Grants AER 75-00187 and ENG 75-14848 to Northwestern University is gratefully acknowledged. The authors are grateful to Drs. S. Nemat-Nasser, L. M. Keer and K. S. Parihar for their many helpful comments and suggestions during the long period of gestation of this paper. The first author first communicated to them the present general variational analysis of stability early in 1976, and benefited substantially from the subsequent discussion with them. This led to the initial draft of this paper in July 1976, which contained the present conditions of stability and numerical results. Discussion of this draft with these colleagues greatly assisted in the preparation of the final manuscript.

References

1. M. C. Smith, R. L. Aamodt, R. M. Potter, D. W. Brown, Proc., 2nd United Nations Geothermal Energy Symposium, San Francisco, California (1975).
2. F. Harlow and W. Pracht, J. of Geophys. Res., 77, 7038 (1972).
3. C. R. B. Lister, Geophysics J. R. Astr. Soc., 39, 465-509 (1974).
4. A. H. Lachenbruch, J. of Geophys. Res., 66, 4273 (1961).
5. B. Cotterell, Intern. J. of Fracture, 2, 526-533 (1966).
6. J. F. Knott, Fundamentals of Fracture Mechanics, J. Wiley, New York (1973).
7. G. C. Sih (ed.), Methods of Analysis and Solutions of Crack Problems, Noordhoff International Publ., Leyden, The Netherlands (1973).
8. J. G. Croll and A. C. Walker, Elements of Structural Stability, John Wiley and Sons, New York (1972).
9. H. Murphy, R. M. Potter, Private Communication to Z. P. Bažant, Los Alamos Scientific Laboratory, Dec. 18, 1975.

Appendix

The fact that in a system of parallel cracks the bifurcation associated with the determinant condition in Eq. 10 is impossible is contingent upon (a) $\partial K_2/\partial a_1$ being negative, and (b) both cracks being at the point of extension. Prior to completing the finite element calculations which confirmed that $\partial K_2/\partial a_1$ is always negative, S. Nemat-Nasser intuitively suggested to the authors that it should always be so. Later it was thought that "it is generally true that an extension of a given crack accompanied by no increase in applied loads would result in a decrease of the stress intensity factor at other active cracks, because such an extension decreases the overall stiffness of the elastic body."[4] Subsequently, however, an example of a cracked structure for which $\partial K_2/\partial a_1$ is positive has been found; hence the sign of $\partial K_2/\partial a_1$ is not certain in advance, for the general case.

To show it, consider a horizontal simply supported continuous beam of constant cross section (of depth H) and two equal spans (of length L), loaded

---

[4] L.M. Keer et al., loc. cit.

in the middle of the left span by downward load P_1 and in the middle of the right span by equal but upward load $P_2 = -P_1$. The bending moments in the middles of the left and right spans are $M_1 = P_1L/4$ and $M_2 = -M_1$. Assume further that there are two vertical cracks, one reaching upward from the bottom of the cross section in the middle of the left span to depth a_1 from the bottom, and the second reaching downward from the top of the cross section in the middle of the right span to depth $a_2 = a_1$ from top. Assume also that $K_1 = K_2 = K_c$. Let now the crack length a_1 be increased by δa_1 while keeping a_2 and the loads constant. Increase of a_1 will cause the left span to become less stiff, and it will cause the left span to deflect downward. In a continuous beam, this must cause the right span to deflect upward, and because a_2 is constant, K_2 must increase, i.e. $\partial K_2/\partial a_1 > 0$. Alternatively, this may be also deduced by noting that the decrease of left span stiffness must cause the bending moments to redistribute so that M_1 would decrease and $|M_2|$ would increase; an increase of $|M_2|$ at constant a_2 must cause K_2 to increase.

Likewise, condition (b), namely that both cracks are on the verge of extension, does not have to always occur. Consider the same beam but with different loads P_1 and P_2 and with both cracks of lengths $a_1 = a_2$ emanating from the bottom of the cross section. Assume now that the beam is slender, so that K_1 and K_2 are proportional to M_1 and M_2, and that $a_1 \ll H$, $a_2 \ll H$, so that crack lengths have negligible effect on the stiffness of the spans. First, let loads $P_1 = P_2 = 1.0$ be applied. This causes equal moments, $M_1 = M_2 = L/4$, and assume that this creates equally long cracks $a_1 = a_2$ which are both critical, $K_1 = K_2 = K_c$. Subsequently, load P_2 is changed to $P_2 = 1.3$ and load P_1 is changed to $P_1 = 0.3$. This causes M_1 to become zero while M_2 remains unchanged ($M_2 = L/4$). So, $P_2 = 1.3$ and $P_1 = 0.3$ gives a state where $K_2 = K_c$ and $K_1 = 0$, crack a_1 being on the verge of extension and crack a_2 being on the verge of closing. For checking stability on this state, one must obviously consider $\delta a_2 \geq 0$ and $\delta a_1 \leq 0$ as the admissible δa_i.

In cases where condition (a) or (b) is reversed, the stability condition det $(W_{ij}) > 0$ cannot be dismissed a priori and must be evaluated to see whether or not it is satisfied for all admissible δa_i.