

ENDOCHRONIC INELASTICITY AND INCREMENTAL PLASTICITY

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Abstract—Endochronic and non-associated plastic formulations are compared by introducing an "inelastic stiffness locus", defined as the locus of all strain increments in the strain space which give the same magnitude of inelastic strain increments. For classical plasticity the locus is a straight line, while for endochronic formulations it is a circle, sphere or a quadratic surface (ellipsoid). Similarly to vertex hardening models and the deformation theory of plasticity, endochronic theory gives inelastic strain for strain increments tangential to the current loading surface, while plasticity gives perfectly elastic response. However, in contrast to vertex hardening, the endochronic inelastic strain for tangential strain increments is normal to the loading surface. Consequently, endochronic theory is stiffer than vertex hardening for this loading direction and is less prone to indicate instability. However, it is softer than plasticity. Among all possible constitutive relations, plasticity (without yield vertex) is least prone to indicate material instability, and so it is the least safe model to assume if test data are inconclusive as far as the type of constitution law is concerned.

Tangential linearization of the endochronic inelasticity is presented. The tensor of tangential moduli, with all of its components, depends continuously on the strain increment direction in the strain space. Endochronic analogs of the loading surface and of kinematic and isotropic hardening rules are indicated, and stress-induced anisotropy of the quadratic form defining intrinsic time increments is formulated. It is shown that for proportional loading an endochronic formulation can be readily converted to an equivalent plasticity formulation. The fracturing material theory in which the loading function depends on strain rather than stress is also analyzed and it is shown that its inelastic stiffness locus is similar as for plasticity.

Implications for material instability, and especially for stability of the response to pulsating loads of small amplitude, are discussed. By contrast to plasticity, but similarly to viscoplasticity, the endochronic inelasticity violates Liapunov-type stability conditions, but it meets a proper continuity condition. Refinements to satisfy both are possible, but questionable if one deals with materials such as geological materials, which are unstable or exhibit strain softening. Introducing unloading and reloading criteria and a certain type of kinematic hardening, the endochronic formulation may be refined so as to model cyclic strain accumulation yet satisfy Drucker's postulate for the hysteresis loops.

1. OBJECTIVE

Viscoplasticity with strain-rate dependent viscosity[1, 2], which has crystallized as endochronic theory[3-19], is now receiving considerable attention and is being employed with remarkable success for modeling the experimentally observed inelastic properties of certain materials, especially those in which the prevailing mechanism of inelastic strain is not plastic yield but microcracking or grain rearrangements with separations, as is characteristic of geological materials (soils, rocks, concrete)[4, 5, 8-10, 13-19]. Recently it has been discovered, however, that certain new, more sophisticated, plasticity formulations are capable of modeling the available experimental data for these materials nearly as well. Apparently, one faces a situation where the problem of identification of the constitutive relation from the test data available at present does not have a unique solution.

Therefore, rather than trying to fit further test data, an attempt will be made to compare the types of three-dimensional response which various formulations give, and to determine what are the *essential* differences between classical incremental plasticity (associated and non-associated), vertex hardening plasticity, and endochronic inelasticity.

A reader who might expect this effort to involve a good deal of thermodynamics must be warned that it will not be so. Application of thermodynamics provides for the constitutive relations important restrictions, which have essentially been worked out both for classical plasticity[20, 21, 11] and endochronic forms of viscoplasticity[1, 2, 6, 7, 11, 12]. However, the information furnished by thermodynamics is quite limited, and rather than further refining the

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rigorosity of thermodynamic treatment it seems to be more profitable to turn attention to those three-dimensional tensorial properties on which thermodynamics yields no information. These are the properties which result from the microstructural mechanism of inelastic deformation and are macroscopically manifested by the shape of the loading surfaces in stress and strain spaces and the tangential stiffness for strain increments in various directions. Analysis of these properties and their use for comparing endochronic and plastic formulations is the main objective of this paper (based on report [10]).

2. PROTOTYPE FORMULATION OF NON-ASSOCIATED INCREMENTAL PLASTICITY

Recently it has become clear that plasticity of many materials, especially geological materials (soils, rocks and concrete), is not associated with the yield surface by means of a normality rule and Drucker's postulate [22-25]. The deviation from normality seems to be due mainly to inelastic dilatancy and internal friction due to hydrostatic pressure. A simple way to handle it is to begin with associated stress-strain relations and then relax normality only as far as necessary, i.e. only as far as hydrostatic pressure p is concerned, as has been done by Rudnicki and Rice [26, 27]. The resulting stress-strain relations may be written as

$$de_{ij} = de_{ij}^e + de_{ij}^{pl}, \quad d\epsilon = d\epsilon^e + d\epsilon^{pl} \quad (1)$$

with

$$de_{ij}^e = \frac{1}{2G} ds_{ij}, \quad d\epsilon^e = \frac{1}{3K} d\sigma \quad (2)$$

$$de_{ij}^{pl} = s_{ij} \frac{d\mu}{\bar{\tau}}, \quad d\epsilon^{pl} = \frac{2}{3}\beta d\mu, \quad \bar{\tau} = \sqrt{\left(\frac{1}{2} s_{km} s_{km}\right)}. \quad (3)$$

For $(d\bar{\tau} + \beta' d\sigma)/(2h) \geq 0$:

$$d\mu = \frac{d\bar{\tau} + \beta' d\sigma}{2h}, \quad d\bar{\tau} = \frac{s_{km} ds_{km}}{2\bar{\tau}}, \quad (4a)$$

for $(d\bar{\tau} + \beta' d\sigma)/(2h) < 0$:

$$d\mu = 0. \quad (4b)$$

Here subscripts i, j, k, m refer to cartesian coordinates x_i ($i = 1, 2, 3$); $s_{ij} = \sigma_{ij} - \delta_{ij} \sigma/3$ = deviator of stress tensor σ_{ij} , $\sigma = \sigma_{kk}/3 = -p$ = hydrostatic stress, δ_{ij} = Kronecker delta, $e_{ij} = \epsilon_{ij} - \delta_{ij} \epsilon/3$ = deviator of (small, linearized) strain tensor ϵ_{ij} , $\epsilon = \epsilon_{kk}/3$ = volumetric strain component, e_{ij}^e , e_{ij}^{pl} , ϵ^e , ϵ^{pl} = elastic and plastic component of e_{ij} and ϵ ; $\bar{\tau}$ = stress intensity; G , K = elastic shear and bulk moduli, h = plastic hardening modulus, β = dilatancy factor, β' = coefficient of internal friction. Parameters h , β and β' are, in general, functions of σ_{ij} and eventually also ϵ_{ij} . When $\beta = \beta'$, the normality rule is satisfied, and for $\beta \neq \beta'$ it is not. If $\beta = \beta' = 0$, eqns (2)-(4) reduce to Prandtl-Reuss relations and are associated with von Mises-type yield surface.

For reader's convenience, a brief sketch of the derivation of the incremental plastic relations (2)-(4) for the case of normality ($\beta = \beta'$), may be given. It is important to realize that incremental plasticity rests on two basic hypotheses, which are reasonable but by no means necessary. One basic hypothesis is the existence of a scalar yield function, F , such that inelastic strain occurs if and only if $dF > 0$ [22] and F is independent of the inelastic strain. Equations (1)-(4) correspond to the form

$$F(\sigma_{ij}, H_k) = \bar{\tau} + g(\sigma) - H_k = 0, \quad \bar{\tau} = \sqrt{\left(\frac{1}{2} s_{ij} s_{ij}\right)} \quad (5)$$

where H_k = hardening parameter. Choosing $(\partial F/\partial H_k) dH_k$ to be negative when loading takes place, and noting that $(\partial F/\partial \sigma_{ij}) d\sigma_{ij} + (\partial F/\partial H_k)/dH_k = 0$, it is obvious that $d\sigma_{ij}(\partial F/\partial \sigma_{ij}) > 0$ when

plastic hardening occurs, and $d\sigma_{ij}(\partial F/\partial\sigma_{ij}) \leq 0$ when it does not. The second basic hypothesis is that the dependence of $d\epsilon_{ij}^{pl}$ upon $d\sigma_{ij}$ is linear [22]. Then, $d\epsilon_{ij}^{pl} \sim d\sigma_{ij}(\partial F/\partial\sigma_{ij})$ or

$$d\epsilon_{ij}^{pl} = g_{ij} \frac{\partial F}{\partial \sigma_{km}} d\sigma_{km} \quad (6)$$

where g_{ij} are some constants. Now, adopting Drucker's postulate $d\sigma_{ij} d\epsilon_{ij}^{pl} \geq 0$ [22–25], and comparing this inequality with the condition $d\sigma_{ij}(\partial F/\partial\sigma_{ij}) \geq 0$ for continued plastic loading, it follows that $d\epsilon_{ij}^{pl} \sim \partial F/\partial\sigma_{ij}$. Furthermore, comparison with eqn (6) yields $g_{ij} \sim \partial F/\partial\sigma_{ij}$, and so one may set $d\epsilon_{ij}^{pl} = (\partial F/\partial\sigma_{ij}) d\mu'$ with $d\mu' = (1/h)(\partial F/\partial\sigma_{km}) d\sigma_{km}$. For the special yield function in eqn (5), this may be written as

$$d\epsilon_{ij}^{pl} = \frac{\partial F}{\partial s_{ij}} d\mu' = s_{ij} \frac{d\mu}{\bar{\tau}}, \quad d\epsilon_{ij}^{pl} = \frac{\partial F}{\partial \sigma} \frac{d\mu'}{3} = \frac{2}{3} \beta' d\mu \quad (7)$$

$$d\mu = \frac{1}{2h} (d\bar{\tau} + \beta' d\sigma) \quad (8)$$

where $\beta' = dg(\sigma)/d\sigma$ = friction coefficient, $d\mu = d\mu'/2$, and h = plastic hardening modulus. The ratio of volumetric and deviatoric plastic strain increments for pure shear is $d\epsilon_{kk}^{pl}/d\gamma^{pl} = d\epsilon_{kk}^{pl}/2 d\epsilon_{ij}^{pl} = \bar{\tau}\beta'/s_{ij} = \beta'$ (because for pure shear $\bar{\tau} = s_{ij}$), which confirms that normality occurs when $\beta = \beta'$.

3. PROTOTYPE FORMULATION OF ENDOCHRONIC INELASTICITY

The basic concept in endochronic formulations is the characterization of inelastic strains in terms of one or several non-decreasing scalar variables whose increments depend on strain increments. This variable, which has been initially called reduced time [1, 2], is now generally known as intrinsic time. This term was introduced by Valanis [3], who was first to apply the concept successfully to complicated nonlinear behavior, particularly cyclic loading and cross-hardening of metals, and coined the Greek term "endochronic". The theory is most properly regarded in the context of viscoplasticity [1–3]; it is obtained as a special case of viscoplasticity with strain-rate dependent viscosity (introduced by Schapery [1]) if one imposes the requirement that for a strain rate approaching infinity the ratio of inelastic and elastic strain increment magnitudes must be neither zero nor infinite (see Appendix). The intrinsic time for time independent behavior may be geometrically interpreted as the length of the path traced by the states of the material in a strain space of suitable metric. A variable of this type has been in use since the early 1950s (Hill, Ilyushin, Rivlin and Pipkin, see [3, 4]). Thermodynamics of the viscoplastic constitutive relations based on intrinsic (or reduced) time has been formulated by Schapery [1, 2], Valanis [3, 7, 12] and others [11].

Endochronic formulations of inelastic behavior lack the concept of yield surface and suggest physical interpretations in terms of damage, microcracking, grain rearrangements and internal friction. Thus, endochronic formulations seem to be more suited for geological materials than for metals, in which the mechanism of inelastic strain is frictionless plastic slip (or dislocations). Therefore, the general form of the practical endochronic constitutive relation [4, 5, 8] which have met with great success in modeling geological materials exhibiting strain-softening, pressure sensitivity and inelastic dilatancy [5, 8, 9, 13–19] is chosen to serve as the prototype endochronic formulation. Restricting attention to time-independent deformations, we may write the constitutive relation in the form of eqn (1) in which

$$d\epsilon_{ij}^{pl} = s_{ij} d\zeta, \quad d\epsilon^{pl} = d\lambda \quad (9)$$

with

$$d\zeta = F_1(\sigma, \epsilon, \zeta) d\xi, \quad d\lambda = F_2(\sigma, \epsilon, \lambda) d\xi, \quad d\xi = \left(\frac{1}{2} d\epsilon_{ii} d\epsilon_{jj} \right)^{1/2}. \quad (10)$$

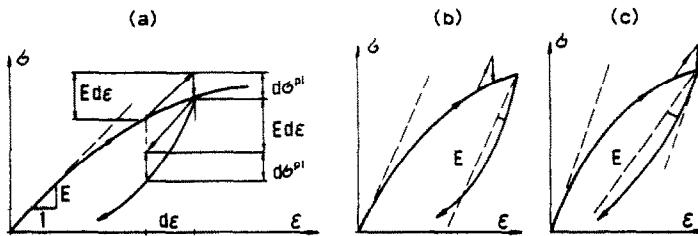


Fig. 1. Loading and unloading for endochronic theory.

Here ζ is called intrinsic time, ξ is called deformation measure, λ = inelastic dilatancy (due to shearing), F_1, F_2 = positive-valued scalar functions called hardening-softening functions.

The most conspicuous feature of the ordinary endochronic formulation (eqns 9 and 10) is that no distinction is made between loading and unloading, provided that at the start of unloading the inelastic stress increment can be assumed to be non-zero and equal to that for continued loading. To illustrate it, consider the case of uniaxial strain, ϵ_{11} , with all other ϵ_{ij} being zero [8]. Then, dropping subscripts 11, $d\xi \sim |d\epsilon|$ and $d\sigma^{pl} \sim \sigma |d\epsilon|$. The incremental relation is $d\sigma = d\sigma/E + d\sigma^{pl}$, which may be rewritten as $d\sigma = E d\epsilon - d\sigma^{pl}$, where $d\sigma^{pl} = E d\epsilon^{pl}$ or $d\sigma^{pl} = E c |d\epsilon|$, c being some constant. Increments $d\sigma^{pl} = E d\epsilon$ and $d\sigma^{pl}$ are depicted in Fig. 1 for positive $d\epsilon$. Consider now that positive $d\epsilon$ is followed by negative $d\epsilon$, i.e. loading is reversed to unloading. Obviously, $d\sigma^{pl}$ changes sign but $d\sigma^{pl}$ does not change sign because it depends on $|d\epsilon|$, as is shown in Fig. 1 [8]. Thus, the irreversibility at unloading, the salient feature of all inelastic behavior, is modeled by the endochronic formulation in a very simple manner, without the need for any inequalities for expressing the unloading criterion, provided that the material response is adequately described by the implied assumption of equality of the inelastic strains for continued loading and for the start of unloading.

4. LOADING FUNCTION IN ENDOCHRONIC THEORY

In associated plasticity, the tensor of inelastic strain increments is derivable from a scalar potential, called loading function or yield function, $F(\sigma_{ij})$, i.e. $d\epsilon_{ij}^{pl} \sim \partial F(\sigma_{ij})/\partial \sigma_{ij}$. Even though in non-associated plasticity and in endochronic theory the concept of yield surface lacks physical foundation, it seems reasonable and useful to retain this concept for the endochronic formulation and continue to speak of loading function. All practical endochronic formulations used thus far satisfy this concept. The deviatoric part of the endochronic relations in eqns (9)–(10), which were shown to agree with extensive test data, is associated with von Mises loading function because $d\epsilon_{ij}^{pl} \sim s_{ij} \sim \partial J_2^0 / \partial s_{ij}$. In the volumetric cross section, the loading function of the endochronic theory is a curve (like that in Fig. 3b) with the slope $\beta' = \partial F/\partial \sigma$ given by the relation $k \partial F / \partial \sigma = d\epsilon_{ij}^{pl} = d\lambda$ in which, from the relation $k \partial F / \partial s_{ij} = d\epsilon_{ij}^{pl} = s_{ij} d\zeta$, one finds $k = 1/d\zeta$. Thus, $\beta' = \partial F / \partial \sigma = d\lambda / d\zeta$, and so the endochronic formulation in eqns (9)–(10) may be written as

$$d\epsilon_{ij}^{pl} = \frac{\partial F(\sigma_{ij})}{\partial \sigma_{ij}} d\zeta, \quad \text{with} \quad F(\sigma_{ij}) = \frac{1}{2} s_{ij} s_{ij} + g(\sigma) - H_1 = 0 \quad (11)$$

$$\frac{dg(\sigma)}{d\sigma} = \beta' = \frac{d\lambda}{d\zeta} = \frac{F_2(\sigma, \epsilon, \lambda)}{F_1(\sigma, \epsilon, \zeta)} \quad (12)$$

H_1 being a parameter independent of σ_{ij} components. Note that here β' inevitably depends not only on current ϵ_{ij} and σ_{ij} but also on its history, and that $dH_1 = -(\partial F / \partial \sigma_{ij}) d\sigma_{ij} / (\partial F / \partial H_1) = s_{ij} ds_{ij} + \beta' d\sigma$.

In the deviatoric strain space, a loading direction which is normal to the loading functions associated with eqns (3) and (9) coincides with proportional (radial) loading paths. Most experimental data pertain to such loading paths or to paths rather close to them. Even the standard triaxial tests are essentially of this nature, because the hydrostatic stress, applied first, causes little inelastic strain, with no directional damage (no stress-induced anisotropy), and the uniaxial load which is subsequently superimposed is itself a proportional (radial) loading.

If attention is restricted to proportional or almost proportional loading, it appears from

numerical studies that the main experimental data available can be fitted reasonably closely by either formulation, i.e. both endochronic theory and plasticity with dilatancy and friction can represent the behavior of concrete under these conditions reasonably well. However, the two formulations are vastly different in nonproportional loading. Of most interest are the strain paths or stress paths which have a sharp corner. Such a corner must always be considered in investigating instabilities (e.g. strain localization) due to material nonlinearity [28, 29]. In case of instability, the strain or stress path can proceed in any direction in the strain or stress space, and generally it will not proceed in the direction of the preceding strain path.

The strain increment $d\epsilon_{ij}$ in the strain space will be called "normal loading" or *straight-ahead loading* when it is normal to the loading surface corresponding to tensor $d\epsilon_{ij}^P$, and "tangential loading" or *loading to the side* when it is tangential to the loading surface. For the loading functions associated with eqns (3) and (9) this corresponds to proportional loading direction and directions normal to it, respectively (see Fig. 2).

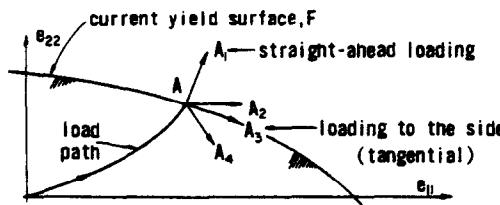


Fig. 2. Various types of load path.

5. INELASTIC STIFFNESS LOCUS

For comparing the endochronic and plastic formulations it is useful to define the following property.

Definition. Inelastic stiffness locus is the locus of all strain increments $d\epsilon_{ij}$ which give for a given initial state the same magnitude $\|d\epsilon_{ij}^P\|$ of the inelastic strain increment tensor $d\epsilon_{ij}^P$. The magnitude (or norm) may be defined as the length of the vector $d\epsilon_{ij}^P$ in a six-dimensional strain space of suitable metric or $\|d\epsilon_{ij}^P\| = \sqrt{[d\epsilon_{ij}^P d\epsilon_{ij}^P + M_1(d\epsilon_{ij}^P)^2]^{1/2}}$ where M_1 = given constant.

Note that $d\epsilon_{ij}^P$ is proportional to the plastic tangential modulus in the $d\epsilon_{ij}$ -direction. Thus, the farther a point is on this locus from the current state in a given direction, the stiffer is the plastic response in that direction.

To determine the locus just defined, it is necessary to express $d\epsilon_{ij}^P$ in terms of $d\epsilon_{ij}$. Consider first the plastic formulation. Equations (1)-(4) may be regarded as a system of linear algebraic equations in which $d\epsilon_{ij}^P$ are the unknowns and $d\epsilon_{ij}$ are given. First, $d\mu$ must be expressed in terms of $d\epsilon_{ij}$. To this end, eqn (4a) may be used to calculate $d\bar{\tau} = 2h d\mu - \beta' d\sigma = 2h d\mu - K\beta'(3d\epsilon - 2\beta d\mu) = 2d\mu(h + K\beta\beta') - 3K\beta'd\epsilon$. Then, from eqns (1)-(3), $ds_{km} = 2G(d\epsilon_{km} - s_{km} d\mu/\bar{\tau})$, and so $d\bar{\tau} = s_{km} ds_{km}/2\bar{\tau} = (s_{km}/2\bar{\tau})2G(d\epsilon_{km} - s_{km} d\mu/\bar{\tau})$. Equating both expressions for $d\bar{\tau}$, one obtains an equation which yields

$$d\mu = \frac{G s_{km} d\epsilon_{km} + \bar{\tau} K\beta' d\epsilon_{kk}}{2\bar{\tau}(h + G + K\beta\beta')} \quad (13)$$

It is now convenient to define inelastic stress increments as $ds_{ij}^P = 2G d\epsilon_{ij}^P$ and $d\sigma_{ij}^P = 3K d\epsilon_{ij}^P$. Then $d\sigma_{ij}^P = ds_{ij}^P + \delta_{ij} d\sigma^P = 2(Gs_{ij} + K\beta\bar{\tau}\delta_{ij})d\mu/\bar{\tau}$ where δ_{ij} = Kronecker delta. Substitution of eqn (13) for $d\mu$ yields

$$d\sigma_{ij}^P = D_{ijkm}^P d\epsilon_{km} \quad (14)$$

with

$$D_{ijkm}^P = \frac{[(G/\bar{\tau})s_{ij} + K\beta\delta_{ij}][(G/\bar{\tau})s_{km} + K\beta'\delta_{km}]}{h + G + K\beta\beta'} \quad (15)$$

These are the tangential moduli for inelastic stress increments. Note that they are symmetric if

and only if $\beta = \beta'$, and that they are different from plastic tangential moduli defined by the relation $d\sigma_{ij} = C_{ijkl}^{pl} d\epsilon_{km}^{pl}$.

It is now obvious that if $D_{ijkl}^{pl} d\epsilon_{km}$ is constant, all components of $d\sigma_{ij}^{pl}$ are constant and then $\|d\sigma_{ij}^{pl}\|$ as well as $\|d\epsilon_{ij}^{pl}\|$ is also constant. Substituting eqn (15) one finds that this happens for

$$s_{km} d\epsilon_{km} = C - \frac{3K\beta'}{G} \bar{\tau} d\epsilon \quad (16)$$

where C is some constant ($s_{km} d\epsilon_{km} = s_{km} d\epsilon_{km}$).

To interpret this result geometrically, consider the material state $(\sigma_{ij}, \epsilon_{ij})$ plotted as point A in superimposed stress and strain spaces in Fig. 3, separately for the deviatoric components and for the volumetric cross section. Also imagine that a space of infinitesimal increments $d\epsilon_{ij}$ is superimposed at point A in Fig. 3. The six components of s_{ij} and $d\epsilon_{ij}$ may be imagined to form vectors s and de . Then, eqn (16) at constant $d\epsilon$ may be written as $s \cdot de = \text{const.}$; this is a scalar product, and the equation means that the projection of vector de upon the direction of s (or s_{ij}) must be constant. This shows that the inelastic stiffness locus is a *plane* in the six-dimensional space of $d\epsilon_{ij}$. In the two-dimensional picture of Fig. 3(a), this locus is given by a *straight line*. The normals to the current yield surface have the direction of s_{ij} , and the inelastic stiffness locus consists of a straight line parallel to the tangent of the yield surface at point A. Similarly, in the volumetric cross section, the inelastic stiffness locus is a straight line parallel to the tangent of the current yield surface (see Fig. 3b).

Let attention be now turned to the endochronic formulation. Here, according to eqn (9), constant values of $d\epsilon_{ij}^{pl}$ are obtained when $d\zeta = \text{const.}$, which corresponds to $d\xi = \text{const.}$ if σ_{ij} and ϵ_{ij} are fixed. Hence, the locus of the end points of all strain increments de or $d\epsilon_{ij}$ which give the same values of inelastic strain increments $d\epsilon_{ij}^{pl}$ is given by the equation

$$de_{ij} d\epsilon_{ij} = \text{const.} \quad (17)$$

or $de \cdot de = \text{const.}$ Consequently, in the deviatoric strain space, the inelastic stiffness locus is a hypersphere around point A, which appears in a two-dimensional picture as a *circle* (see Fig. 4a). Due to hardening and softening functions of the endochronic theory, the diameter of this infinitesimal circle varies as it is dragged through the strain space, but the shape of the locus always remains a circle. In the full strain space, eqn (17) represents a hypercylinder, and in the two-dimensional volumetric cross sections (Fig. 4) the inelastic stiffness locus appears as a set of two parallel straight lines.

Note that for both plastic and endochronic formulations, not only $\|d\epsilon_{ij}^{pl}\|$ but all components of $d\epsilon_{ij}^{pl}$ are the same for all vectors $d\epsilon_{ij}$ ending on the inelastic stiffness locus.

Consider now the dependence of the tangent modulus of inelastic stress

$$H = \|ds_{ij}^{pl}\| / \|d\epsilon_{ij}^{pl}\| \quad (18)$$

upon the $d\epsilon_{ij}$ -direction, characterized by angle α in Figs. (3)–(5). By definition, $\|ds_{ij}^{pl}\|$ is constant for all vectors $d\epsilon_{ij}$ on the inelastic stiffness locus, and so $1/H$ is proportional to the distance, $\|d\epsilon_{ij}\|$, from point A to the inelastic stiffness locus along the α -direction. The plots of $1/H$ vs α are shown in Fig. 6.

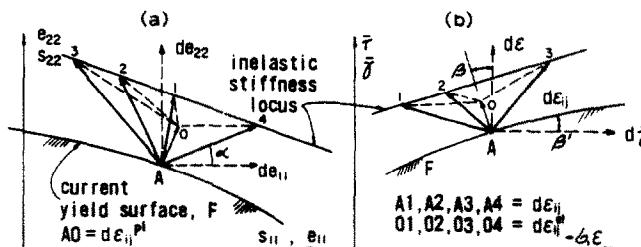


Fig. 3. Inelastic stiffness locus for plasticity theory.

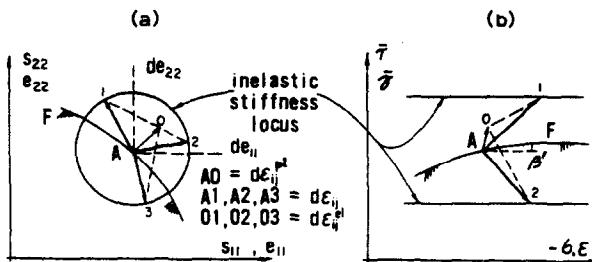


Fig. 4. Inelastic stiffness locus for endochronic theory.

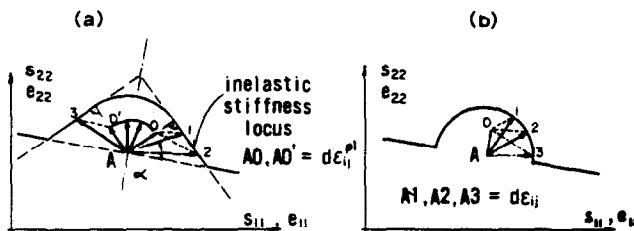


Fig. 5. Inelastic stiffness locus for vertex hardening and endochronic theory with unloading criterion.

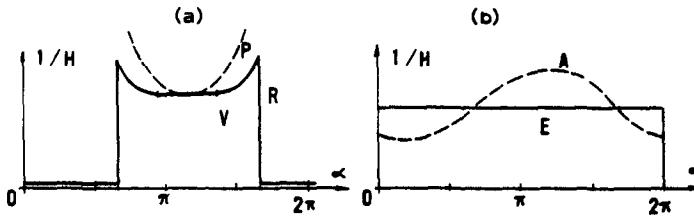


Fig. 6. Inelastic tangent modulus as function of loading direction; for classical plasticity (P), vertex hardening from Fig. 5(a) (V) and from Fig. 7(b) (R), Endochronic theory (E), endochronic theory with stress-induced anisotropy (A).

The inelastic stiffness locus reveals the fundamental difference between plastic and endochronic formulations. If both are fitted to the same data on proportional loading, then the plastic formulation is stiffer for loading to the side. For the tangential loading direction, plasticity gives perfectly elastic response ($H = 0$), while endochronic theory gives inelastic response ($H > 0$). Nevertheless, it must be realized that in both these formulations the inelastic strain increment de_{ij}^p for all loading directions of de_{ij} is always in the straight-ahead direction, given by the normality rule (flow rule), and the components of de_{ij} in the tangential direction are purely elastic; see the vectors de_{ij}^p and de_{ij}^e shown in Figs. 3 and 4.

6. RELATIONSHIP TO VERTEX HARDENING EFFECTS

In recent developments of plasticity theory, the creation of corners (vertices) on the yield surface, called "vertex hardening", has received considerable attention. According to classical plastic formulations (eqns 1–4) the inelastic strain is created only by the normal (straight-ahead) component of de_{ij} , whereas the tangential (loading to the side) component causes no further inelastic strain, with the consequence that the response for load increments to the side is overall much stiffer than it is for straight-ahead loading. For pure loading to the side (Fig. 2), no inelastic strain is produced at all. This feature has been recognized to conflict with the predictions of microstructural polycrystalline models of plasticity, which all indicate that the "loading to the side" should also produce inelastic strain [30, 31].

To correct this defect various forms of vertex hardening models have recently been introduced. In some of them, the yield surface is assumed to form a vertex (corner) at the current state point on the loading surface, which indicates the inelastic stiffness locus to have the shape shown in Fig. 5(a). A different type of vertex hardening has recently been proposed by Rudnicki and Rice [26]; they considered linear incremental equations in which the expression for de_{ij}^p from eqn (3) is augmented by the term $(\bar{\tau} ds_{ij} - s_{ij} d\bar{\tau})/2h_1\bar{\tau}$ where h_1 = plastic modulus for loading to the side; this term is derived from the requirements that it must vanish for

straight-ahead loading ($ds_{ij} \sim s_{ij}$) and must be linear in ds_{ij} . When unloading takes place, this term is omitted (along with the straight-ahead term $s_{ij} d\mu/\bar{\tau}$). With the addition of the foregoing term, the formulation still retains the linear form, which is formally identical to the original form, provided that G , h , β , β' and K are replaced by the following parameters [26]:

$$\begin{aligned}\tilde{G} &= (1/G + 1/h_1)^{-1}, & \tilde{h} &= (1/h - 1/h_1)^{-1}, & \tilde{\beta} &= \beta\tilde{h}/h, \\ \tilde{\beta}' &= \beta'\tilde{h}/h, & \tilde{K} &= [1/K + \beta\beta'/(h_1 - h)]^{-1}.\end{aligned}\quad (19)$$

It is now clear that Rudnicki and Rice's vertex hardening still leads to the linear incremental relation (14), in which however, D_{ijklm}^{pl} is replaced by \tilde{D}_{ijklm} and is expressed in terms of \tilde{G} , \tilde{K} , \tilde{h} , $\tilde{\beta}$ and $\tilde{\beta}'$. However, the true elastic stress increments are not $ds_{ij} = 2\tilde{G} de_{ij}$ but $2G de_{ij}$. Noting that the elastic moduli are given as $D_{ijklm}^e = 2G\delta_{ik}\delta_{jm} + (K - 2G/3)\delta_{ij}\delta_{km}$, one must remove from $\tilde{D}_{ijklm} de_{km}$ the false elastic stress increments $\tilde{D}_{ijklm}^e de_{km}$ and add the correct elastic stress increments $D_{ijklm}^e de_{km}$, i.e.

$$d\sigma_{ij}^{pl} = (\tilde{D}_{ijklm}^{pl} - \tilde{D}_{ijklm}^e + D_{ijklm}^e) de_{km} \quad (20)$$

According to eqn (15), it is now necessary to replace eqn (16) for the inelastic stiffness locus by the equation

$$\left\| C_{ij} \left(s_{km} de_{km} + \frac{3\tilde{K}}{\tilde{G}} \tilde{\beta}' \bar{\tau} de \right) - (\tilde{D}_{ijklm}^e - D_{ijklm}^e) de_{km} \right\| = \text{const.} \quad (21)$$

where C_{ij} are certain constants for a given stress state. It is apparent that, due to the last term in eqn (21), the projection of vector de upon vector ds is no longer a constant but depends on the loading direction de . So, the inelastic stiffness locus can no longer be a straight line. The expression within $\|\cdot\cdot\|$ is linear, and thus eqn (21) may be written in the form of a quadratic equation for de_{km} components. Therefore, Rudnicki and Rice's vertex hardening model [26] gives the inelastic stiffness locus in the form of a quadratic curve; this curve must intersect the straight-ahead direction orthogonally (see Figs. 5b or 7b), and its curvature is a function of modulus h_1 for loading to the side. (It must be pointed out, however, that this vertex hardening model was intended [26] only for loading directions which are close to the straight-ahead direction.)

There exists one essential difference from the previous cases. For classical plasticity as well as endochronic theory, not only the magnitude of de_{ij}^{pl} but all its components are the same for all vectors de_{ij} ending on the inelastic stiffness locus, while for Rudnicki and Rice's vertex hardening, only the magnitude is the same while the individual components of de_{ij}^{pl} vary when moving along the locus. This means that the direction of de_{ij}^{pl} depends on the direction of de_{ij} , while for endochronic and plastic formulations the direction of de_{ij}^{pl} is unaffected by the direction of de_{ij} . On the other hand, by introducing modified elastic moduli the inelastic stiffness locus for Rudnicki and Rice's vertex hardening can be transformed to a straight line, whereas for the endochronic formulation this is impossible. However, the endochronic formulations and vertex hardening formulations share one most important property—namely, for both the loading to the side creates inelastic strain. This property, for example, made it possible for Valanis to model "cross-hardening of metals", such as the effect of plastic twist on subsequent axial extension diagrams, which was the earliest success of the endochronic formulation [3].

The dependence of tangent modulus H for inelastic stress is depicted in Fig. 6 for vertex hardening from Fig. 5(a). This dependence is not smooth, while for the endochronic theory as well as Rudnicki and Rice's vertex hardening it is smooth. This might be preferable also for iterative numerical solutions of structural problems. In contrast to both classical plastic and endochronic formulations, the inelastic strain increments are generally not in the straight-ahead direction and include inelastic components oriented to the side (Fig. 5a). The tangent modulus $d\sigma_{ij}/de_{ij}$ (no sum) for loading to the side is a reduced modulus, while for endochronic and plasticity theories it equals the unreduced elastic modulus for that direction.

7. STRESS-INDUCED ANISOTROPY OF INTRINSIC TIME

In the light of the preceding discussion it seems appropriate to consider a generalization of the endochronic theory in which the inelastic stiffness locus is not restricted to a circular (hyperspherical) shape. Indeed, it would be purely by chance if for some real material the shape were exactly circular. Modified shapes can be achieved by replacing $d\xi$ from eqn (11) with

$$d\xi = \left(\frac{1}{2} p_{ijklm}(\sigma) d\epsilon_{ij} d\epsilon_{lm} \right)^{1/2} \quad (22)$$

in which $p_{ijklm}(\sigma)$ are coefficients which are not constant but depend on stress tensor $[\sigma_{ij}] = \bar{\sigma}$. In fact, since there is no reason for p_{ijklm} to be independent of σ , one must expect that p_{ijklm} depends on σ . For an initially anisotropic material, eqn (22) but with constant p_{ijklm} was proposed by Valanis [3]. A complete anisotropic formulation has been developed for clays [19].

For isotropic materials, p_{ijklm} must form an isotropic tensorial polynomial in σ , and for $\sigma = 0$ coefficients p_{ijklm} must form a constant isotropic tensor (as is the case for eqn 10). Thus, coefficients p_{ijklm} exhibit stress-induced anisotropy, in the sense that the quadratic form $p_{ijklm} d\epsilon_{ij} d\epsilon_{lm}$ is invariant with regard to the direction of $d\epsilon_{ij}$ only if the material is stress-free. The simplest candidate for practical characterization of a material will be the case where the fourth-order tensor p_{ijklm} is given as a linear isotropic tensorial function of σ_{ij} . Then,

$$\begin{aligned} 2d\xi^2 &= d\epsilon_{ij} d\epsilon_{ij} + p_0 d\epsilon_{ii} d\epsilon_{jj} + p_1 \sigma_{ij} d\epsilon_{ik} d\epsilon_{jk} + p_2 \sigma_{ij} d\epsilon_{ij} d\epsilon_{kk} \\ &\quad + p_3 \sigma_{kk} d\epsilon_{ij} d\epsilon_{ij} + p_4 \sigma_{kk} d\epsilon_{ii} d\epsilon_{jj} \end{aligned} \quad (23)$$

or, as a special case,

$$\begin{aligned} 2d\xi^2 &= d\epsilon_{ij} d\epsilon_{ij} + p_1 s_{ij} d\epsilon_{ik} d\epsilon_{jk} \\ &= (1 + p_1 s_{11}) d\epsilon_1^2 + (1 + p_1 s_{22}) d\epsilon_2^2 + (1 + p_1 s_{33}) d\epsilon_3^2 \end{aligned} \quad (24)$$

where p_0, \dots, p_4 = constants, and $d\epsilon_1, d\epsilon_2, d\epsilon_3$ = principal deviatoric strain increments. The first expression can be shown to correspond to the most general symmetric form of a fourth-order tensor, p_{ijklm} , linear in σ_{ij} (as known from hypoelasticity [32]). The second expression, eqn (24), is the most general form which is independent of volumetric components $d\epsilon_{kk}$ and σ_{kk} , as might be reasonable to assume for many materials (see the arguments in Refs. [3] and [4]). With eqns (23) or (24), the inelastic stiffness locus becomes a quadratic surface, which would appear in any two-dimensional cross section of strain space as an ellipse, parabola or hyperbola (Fig. 7a). However, the latter case in which the quadratic form in eqn (24) becomes hyperbolic is inadmissible for it would give imaginary $d\xi$. As a remedy, $d\xi^2$ would have to be set equal to zero whenever it would be obtained as negative, which is equivalent to imposing an unloading criterion (Fig. 7b). Nevertheless, it is possible to choose such p_1 that ensures ellipticity of eqn (24) for all stress states expected to be sustained by the material. This is achieved by choosing $1 + p_1 |s_{\max}| \geq 0$ or $p_1 \leq -1/|s_{\max}|$ where s_{\max} is the principal value of s_{ij} which is largest in absolute value among all s_{ij} expected to occur.

Equation (24) describes an ellipse located symmetrically about point A. It gives different

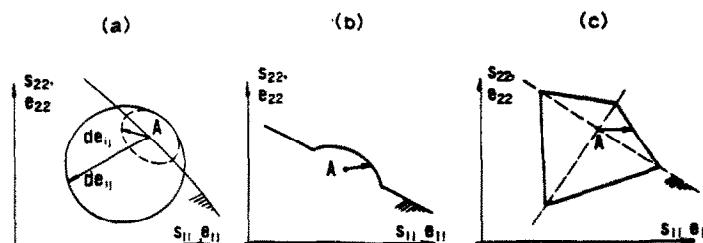


Fig. 7. Inelastic stiffness locus for endochronic theory with stress-induced anisotropy (a); with unloading criterion (b); and with piecewise-linear intrinsic time increments (c).

plastic hardening moduli for loadings to the side and straight ahead, but the same moduli for straight-ahead loading and unloading. The latter feature is questionable, and it can be removed by a different type of stress-induced anisotropy:

$$d\xi = \left[\frac{1}{2} (de_{ij} + p_0 s_{ij} d\xi') (de_{ij} + p_0 s_{ij} d\xi') \right]^{1/2}, \quad d\xi' = \left[\frac{1}{2} de_{ij} de_{ij} \right]^{1/2}. \quad (25)$$

For this expression, the inelastic stiffness locus is a circle which is not centric about point A but is shifted towards the origin (Fig. 7a), giving smaller inelastic strain for unloading as compared to loading.

Alternatively, of course, $d\xi$ could be defined so as to give a locus of de_{ij} that is piecewise linear (hyperpolyhedron). An example is the inelastic stiffness locus in Fig. 7(c), for which, e.g. $d\xi = a|de_1| + b|de_2| + c|de_3|$, where de_1 , de_2 and de_3 must be the principal deviator strain increments in order to satisfy the tensorial invariance restrictions, and a , b and c depend on s_{ij} . In such a case, the inelastic stiffness locus becomes similar to that for certain vertex hardening models. Conversely, it is possible to construct plasticity-type formulations with vertex hardening for which the inelastic stiffness locus approaches that for the usual endochronic formulation (eqns 9 and 10) as closely as desired. This is obtained when the set of all orientations of de_{ij} (directions α), is subdivided into many cones (hypercones in the full strain space, and angular segments in (de_{11}, de_{22}) space). Within each directional cone, linear incremental equations are used, giving a piece-wise linear inelastic stiffness locus (describing a hyperpolyhedron). In the limit for the number of direction cones approaching infinity, this locus approaches a smooth surface characteristic of the endochronic theory.

8. REMARKS ON MATERIAL STABILITY

One property which is intimately connected with loading to the side and vertex hardening is the question of material stability and unstable strain localization [28, 29, 26]. Due to the fact that loading to the side produces inelastic strain, the material response to the side is "softer" than it is for the classical plastic formulation, and this can be expected to have a destabilizing effect [26, 27]. For Rudnicki-Rice type vertex hardening, which gives inelastic strain for vectors de_{ij} that are parallel to the yield surface, Drucker's postulate is not satisfied and stability of the material is not guaranteed. While some materials are stable, most materials must indeed be expected to violate Drucker's postulate and the normality rule at sufficiently large strain, and permit material instabilities [26, 29]. An important example is the class of geological materials, such as sands, clays, rocks and concrete. In these materials, the inelastic strain depends on friction, and in such a case the normality rule and Drucker's stability postulate do not apply [20, 33]. Micro-fracturing in these materials, and the inherent dilatancy, are undoubtedly also sources of possible material instabilities.

Recently, it has been shown that these phenomena give rise to behavior which is approximately modeled by vertex hardening, and that the vertex hardening has a profound destabilizing effect, promoting instabilities in the form of a localization of a strain in a narrow band [26, 27, 29].

Material instability is also caused by strain-softening [34], which is known to exist in concrete, rock and soils, as recent tests in tension, compression and torsion indicate. Strain-softening can only be observed on specimens of micro-inhomogeneous material which are sufficiently small to prevent unstable strain-localization and are loaded by a sufficiently stiff displacement-controlled testing machine. Strain-softening is not allowed by Drucker's postulate [23], but is admitted by an analogous approach, called fracturing material theory [35-37]. Agreement with experimental data on strain-softening has so far been obtained only with the endochronic formulation [8].

Thus, it is clear that for materials which do exhibit unstable strain localization, or strain-softening, such as geological materials, the endochronic formulation, compared to plasticity with a smooth yield surface, stands at the proper place of the scene—it does allow plastic strain at loading to the side, similarly to the vertex hardening models, and it does allow strain-softening. With classical plastic formulations satisfying Drucker's stability postulate, such effects, if they exist, are inevitably missed.

On the basis of microstructural polycrystalline models [31], it has been found that already

for a relatively small strain the current zero-offset yield surface, representing an envelope of all points which can be reached from the current state without causing inelastic strain, shrinks almost to a point. The point is equivalent to an infinitesimal circle, and precisely this happens in endochronic theory. This fact serves as a physical justification of the endochronic theory.

The fact that for loading to the side the endochronic formulation gives generally a softer response (lower stiffness) than plasticity means that the model is more prone to indicate instability. Thus, the endochronic formulation is on the side of safety in case of stability predictions, whereas classical plasticity is, among all possible inelastic constitutive relations, the least conservative and the least safe model that can be assumed if test data are inconclusive as far as the choice of the type of constitutive relation is concerned. On the other hand, vertex hardening models are still more prone to indicate instability because for tangential loading they exhibit inelastic strain components in the tangential direction while the endochronic inelastic strain is entirely in the straight-ahead direction. Thus, it might be appropriate to incorporate into endochronic theory some vertex hardening features, e.g. the model of Rudnicki and Rice [26].

9. RELATIONSHIP TO DEFORMATION THEORY OF PLASTICITY

A simple prototype of Hencky's deformation theory of plasticity is given by Nádai's stress-strain relation [24, 25]:

$$s_{ij} = f(J_2'') e_{ij} \quad (26)$$

in which $J_2'' = (1/2)s_{ij}s_{ij} = \bar{\tau}^2$ = second invariant of stress tensor σ_{ij} . Prager and others have shown that this type of formulation has certain serious deficiencies, such as independence from the loading history [24, 25]. On the other hand, from experiments it is known that the deformation theory happens to give better predictions than the incremental theory of plasticity (eqns 1-4) in many cases, one of which is loading to the side [38].

To discuss loading to the side, eqn (26) may be differentiated:

$$ds_{ij} = f(J_2'') de_{ij} + e_{ij} \frac{\partial f(J_2'')}{\partial s_{km}} ds_{km}. \quad (27)$$

Consider now a state in which s_{11} and e_{11} are non-zero, all other s_{ij} and e_{ij} being zero. The straight-ahead (radial, proportional) loading is here represented by de_{11} , and the loading to the side is represented, e.g. by de_{22} or by de_{12} . The corresponding stress increments, ds_{11} or ds_{12} , are obtained from eqn (27) as $ds_{22} = f(J_2'') de_{22}$ and $ds_{12} = f(J_2'') de_{12}$. Thus, the tangent modulus for loading to the side is less than the elastic modulus and equals the current secant elastic modulus [38].

In the present context, the foregoing result means that in the deformation theory there exists inelastic strain for loading to the side [38], which is a type of loading for which the deformation theory often gives good agreement with experiment. This is in similarity to vertex hardening, and partly also to the endochronic formulation, and in contrast to classical plastic formulations (eqns 1-4), which give purely elastic response at loading to the side.

10. ENDOCHRONIC KINEMATIC HARDENING AND OTHER LOADING FUNCTIONS

The endochronic formulations used so far (such as eqns 9 and 10) correspond to isotropic hardening, and so does the plastic formulation in eqns (1)-(4). This is because $de_i^T \sim s_{ij} \sim \partial J_2'' / \partial s_{ij}$ where $J_2'' = \text{const.}$ characterizes in plasticity theory a yield surface which is always centered at the origin of stress space (Fig. 8) and dilates while retaining the same shape.

So, it may be of interest to identify a counterpart of anisotropic hardening rules known from plasticity, especially kinematic hardening. Here, the yield surface not only dilates but also moves as a rigid body. Considering, e.g. Prager's kinematic hardening rule [24, 25], an analogous generalization of eqns (9) and (10) would be obtained by deriving de_i^T from a loading function F which, in addition to expanding radially (isotropic hardening) also moves as a rigid body (kinematic hardening). Thus, adhering to von Mises-type loading function for deviator defor-

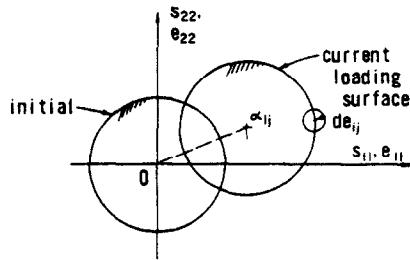


Fig. 8. Endochronic kinematic hardening.

mations, the endochronic loading function from eqn (11) may be generalized as

$$F(\sigma_{ij}) = \left[\frac{1}{2} (s_{ij} - \alpha_{ij})(s_{ij} - \alpha_{ij}) \right]^q + g(\sigma - \alpha_0) - H_1 = 0 \quad (28)$$

and according to eqn (11), eqns (9) for $q = 1$ are generalized as

$$d\epsilon_{ij}^p = (s_{ij} - \alpha_{ij}) d\zeta, \quad d\epsilon^{pi} = d\lambda = \frac{dg(\sigma - \alpha_0)}{d\sigma} d\zeta. \quad (29)$$

Coefficients α_{ij} , α_0 (with $\alpha_{ik} = 0$) indicate the current center of the loading surface.

One could, of course, further speculate on the rules for the increments $d\alpha_{ij}$ and $d\alpha_0$ as functions of $d\epsilon_{ij}$, $d\sigma_{ij}$ and $d\zeta$. For example, similarly to Shield and Ziegler's hardening rule in plasticity [24, 25],

$$d\alpha_{ij} = k_1 ds_{ij}^p = 2Gk_1 d\epsilon_{ij}^p, \quad d\alpha_0 = k_0 d\sigma^{pi} = 3Kk_0 d\epsilon^{pi} \quad (30)$$

where k_0 and k_1 are constants and $0 \leq k_1 \leq 1$ may be expected. (According to a private communication by C. L. Shieh of Northwestern University, the use of $k_1 \approx 0.15$ and $k_0 = 0$ distinctly improves the fits of asymmetric hysteresis loops for highly strained concrete.) Equation (30) yields pure isotropic hardening for $k_0 = 0$.

For the endochronic formulations corresponding to a Maxwell chain model [8, 14], a corresponding generalization would be to use $d\epsilon_{ij\mu}^p = (s_{ij\mu} - 2G_\mu k_1 d\epsilon_{ij}) d\zeta_\mu$, subscript μ referring to relaxation time τ_μ .

Refinements within the isotropic loading functions are also possible. For example, it seems that a somewhat improved description of concrete is possible with $F(\sigma_{ij}) = J_2 + cJ_3^{2/3} + g(\sigma) - H_1$ where $J_3 = s_{mn}s_{np}s_{pm}/3 = \text{third invariant of } s_{ij}$. Equation (11) then yields

$$d\epsilon_{ij}^p = (s_{ij} + cs_{ik}s_{kj}) d\zeta. \quad (31)$$

However for a clear answer one needs more accurate test data than are available at present.

In endochronic theory there is, however, one important practical difference in hardening rules as compared to plasticity theory. This is due to the fact that in plasticity there exists the property that all states within the current yield (loading) surface can be reached without inelastic straining, while in endochronic theory no state can be reached in this manner. Therefore, points of the loading surface which are at a finite distance from the current state A (Fig. 9) are irrelevant for the endochronic formulation. The only relevant property of the loading surface is the local curvature of the loading surface at the current state A . This curvature is reflected in the current location α_{ij} of the center of the loading surface. In the light of these considerations, it seems that the absence of yield surface in the plasticity sense might be a useful and simplifying feature of the endochronic theory. It makes it possible to cease worrying about the entire current loading surface and reduces attention to the local properties of the current loading surface near the current state.

10. TANGENTIAL LINEARIZATION OF ENDOCHRONIC FORMULATION

The stress-strain relations of incremental plasticity are linear in stress and strain increments, and therefore it is possible to relate the increments $d\sigma_{ij}$ and $d\epsilon_{ij}$ by a matrix of tangential

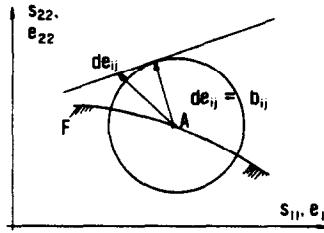


Fig. 9. Tangential linearization of the inelastic stiffness locus for the endochronic theory.

moduli, D_{ijklm} ; i.e. [24]

$$d\sigma_{ij} = D_{ijklm} d\epsilon_{km}. \quad (32)$$

For the plastic formulation in eqns (1)–(4), $D_{ijklm} = D_{ijklm}^e - D_{ijklm}^{pl}$, where $D_{ijklm}^e = 2G\delta_{ik}\delta_{jm} + (K - 2G/3)\delta_{ij}\delta_{km}$, and D_{ijklm}^{pl} is given by eqn (15). As will be seen, for the endochronic formulation moduli D_{ijklm} can be expressed only if the direction of vector $d\epsilon$ (or $d\epsilon_{ij}$) is known.

In analyzing material instabilities, it may often be necessary to linearize the incremental stress-strain relation so as to obtain an eigenvalue problem. For this purpose, the curved (circular) inelastic stiffness locus of the endochronic theory must be replaced by a plane (or a straight line) which is tangential to the curved locus at the point of assumed strain increment direction, $d\epsilon_{ij} = b_{ij}$; see Fig. 9. The linearized formulation will then be equivalent for all $d\epsilon_{ij}$ directions which are sufficiently close to $d\epsilon_{ij} = b_{ij}$. So one must replace $(d\epsilon_{ij} d\epsilon_{ij}/2)^{1/2}$ by a linear expression which represents, for fixed $d\epsilon$, a plane that is normal to vector b_{ij} in the six-dimensional strain space; i.e. $d\xi = kb \cdot d\epsilon$ or $d\xi = kb_{ij} d\epsilon_{ij}$. Constant k must be such that for $d\epsilon_{ij} = b_{ij}$ the correct value of $d\xi$ be obtained. This requires that $d\xi = (b_{km} b_{km}/2)^{1/2} = kb_{km} b_{km}$, which yields

$$d\xi = B_{ij} d\epsilon_{ij}, \quad \text{with} \quad B_{ij} = \frac{b_{ij}}{\sqrt{(2b_{km} b_{km})}}. \quad (33)$$

The tangential linearization can, of course, be also obtained without resorting to geometrical considerations. Let $d\epsilon_{ij} = b_{ij} + p_{ij}$ where tensor p_{ij} is small compared to b_{ij} , in the sense that $|b_{ij}p_{ij}| \ll b_{ij}b_{ij}$. Noting that $(1 + \delta)^{1/2} = 1 + \delta/2$ if $\delta \ll 1$, one may arrive at eqn (33) by the following transformations:

$$\begin{aligned} d\xi &= \left(\frac{1}{2} d\epsilon_{ij} d\epsilon_{ij} \right)^{1/2} = \left[\frac{1}{2} (b_{ij} + p_{ij})(b_{ij} + p_{ij}) \right]^{1/2} \\ &\approx \left(\frac{1}{2} b_{ij}b_{ij} + b_{ij}p_{ij} \right)^{1/2} = \left[\frac{1}{2} b_{km}b_{km} \left(1 + \frac{2b_{ij}p_{ij}}{b_{rs}b_{rs}} \right) \right]^{1/2} \\ &\approx \left(\frac{1}{2} b_{km}b_{km} \right)^{1/2} \left(1 + \frac{b_{ij}p_{ij}}{b_{rs}b_{rs}} \right) = \frac{b_{ij}b_{ij} + b_{ij}p_{ij}}{(2b_{km}b_{km})^{1/2}} \\ &= (2b_{km}b_{km})^{-1/2} b_{ij} d\epsilon_{ij} = B_{ij} d\epsilon_{ij}. \end{aligned} \quad (33a)$$

A similar linearization may be applied in the general endochronic theory to the expression $d\xi = (D_{ijklm} d\epsilon_{ij} d\epsilon_{kl})^{1/2}$.

Substituting eqn (33) into the stress-strain relations of endochronic theory, the formulation becomes equivalent to plasticity without normality. If eqn (33) is substituted in eqn (10) or (11), one obtains a plasticity formulation which is identical to the endochronic formulation for the cases of *proportional* (radial) loading, $e_{ij} = A_{ij}t$. Here

$$B_{ij} = \frac{A_{ij}}{\sqrt{(2A_{km}A_{km})}} \quad \text{for} \quad e_{ij} = A_{ij}t \quad (34)$$

where A_{ij} are given constants, and B_{ij} is also a constant. Furthermore, assuming that the test data available are essentially of the proportional loading type, for which $b_{ij} \sim e_{ij}$, the substitution of

$$d\xi = \frac{e_{ij} de_{ij}}{\sqrt{(2e_{km} e_{km})}} \quad \text{or} \quad d\xi = \beta_0 \sqrt{\left(\frac{1}{2} de_{ij} de_{ij}\right) + \frac{1-\beta_0}{\sqrt{(2e_{km} e_{km})}} e_{ij} de_{ij}} \quad (35)$$

in the endochronic formulation should allow (for any β_0) an equally good fit of test data. Moreover, parameter β_0 in this expression allows the ratios of inelastic strain at loading and unloading to be controlled.

Having linearized the incremental stress-strain relations, it is possible to put them in the form of eqn (32) and express the fourth-order tensor of tangential moduli D_{ijklm} . For this purpose, eqn (33) for $d\xi$ may be substituted in eqns (9) and (10), yielding

$$\begin{aligned} ds_{ij} &= 2G de_{ij} - 2GF_1 s_{ij} B_{rs} de_{rs}, \\ d\sigma_{nn} &= 3K de_{nn} - 9KF_2 B_{rs} de_{rs}. \end{aligned} \quad (36)$$

Insertion of $d\sigma_{ij} = ds_{ij} + \delta_{ij} d\sigma_{nn}/3$ and rearrangement yields

$$d\sigma_{ij} = (2G\delta_{ik}\delta_{jk} - 2GF_1 s_{ij} B_{rs} - 3KF_2 \delta_{ij} B_{rs}) de_{rs} + K\delta_{ij}\delta_{km} de_{km}. \quad (37)$$

Subsequently, introducing $de_{rs} = de_{rs} - \delta_{rs} de_{pp}/3 = (\delta_{kr}\delta_{rs} - \delta_{km}\delta_{rs}/3) de_{km}$, one obtains an equation of the form $d\sigma_{ij} = D_{ijklm} de_{km}$ (eqn 32) in which

$$D_{ijklm} = 2G\delta_{ik}\delta_{jk} + (K - 2G/3)\delta_{ij}\delta_{km} - (2GF_1 s_{ij} + 3KF_2 \delta_{ij})(B_{km} - 1/3B_{nn}\delta_{km}). \quad (38)$$

To illustrate the linearization, consider a simple endochronic formulation for a material with only two stress-strain components:

$$\begin{aligned} d\sigma_{11} &= E_{11} de_{11} + E_{12} de_{22} + F\sigma_{11}\sqrt{(de_{11}^2 + de_{22}^2)}, \\ d\sigma_{22} &= E_{21} de_{11} + E_{22} de_{22} + F\sigma_{22}\sqrt{(de_{11}^2 + de_{22}^2)}, \end{aligned} \quad (39)$$

where E_{11} , E_{22} , E_{12} , E_{21} are elastic moduli, F = constant, and the square-root expression corresponds to $d\xi$ from eqn (10). Assuming that the strain direction is $(1, 0)$ or $de_{11} > 0$, $de_{22} = 0$, one has $B_{11} = 1$, $B_{22} = 0$ and $(de_{11}^2 + de_{22}^2)^{1/2} \approx B_{11} de_{11} = de_{11}$ near the assumed direction. Equation (39) may then be brought to the form

$$\begin{aligned} d\sigma_{11} &= (E_{11} + FB_{11}\sigma_{11}) de_{11} + E_{12} de_{22}, \\ d\sigma_{22} &= (E_{21} + FB_{11}\sigma_{22}) de_{11} + E_{22} de_{22}. \end{aligned} \quad (40)$$

On the other hand, if the strain direction is assumed as $(0, 1)$ or $de_{22} > 0$, $de_{11} = 0$, one has $B_{11} = 0$, $B_{22} = 1$ and $(de_{11}^2 + de_{22}^2)^{1/2} \approx B_{22} de_{22} = de_{22}$ near this assumed direction. Equation (39) then becomes

$$\begin{aligned} d\sigma_{11} &= E_{11} de_{11} + (E_{12} + FB_{22}\sigma_{11}) de_{22}, \\ d\sigma_{22} &= E_{21} de_{11} + (E_{22} + FB_{22}\sigma_{22}) de_{22}. \end{aligned} \quad (41)$$

When the loading path is smooth, then the direction de_{ij} in each loading increment can be based on the direction in the preceding increment. However, when the loading path forms a sharp corner (paths AA_1 , AA_2 , AA_3 in Fig. 2), and this must always be assumed in analyzing material instability, then the direction b_{ij} of de_{ij} is unknown.

The dependence of D_{ijklm} on the unknown direction of de_{ij} will undoubtedly cause difficulties

in numerical analysis of material instabilities by finite elements. However, for the plastic formulations the situation is partly similar, in that the matrix D_{ijklm} is different for loading and unloading. For some types of vertex hardening, different matrix D_{ijkl} applies for several segments (cones) of directions: unloading, loading straight ahead, loading to the left in (e_{11}, e_{22}) -plane, loading to the right in (e_{11}, e_{22}) -plane, etc.

12. STABILITY, UNIQUENESS AND CYCLIC LOADING

It has been known since their inception that the endochronic formulations violate the normality rule and Drucker's stability postulate. On the basis of this fact it has been suspected in a recent critical study[39] of the endochronic formulations that they could lead to numerical difficulties, especially in cyclic loading, and it was concluded that the endochronic formulations may, therefore, be unsuitable for numerical structural analysis. In this respect it must be noted, however, that the violation of Drucker's postulate *per se* cannot be objected and is even proper. It is well established that unstable materials and strain-softening materials do exist and are quite common. It is, in fact, of main interest to detect situations when this is not so. Various studies of unstable strain localization and of vertex hardening are motivated by efforts to reveal material instabilities. It is the purpose of structural analysis to predict such phenomena. When an instability is encountered, the numerical algorithm cannot be stable, and convergence cannot take place. Thus, the aforementioned numerical difficulties might often be just an indication that material instability has been reached. For materials which are suspected of developing unstable strain localization, or which are known to exhibit strain-softening, it is actually imperative *not* to use a formulation which satisfies Drucker's postulate, or else real instabilities could be left undetected in the numerical calculation. In view of this, and because endochronic formulations are "softer for loading to the side", they will yield more conservative (safer) designs than plasticity formulations.

However, it must be admitted that there is at present little experience with the use of endochronic theory in finite element codes. Some numerical difficulties which have nothing to do with actual material instability, i.e. with the question of validity of Drucker's postulate, might be found, and methods to cope with them will then have to be investigated.

The feature of the endochronic formulation which has been repeatedly criticized in discussions at technical meetings and is also elaborated upon in Ref. [39] is the fact that inelastic strain can be getting continuously accumulated without bounds if a cyclic loading of arbitrarily small amplitude s is superimposed on constant stress σ_0 , with the result that instability of response and lack of uniqueness may occur. However, the choice of the precise nature of the stability and uniqueness condition is debatable, and so a reexamination is in order. Like in Ref. [39], let attention be restricted to uniaxial behavior, and consider the uniaxial endochronic formulation[8, 13]:

$$d\epsilon = \frac{d\sigma}{E} + \frac{\sigma}{E} d\zeta, \quad d\zeta = \frac{d\eta}{f(\eta)}, \quad d\eta = F(\epsilon) d\zeta, \quad d\zeta = |\epsilon| \quad (42)$$

in which $f(\eta)$ = hardening function[3, 4, 8] and $F(\epsilon)$ = softening function[4, 8], which are non-decreasing continuous functions of η and ϵ , respectively, and σ, ϵ now are the uniaxial stress and strain. (The subsequent analysis could also be applied to the endochronic formulation given by eqn (42) with $d\zeta = |\sigma|$, which was introduced in 1969 on pp. 70-71 of Ref. [40].)

Consider now a pulsating load in which the stress is prescribed in the form $\sigma = \sigma_0 + s \sin \omega t$, where t is a loading parameter. Let the strain produced by static load σ_0 at $t = 0$ and $0 < s \ll \sigma_0$ be denoted as ϵ_0 . Because for $s \rightarrow 0$ the pulsating stress is physically equivalent to static stress σ_0 , uniqueness and stability requires that the response $\epsilon(t)$ approach in some sense the value ϵ_0 produced by static stress σ_0 alone. However, it is arguable precisely in which sense this approach must take place. The following three different conditions of uniqueness and stability may be considered:

$$|\epsilon(t) - \epsilon_0| < \delta \quad \text{for all } t \text{ and some (sufficiently small) } s \quad (43a)$$

$$|\epsilon(t) - \epsilon_0| < \infty \quad \text{for all } t \quad (43b)$$

$$\lim_{s \rightarrow 0} [\epsilon(t) - \epsilon_0] = 0 \quad \text{for any given } t \quad (43c)$$

where δ is a given arbitrarily small positive number and ϵ_0 is the value of ϵ for $\sigma = \sigma_0$ at $t = 0$. The first condition, eqns (43a), is a strict stability condition of Liapunov type, like that used in dynamics. The second condition is a weaker one, and the last one is merely a continuity condition, which is required for the problem to be properly posed and is the most reasonable condition from the physical point of view.

Although none of the endochronic formulations guarantees fulfilment of the strict condition, eqn (43a), unless an unloading criterion is introduced, all endochronic formulations satisfy eqn (43c) and some also satisfy eqn (43b). To show it, integrate eqn (42). It is easy to check that, for $\sigma_0 > s > 0$,

$$\frac{\sigma_0 - s}{E} \Phi_t < \epsilon(t) - \epsilon_0 < \frac{\sigma_0 + s}{E} \Phi_t \quad \text{with} \quad \Phi_t = \int_0^t \frac{d\eta}{f(\eta)} \quad (44)$$

in which Φ_t is a continuous non-decreasing function of $\eta = \eta(t)$. For the pulsating load given

$$\eta_0 + k_0 s F(\epsilon_0) t < \eta < \eta_0 + k_1 s F(\epsilon_0) t \quad (45a)$$

where

$$k_0 = \left(\frac{1}{E_u} + \frac{1}{E_r} \right) \frac{2\omega}{\pi}, \quad k_1 = \left(\frac{1}{E} + \frac{1}{E_r} \right) \frac{2\omega}{\pi} \quad (45b)$$

in which $\eta_0 = \text{constant}$, E_r is the initial tangent modulus at $t = 0$, and E_u is the initial unloading modulus at $t = 0$. The inequality ensues from the fact that as pulsation goes on the hardening function causes the current tangent loading modulus to increase and the current unloading modulus E_u to decrease, while always $E_r < E < E_u$ (see Fig. 1).

Since $\Phi_t < f_1(st)$ where f_1 is an increasing continuous function, with $f_1(0) = 0$ and bounded for finite st , it follows that $\lim \Phi_t$ for $s \rightarrow 0$ at any fixed t is 0. According to eqn (44), this means that the weak condition in eqn (43c) is always satisfied.

According to eqn (45a), $\lim \eta$ for $t \rightarrow \infty$ is ∞ , and, in consequence, the stronger stability condition in eqn (43b) is satisfied if $\lim \Phi_t$ for $t \rightarrow \infty$ is bounded, i.e. if the inverse of the hardening function $f(\eta)$ is integrable up to ∞ . This is not true for Valanis' hardening function [3] $f(\eta) = 1 + \beta_1 \eta$, for which $\Phi_t = (1/\beta_1) \ln(1 + \beta_1 \eta) + \text{constant}$. Equation (43b) is satisfied, however, if

$$f(\eta) \leq A\eta^m, \quad m > 1, \quad \text{for sufficiently large } \eta, \quad (46)$$

where A is some constant. This is true, for instance, if

$$f(\eta) = 1 + \beta_1 \eta + \beta_2 \eta^2 \quad (\beta_1, \beta_2 > 0) \quad (47)$$

which is the function that has allowed improvement in fits of cyclic test data [8].

The strict stability condition (eqn 43a) could be satisfied only if $\lim \Phi_t$ for $t \rightarrow \infty$ and fixed s approached 0 as $s \rightarrow 0$. This would require that $\lim \eta$ for $t \rightarrow \infty$ approached $\eta(0)$ as $s \rightarrow 0$, and according to eqn (45a) this is never true because $\lim \eta$ for $t \rightarrow \infty$ is ∞ .

Another case studied in Ref. [39] was the pulsating strain. Thus, consider that $\epsilon(t) = \epsilon_0 + e \sin \omega t$, with $0 < e \ll \epsilon_0$. Rewriting eqn (42) as $d\sigma + \sigma d\xi = E d\epsilon$, it is seen that $\Delta\sigma + \sigma_0 \Delta\xi \approx 0$ for each pulsation cycle, and so (for $e \ll \epsilon_0$) the response is stress relaxation of the form:

$$\sigma(t) \approx \sigma_0 \exp(-\Phi_t) \quad (48)$$

in which Φ_t is again given by eqn (44) and $\xi_t = ket$, with $k = 2\omega/\pi$. Further arguments are similar as before and the conclusions are the same, except that $\sigma(t)$ always satisfies not only the condition of the type of eqn (43c), but it always satisfies also the stronger condition, eqn (43b).

For comparison, consider now a typical, classical formulation of viscoplasticity. Its simple uniaxial form may be written as $d\epsilon = d\sigma/E + \psi(\sigma)\sigma dt$ with $\psi(\sigma) = 1 + k\sigma^2$, $k > 0$. The response to static stress $\sigma(t) = \sigma_0$ applied suddenly at $t = 0$ is $\epsilon(t) = \epsilon_0 + \psi(\sigma_0)t$, with $\epsilon_0 = \sigma_0/E$. Assuming that $\sigma > 0$, $\epsilon > 0$, the response to pulsating stress $\sigma(t) = \sigma_0 + s(1 + \sin \omega t)$, $s > 0$, satisfies the inequality:

$$\epsilon_0 + \psi(\sigma_0 + s)t < \epsilon(t) < \epsilon_0 + \psi(\sigma_0 + 2s)t. \quad (49)$$

Thus, the difference between the responses to static and pulsating load is $\epsilon(t) - \epsilon_0 = at$, where $0 < \psi(\sigma_0 + s) - \psi(\sigma_0) < a < \psi(\sigma_0 + 2s) - \psi(\sigma_0)$. Since $\epsilon(t) - \epsilon_0$ grows beyond any bound, only the condition in eqn (43c) is satisfied. However, the stronger conditions in eqns (43a) and (43b) are not satisfied. If this is admissible in classical viscoplasticity, it must be admissible in endochronic inelasticity.

In viscoplasticity, the strict stability condition in eqn (43a) becomes satisfied in the limit of infinitely rapid deformation, i.e. if $\omega \rightarrow \infty$, making the response perfectly elastic. If desired, the endochronic theory, too, can be formulated so as to make the response for $\omega \rightarrow \infty$ as close to perfect elasticity as one pleases. To this end, it suffices to use a time-dependent endochronic formulation associated with a Maxwell chain model[8, 14] whose shortest relaxation time is sufficiently short compared to the oscillation period.

Fulfillment of the strict stability condition in eqn (43a), if deemed desirable, can be achieved also in other ways. One other way is to incorporate into the endochronic formulation an unloading criterion. This is not at all against the spirit of endochronic theory, as the absence of the unloading criterion is not the essential feature of endochronic theory, anyway. (Rather, it is the curvature of the inelastic stiffness locus.) As an alternative way, eqn (43a) may be satisfied by making hardening function $f(\eta)$ depend on the energy dissipated up to current time, D , is such a way that a certain value of D depending on I_1'') could not be exceeded.

The violation of uniqueness and stability requirements was suggested in Ref. [39] to give rise to serious numerical difficulties and preclude the use of endochronic formulations in practical numerical analysis of structures. However, within the context of eqn (43c), the physically reasonable condition, this could be true only if ω were arbitrarily large; but this is impossible, because ω can never exceed the first fundamental frequency of the grid used (not even for a step load history). In a continuum, ω can, of course, be arbitrarily large, but then a time-dependent endochronic formulation based on Maxwell chain with a sufficiently short first relaxation time should properly be used. Moreover, the period of oscillation of the grid due to numerical error would be very short, probably shorter than the time step used, in which case the oscillation could not be reflected in numerical solution.

Therefore, the claim of innate numerical unsuitability of endochronic formulations[39] appears to be an exaggeration.

13. UNLOADING, RELOADING AND NON-VISCOUS HYSTERESIS

By contrast to plasticity, the response curves for the ordinary endochronic formulations (eqns 9 and 10) exhibit at the start of unloading a slope that exceeds the current elastic modulus E . For metals, experimental data clearly contradict such behavior. However, for geological materials the interpretation of experimental data is not clear because the current elastic modulus E gets reduced by microcracking, as compared to the initial modulus E_0 , which causes that the unloading slope which does not exceed E_0 may or may not be higher than E , depending on the value of E .

However, if a reduction of E due to microcracking is not considered, or if it is too mild, it may be appropriate to introduce an expedient combination with plastic formulations, in which an endochronic unloading criterion is postulated and strain $d\epsilon_{ij}^{pl}$ is either reduced or completely canceled whenever unloading occurs. Some unloading criteria have already been suggested for endochronic formulations: $dJ_2(\epsilon) < 0$ [8], and $s_{ij}^{pl} d\epsilon_{ij}^{pl} < 0$ [11]. Valanis' concept of internal barriers[12] is also a form of an unloading criterion. When $d\xi$ is taken as zero for unloading, the inelastic stiffness locus for the endochronic theory assumes the shape of a "bulge", as shown in Figs. 5(b) or 7(b). With such a criterion the endochronic formulation can be made to satisfy Drucker's postulate.

Consider now again cyclic loading, directing particular attention to the work ΔW dissipated due to non-viscous (rate-independent) hysteresis during unload-reload stress cycles of arbitrarily small amplitude s . This work consists only of second order terms $\Delta W = d\sigma_{ij} d\epsilon_{ij}^{pl}/2$ because the first order terms $\sigma_{ij} d\epsilon_{ij}^{pl}$ cancel with the work of applied loads, owing to the principle of virtual work. Although there is no fundamental necessity to satisfy the condition $\Delta W \geq 0$ (Drucker's postulate), it may be reasonable to do so, especially at lower stress levels, unless there is some good reason against it (e.g. when a release of frictionally blocked elastic energy is expected due to friction reduction). The ordinary endochronic formulations (eqns 9 and 10) violate this condition, because for the unload-reload cycle ΔW is negative (being represented by the cross-hatched area 123 in Fig. 10)[39]. Classical plasticity, as well as the afore-mentioned endochronic formulations with unloading criterion, gives $\Delta W = 0$. This satisfies Drucker's postulate but does not permit representation of cyclic strain accumulation (cyclic creep, "ratcheting"). This is a rather important phenomenon, whose mechanism cannot be explained by plastic slip (and calls for other effects, such as the "ratchet effect"[40-42]). So all existing formulations are inadequate.

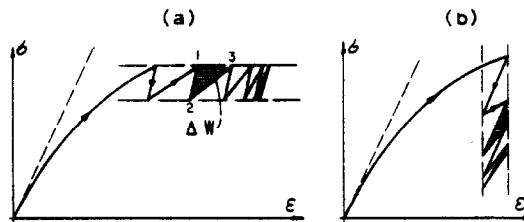


Fig. 10. Small cycles stress or strain superimposed on static stress or strain.

To model cyclic strain accumulation, the end point of the cycle (point 3 in Fig. 10a) must be to the right of the starting point of the cycle (point 1 in Fig. 10a). Obviously, the only way to obtain this and yet satisfy $\Delta W \geq 0$ for arbitrarily small amplitudes s is to meet these conditions:

- (1) At the start of unloading (point 1), as well as at the start of reloading (point 2), $d\epsilon_{ij}''$ may not be of the opposite sign as $d\epsilon_{ij}'$.
- (2) Inelastic strain of the same sign as $d\epsilon_{ij}'$ must be produced right after the start of unloading and again right after the start of reloading.
- (3) During reloading the inelastic strains must be more pronounced than during unloading, but less pronounced than during virgin loading.

These conditions can be met as follows. Conditions 1 and 2 require the use of kinematic hardening, such that α_{ij} (center of loading surface) (eqn 29) is set equal to s_{ij} at the start of deviatoric unloading and again at the start of deviatoric reloading; and similarly α_0 is set equal to σ_0 at the start of volumetric unloading or reloading. An unloading-reloading criterion must, therefore, be introduced. This criterion can neither involve only stresses, for strain-softening may not be interpreted as unloading, nor can it involve only strains, for in symmetric hysteresis loops the return branch would make the transition to virgin loading too late. This fact, along with the fact that Drucker's postulate[22] is concerned with work, suggests that the criterion be expressed in terms of internal volumetric and deviatoric work W and W' , defined as

$$dW = 3\sigma d\epsilon, \quad dW' = s_{ij} d\epsilon_{ij}. \quad (50)$$

The inelastic strains from eqn (9) may be redefined as

$$d\epsilon^{pl} = (\sigma - \alpha_0)c d\lambda, \quad d\epsilon_{ij}^{pl} = (s_{ij} - \alpha_{ij})c' d\zeta \quad (51)$$

in which

$$(1) \text{ for } dW \geq 0 \text{ and } W = W_0: \quad c = 1 \\ \text{for } dW' \geq 0 \text{ and } W' = W'_0: \quad c' = 1 \quad (\text{virgin loading}) \quad (52a)$$

$$(2) \text{ for } dW < 0: \quad c = c_u \\ \text{for } dW' < 0: \quad c' = c'_u \quad (\text{unloading}) \quad (52b)$$

- (3) for $dW \geq 0$ and $W < W_0$: $c = c_r$,
 for $dW' \geq 0$ and $W' < W'_0$: $c = c'_r$ (reloading) (52c)

with $1 \leq c_r < c_u$, $1 \leq c'_r < c'_u$; W_0 and W'_0 are the maximum values of W and W' attained up to the current time. Coefficients α_{ij} or α_0 are set equal s_{ij} whenever ds_{ij}^{pl}/de_{ij} or $d\sigma^{pl}/d\epsilon$ becomes negative, which assures that each of these expressions always remains positive (Il'iushin's postulate). (According to a private communication by C. L. Shieh, this formulation works quite well for cyclic loading of concrete as well as sand.)

Consider now the uniaxial equivalent of eqns (50) and (51):

$$d\epsilon = \frac{d\sigma}{E} + \frac{\sigma - \alpha}{E} \phi c d\zeta, \quad \phi = (\sigma - \alpha)^{m-1}, \quad d\zeta = \frac{d\eta}{f(\eta)}, \quad d\eta = F(\epsilon)|d\epsilon| \quad (53)$$

where α is the current center of yield surface, corresponding to α_{ij} , α_0 , and function ϕ is added as it is useful to control the dissipated work during the cycle. (This form has a multiaxial counterpart in setting $q = (m+1)/2$ in eqn (28) for the loading function.) The second order work dissipated during an unload-reload cycle is $\Delta W = \Delta W_1 - \Delta W_2$ where ΔW_1 and ΔW_2 are the areas shown in Fig. 11. For a very small amplitude s , ΔW may be easily calculated from eqn (53), because F , f , and E may be considered constant during the cycle. Equation (53) for the unloading branch as well as the reloading branch may be written as $d\bar{\sigma} = E(1 - a\bar{\sigma}^m)d\bar{\epsilon}$ in which $\bar{\sigma} = \sigma - \alpha_0$, $\bar{\epsilon} = \epsilon - \epsilon_0$, $a = cF/fE$ at the beginning of the cycle, ϵ_0 = strain at the beginning of the unloading branch. For small $\bar{\sigma}$ this equation is equivalent to $d\bar{\epsilon} = (1 + a\bar{\sigma}^m)d\bar{\sigma}/E$. The area \bar{W} above the unloading curve or below the reloading curve satisfies the equation $d\bar{W} = \bar{\sigma} d\bar{\epsilon} = (1 + a\bar{\sigma}^m)\bar{\sigma} d\bar{\sigma}/E$. Integration of these two equations yields

$$\bar{\epsilon} = \frac{\bar{\sigma}}{E} \left(1 + \frac{a}{m+1} \bar{\sigma}^m \right), \quad \bar{W} = \frac{\bar{\sigma}^2}{2E} \left(1 + \frac{2a}{m+2} \bar{\sigma}^m \right). \quad (54)$$

Superimposing the strains and the areas for the unloading and reloading branches, one finds that the net strain increment and the dissipated work for the unload-reload cycle (Fig. 11) are

$$\Delta\epsilon = \frac{F(c_r - c_u)}{E(m+1)f} (2s)^{m+1}, \quad \frac{\Delta W}{W_1} = \frac{2F}{(m+2)f} \left(c_u - \frac{c_r}{m+1} \right) (2s)^m \quad (55)$$

in which $W_1 = (2s)^2/2E$ = elastic work of $\bar{\sigma}$ during reloading.

It is now evident that cyclic strain accumulation will occur if $c_u < c_r$. To also ensure that $\Delta W \geq 0$, it is necessary that

$$c_u < c_r \leq (m+1)c_u. \quad (56)$$

This is the condition under which an inelastic constitutive law exhibits cyclic non-viscous strain accumulation, yet satisfies Drucker's postulate. This condition is of general validity, because for small enough s the loop can always be approximated by power curves, for any constitutive law.

The ratio $\Delta W/W_1$ characterizes non-viscous hysteretic internal damping. Equation (55) for

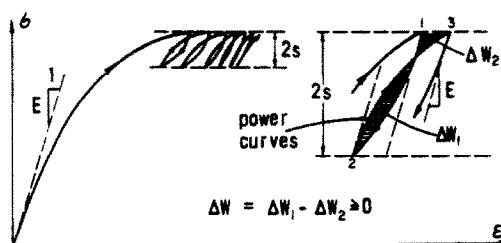


Fig. 11. Cyclic creep with small hysteresis loops for positive energy dissipation.

$\Delta\epsilon$ seems to give best agreement with Whaley and Neville's test data on cyclic creep of concrete if $m = 2/3$, although $m = 1$ is also acceptable.

It can be shown in the same manner as before that the present formulation again satisfies only continuity condition (43a), and possibly also (43b), but not the Liapunov-type condition (43a), despite the fact that Drucker's postulate is not violated by the cyclic response. The opposite case, in which a frictional material is Liapunov-stable yet violates Drucker's postulate, has also been pointed out [33]. Thus it must be concluded that Drucker's stability postulate is not a reliable indicator of Liapunov-type stability.

The preceding treatment of hysteresis can be readily adapted to plastic formulations using the tangential linearization (eqn 33).

14. INELASTIC STIFFNESS LOCUS FOR FRACTURING MATERIAL

The strain-softening, i.e. the decline of stress at increasing strain, as commonly observed in geological materials, is very hard to model in terms of plasticity theory. Even the modeling of a limit state, in which $d\sigma/d\epsilon \rightarrow 0$, is difficult. One suitable formulation for limit states and the subsequent strain-softening is the endochronic formulation. However, another possible formulation which is similar to plasticity has been recently proposed simultaneously by Dougill [35, 36] and by Naghdi and Trapp [37]. Dougill assumes that $\sigma_{ij} = D_{ijklm}\epsilon_{lm}$ where D_{ijklm} are the elastic moduli which decrease as the strain grows, $D_{ijklm} = D_{ijklm}(\epsilon_{ij})$, so as to model the effect of microcracking in an inhomogeneous material. By differentiation this yields

$$d\sigma_{ij} = D_{ijklm} d\epsilon_{lm} - d\sigma'_{ij}, \quad d\sigma'_{ij} = -dD_{ijklm} \epsilon_{lm}. \quad (57)$$

The main question now is how to determine dD_{ijklm} . To this end, Dougill postulates a "fracture surface" $F(\epsilon_{ij}, H_k) = 0$, which is defined as an envelope of all states ϵ_{ij} which can be reached from the current state without further fracturing (microcracking); H_k are fracturing parameters. Choosing $(\partial F/\partial H_k) dH_k$ to be negative when fracturing occurs and noting that $dF = (\partial F/\partial \epsilon_{ij}) d\epsilon_{ij} + (\partial F/\partial H_k) dH_k = 0$, it is clear that

$$(\partial F/\partial \epsilon_{ij}) d\epsilon_{ij} > 0 \quad (58)$$

when fracturing occurs and $\partial F/\partial \epsilon_{ij} \leq 0$ when it does not. Consequently

$$d\sigma'_{ij} = g_{ij} \frac{\partial F}{\partial \epsilon_{km}} d\epsilon_{km} \quad (59)$$

where g_{ij} are some constants. Now, imposing Il'iushin's postulate [35, 37], $d\sigma'_{ij} d\epsilon_{ij} \geq 0$, which is a complementary form of Drucker's stability postulate, and comparing this with eqn (58), it follows that $d\sigma'_{ij} \sim -\partial F/\partial \epsilon_{ij}$. Furthermore, comparison with eqn (59) yields $g_{ij} \sim -\partial F/\partial \epsilon_{ij}$, and so one may set

$$d\sigma'_{ij} = \frac{\partial F}{\partial \epsilon_{ij}} \phi d\kappa, \quad d\kappa = \frac{\partial F}{\partial \epsilon_{km}} d\epsilon_{km}. \quad (60a,b)$$

For a dilatant pressure-sensitive material, a suitable form for F is here proposed to be $F(\epsilon_{ij}) = \bar{\gamma} + h(\epsilon) - H_1$, with $\bar{\gamma} = (\epsilon_{ij}\epsilon_{ij}/2)^{1/2}$. For this case eqns (60a,b) yield

$$ds'_{ij} = \phi \epsilon_{ij} \frac{d\kappa}{2\bar{\gamma}}, \quad d\sigma'_{ij} = \frac{1}{2} \alpha \phi d\kappa \quad (61)$$

$$d\kappa = d\bar{\gamma} + \alpha' d\epsilon = \frac{\epsilon_{km} d\epsilon_{km}}{2\bar{\gamma}} + \alpha' d\epsilon \quad (62)$$

where ϕ is an empirical coefficient, which can depend on σ_{ij} and ϵ_{ij} , $\alpha' = dh(\epsilon)/d\epsilon$ and $\alpha = \alpha'$ according to eqns (60a,b). Similarly as for the effect of internal friction on plastic shear, it is possible that shear fracturing depends on volume change or ϵ . Then, $\alpha \neq \alpha'$, and the normality

rule, expressed in eqn (60a), no longer applies for volume changes. For cyclic loading it seems appropriate to introduce kinematic hardening by replacing ϵ_{ij} with $\epsilon_{ij} - \alpha_{ij}$ in the expression for F .

To compare the fracturing material formulation with other formulations, it is of interest to determine the inelastic stiffness locus, defined again as the locus of the end points of all vectors $d\epsilon_{ij}$ which give the same $d\sigma_{ij}^H$. According to eqn (60a) it is necessary that $d\kappa$ be constant. Hence, according to eqn (60b), the plastic stiffness locus is a plane (hyperplane), which appears as a straight line in a two-dimensional picture. The plane is oriented tangentially to the current fracture surface except in case of eqn (62) where for $\alpha' \neq \alpha$ normality is not satisfied in the volumetric cross section.

Thus, the inelastic stiffness locus is the same as in plasticity theory, and the difference consists only in the facts that hardening is governed by strain rather than stress and that elastic moduli decrease rather than being constant. So, the fracturing material theory as described here exhibits no inelastic strain for loading to the side, which must be questioned when material instability is to be investigated.

15. CONCLUSIONS

(1) A meaningful way to compare the endochronic and classical plastic formulations is by studying the locus of all strain increment vectors which give the same magnitude of inelastic strain increment, called inelastic stiffness locus.

(2) For classical plasticity the inelastic stiffness locus is a straight line, whereas for the endochronic theory it is a circle or a sphere, and for its refined versions it is an ellipse (ellipsoid) or a bulge on a line. The basic difference of endochronic formulations from incremental plastic formulations consists in the fact that the inelastic stiffness locus is curved, rather than straight. This causes that for a strain increment which is tangent to the current loading surface (loading to the side) the endochronic theory exhibits plastic strain, making the response softer, while the associated or non-associated plasticity gives purely elastic response. The latter is in contradiction to the prediction of microstructural polycrystalline models, which show that the current yield surface may shrink almost to a point or infinitesimal circle, as in endochronic theory.

(3) Endochronic theory is similar to the vertex hardening models and to the deformation theory of plasticity in that the inelastic stiffness locus is not a straight line and that inelastic strain accompanies strain increments tangential to the loading surface, while in plasticity the response is elastic. Therefore, among all inelastic theories, classical plasticity is least prone to indicate material instability.

(4) In endochronic theory as well as classical plasticity, the inelastic strain is always normal to the loading surface, while in vertex hardening models it is not. Thus, endochronic formulation is stiffer than vertex hardening for strain directions parallel to the loading surface, and so it is less prone to indicate material instability.

(5) The decision whether the plastic or the endochronic formulation is correct is solely up to the test data or a microstructural model for a given material. The classical plasticity formulation with normality rule is the least safe assumption when the material is expected to exhibit instabilities (unstable strain-localization) or strain-softening, and when one is interested in finding these instabilities. The experiments which are most relevant for making the choice between these two theories are not only unloading and cyclic loading but also loading to the side of the previous path in the strain space, and loading into the strain-softening range.

(6) For the loading surface and the hardening rules of plasticity, such as isotropic and kinematic hardening, one can define their counterparts in endochronic formulations; but these are relevant only as far as the "local" hardening rule near the current state is concerned.

(7) It is reasonable to expect that the intrinsic time increments $d\xi$ exhibit stress-induced anisotropy, such that the quadratic form defining $d\xi$ consists of invariants of $d\epsilon_{ij}$ only if the material is stress-free. This type of stress-induced anisotropy distorts a circular inelastic stiffness locus into an ellipse or an eccentric circle.

(8) While in plasticity theory it is possible to formulate the matrix of tangential moduli, one for loading and one for unloading, in endochronic theory the matrix of tangential moduli can only be expressed if the direction of the strain increment vector is known. Otherwise, the entire

matrix of tangential moduli continuously depends on this direction, which is unknown when material instability is to be analyzed. For proportional loading, the endochronic formulation can be converted to equivalent non-associated plasticity formulation.

(9) By contrast to plasticity, but similarly to viscoplasticity, the ordinary endochronic formulations do not satisfy a stability condition of Liapunov-type, as is revealed by studying the response to pulsating loads of small amplitude. This is because Drucker's stability postulate is violated, and it must be so for materials which exhibit internal friction and microcracking. Physically, however, only a continuity condition is justified, and this condition is satisfied by endochronic formulations provided that the frequency of oscillating stress is bounded (which is always true for finite element grids). Various refinements are possible to make endochronic formulations satisfy the stronger stability condition and/or prevent unbounded accumulation of inelastic deformation during cyclic loading.

(10) Introducing unloading and reloading criteria and kinematic hardening such that the center of the loading surface is moved to the current stress point whenever loading reverses to unloading or *vice versa*, the endochronic formulation can be made to satisfy Drucker's postulate for hysteresis loops, while at the same time not guaranteeing Liapunov stability.

(11) The fracturing material theory in which the loading surface depends on strain rather than stress is similar to plasticity in that the inelastic stiffness locus is also a straight line.

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APPENDIX

ENDOCHRONIC THEORY AS VISCOPLASTICITY WITH STRAIN-RATE DEPENDENT VISCOSITY

Consider the viscoplastic constitutive relation

$$d\epsilon_{ij} = D_{ijklm} d\sigma_{km} + d\epsilon''_{ij}, \quad d\epsilon''_{ij} = \frac{\partial F}{\partial \sigma_{ij}} dz, \quad dz = \frac{dt}{a(\sigma, \dot{\epsilon})} \quad (63)$$

in which t = time; z was initially called reduced time [1, 2] and is now better known as intrinsic time [3]. In classical viscoplasticity, the viscosity coefficient, a , is a function of σ and possibly also $\dot{\epsilon}$. However, as suggested by Schapery [1, 2], generally a must be considered to depend also on the strain rates $\dot{\epsilon}_{ij}$, which may be assumed in the form $a = a_1(\sigma, \dot{\epsilon})a_2(\dot{\epsilon})$. If the inelastic strain develops gradually, function $a_2(\dot{\epsilon})$ may be expected to be continuous and smooth. Then a Taylor series expansion is admissible:

$$[a_2(\dot{\epsilon})]^{-r} = p_0 + p_1 \dot{\epsilon}_{ij} + p_{ijklm} \dot{\epsilon}_{ij} \dot{\epsilon}_{km} + p_{ijklmpq} \dot{\epsilon}_{ij} \dot{\epsilon}_{km} \dot{\epsilon}_{pq} + \dots \quad (64)$$

where r = some exponent to be determined later. The series will be truncated after the cubic terms. The linear and cubic terms must be, however, discarded ($p_{ij} = p_{ijklmpq} = 0$) because they would violate the condition that a must decrease as $|\dot{\epsilon}_{ij}|$ increases.

Furthermore, it is of interest to examine the limit case for infinitely high strain rate, $|\dot{\epsilon}| \rightarrow \infty$. From eqn (64):

$$\frac{|d\epsilon''|}{|d\epsilon|} = \left\| \frac{\partial F}{\partial \sigma_{ij}} \right\| \left(\frac{p_0}{|\dot{\epsilon}|^r} + \frac{p_{ijklm}}{d\epsilon_{pq}} \frac{d\epsilon_{ij}}{d\epsilon_{pq}} |\dot{\epsilon}|^{2-r} \right)^{1/r}. \quad (65)$$

On physical grounds, for $|\dot{\epsilon}| \rightarrow \infty$ this ratio must tend neither to infinity nor to zero. The latter case, which represents perfectly elastic instantaneous response, is obtained for $2 - r < 0$. The former case is obtained for $2 - r > 0$. Therefore, the only possibility left is $2 - r = 0$ or $r = 2$. Equation (64) may then be rewritten in the form

$$dz = \frac{dt}{a} = \frac{dt}{a_1(\sigma, \dot{\epsilon})} (p_0 + p_{ijklm} \dot{\epsilon}_{ij} \dot{\epsilon}_{km})^{1/2} = \frac{1}{a_1(\sigma, \dot{\epsilon})} (p_0 dt^2 + p_{ijklm} d\epsilon_{ij} d\epsilon_{km})^{1/2}. \quad (66)$$

Note that for a certain choice of p_{ijklm} , the reduced time coefficient a is a non-negative function of the total octahedral strain rate, as suggested in 1968 by Schapery (p. 279 of Ref. [1]). The particular square root-type form, deduced here (and

in Ref. [8]) from physically reasonable conditions, was the starting assumption of Valanis[3]. Equation (66) may be rewritten as

$$dx = \left[\left(\frac{d\zeta}{Z_1} \right)^2 + \left(\frac{dt}{\tau_1} \right)^2 \right]^{1/2}, \quad d\zeta = (P_{ijklm} d\epsilon_{ij} d\epsilon_{klm})^{1/2} \quad (67)$$

in which $1/\tau_1 = \sqrt{(p_0)/a_1(\sigma, \epsilon)}$. $P_{ijklm} = Z_1^2 p_{ijklm}/a_1^2$. $Z_1 = \text{constant}$; τ_1 is a characteristic retardation time whose dependence on σ and ϵ models classical viscoplastic behavior. For rapid deformations, $d\zeta/dt \rightarrow \infty$, dt drops from eqn (67) and $z = \zeta/Z_1$, which makes eqns (67) and (63) equivalent to eqns (9) and (10). Coefficients P_{ijklm} are variable, which may be most simply described by a scalar hardening function of ζ , as proposed first by Valanis[3].

The foregoing analysis shows that endochronic theory is a special case of general viscoplasticity and the intrinsic time is equivalent to the reduced time used in viscoplasticity.