Uncertainty Analysis of Creep and Shrinkage Effects in Concrete Structures

by Henrik O. Madsen and Zdeněk P. Bažant

A simple probabilistic model, which allows calculation of simple statistics for shrinkage and creep effects for structural elements, is presented. The statistics can be, in particular, the mean value function and the covariance function (including the variance function), which seem to be the most interesting statistics for serviceability analysis. Any deterministic creep and shrinkage formula can be the basis for probabilistic creep and shrinkage models. The formulas are randomized by introducing the entering parameters as random variables and by further introducing random model uncertainty factors to characterize the incompleteness or inadequacy of the deterministic formulas. Statistics for the model's uncertainty factors are derived by a comparison between available test data and predictions for these tests by the formulas. The creep and shrinkage formulas developed by Bažant and Panula (BP model) are chosen.

Structural analysis can be carried out by any conventional deterministic method. In this paper, the analysis is performed by a matrix method which gives essentially exact results if the time discretization is close enough. A large number of examples of practical relevance demonstrates the potential of the method. Some of the examples show that structural effects which in a deterministic analysis almost vanish can have a very large uncertainty, which should be taken into account in practical design.

Keywords: concrete construction; creep properties; mathematical models; probability theory; reliability; shrinkage; structural analysis; structural members; viscoelasticity.

The analysis of uncertainties in the prediction of shrinkage and creep effects in concrete structures has gained wide interest in recent years. This is quite natural since concrete strength and elasticity are already being treated as random properties while actually the scatter in these properties is not as large as the scatter in shrinkage and creep properties. Although many sources contribute, there are basically two types of uncertainty related to the shrinkage and creep processes. These are referred to as the external uncertainty and the internal uncertainty.¹ The external or parameter uncertainty arises from the uncertainty in the influencing parameters such as those representing environment and concrete composition. The internal uncertainty is that inherent in the microscopic creep processes or creep mechanisms. The external uncertainty is generally believed to be the main type. Models for the description of internal uncertainty are proposed in References 1 and 3. However, only a few test results exist for validation of these models and for parameter estimation.

If no internal uncertainty was present, deterministic relations between the environmental and concrete composition parameters, and shrinkage and creep should in principle exist. Due to the complexity of the phenomena these relations have not quite been determined so far. Many approximate relations have been proposed on the basis of physical considerations and test results. No matter which relation is chosen, a model uncertainty is, however, introduced and must be taken into account.

The proposed creep and shrinkage formulas are based on test results obtained with fairly small test specimens, usually loaded uniaxially in pure compression. In structural analysis concrete is usually considered homogeneous although there exist within the cross sections large variations in pore relative humidity, temperature, and degree of hydration, giving rise to self-equilibrated stresses, microcracking, and cracking. The creep and shrinkage formulas can therefore give only some averaged properties of the cross section and an additional uncertainty is thus introduced.

Other important sources of uncertainty come from structural analysis. For practical reasons a linear creep law is normally adopted so that the principle of superposition can be used. This may often be a good approximation for service stresses less than 40 to 50 percent of the ultimate compressive strength, but this still adds to the uncertainty of the predictions.

Even assuming homogeneity of concrete, linearity of the creep law, and validity of the ordinary bending theory, calculation of structural effects may still be too complicated. In addition to the highly accurate numerical methods based on time step integration, various simplified methods are therefore used; these especially in-

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include the effective modulus method and the rate-of-flow method. Additional uncertainty caused by using one of these simplified methods is reported in Reference 4.

This paper investigates the external uncertainty in the shrinkage and creep functions and the model uncertainties related to choosing one specific set of shrinkage and creep formulas. A practical method for calculating simple statistics for shrinkage and creep effects for structural elements is presented. The probabilistic method adopted is very simple and, in principle, only requires averaging of a certain number of results from deterministic calculations.

The main objective is to quantify the effect that the correlation between the parameters in different cross sections of the same concrete structure has on the total uncertainty. This effect is illustrated by a number of relevant examples.

Loads, geometrical parameters, and steel parameters are assumed to be deterministic and known; only the influence of varying concrete and environmental parameters is analyzed.

**SHRINKAGE AND CREEP FORMULAS**

Many prediction formulas for shrinkage and creep have been suggested both in literature and in national and international codes. A common feature of all the formulas is that none of them predict creep and shrinkage perfectly, although some formulas are definitely more accurate than others. The amount of information on the input parameters such as environmental conditions and concrete composition varies considerably from formula to formula. This is not bad because ideally a hierarchy of formulas should be available, demanding increasing amounts of information on the input parameters and in return giving more reliable predictions.

In this study the formulas developed by Bažant and Panula (BP formulas) are chosen. In comparisons of test results with predictions from different formulas, these formulas showed the best agreement. The BP formulas are to a greater extent based on physical considerations than other formulas. Two main advantages of the BP formulas are that they are equally reliable for a large variety of concrete compositions and environmental conditions and that they predict future developments of creep and shrinkage with good accuracy if one or two short-time measurements have been made.

The BP formulas are given in detail in Reference 5. In short, they are for shrinkage

\[ \varepsilon_{sh}(t, t_0) = \varepsilon_{shm} k_s S(t, t_0) \]  

and for creep

\[ j(t, t') = \frac{1}{E_o} + C_d(t, t') \]

\[ + C_C(t, t', t_0) - C_C(t, t', t_0) \]

Here \( \varepsilon_{sh} \) = shrinkage strain, \( t_0 \) = age of concrete when drying begins; \( j(t, t') \) = compliance function (also called the creep function) = strain at age \( t \) caused by a uniaxial unit stress acting since concrete age \( t' \); \( E_o \) = asymptotic modulus (= 1.5 \( E \), where \( E \) = conventional elastic modulus of concrete); \( C_o, C_d, C_p \) = functions describing basic creep (creep at constant temperature and humidity), increase of creep during drying, and decrease of creep after drying; \( C_C(t, t', t_0) = (\phi \cdot E_o) (t' - m + \alpha (t - t'))^n \) (double power law) where \( \phi, m, n, \) and \( \alpha = \) parameters, depending on the type of concrete; \( C_d \) = function similar to \( S \); and \( C_p \) = function similar to \( C_o, k_s = \) function of relative humidity \( h \), defined as \( 1 - h^3 \) if \( h \leq 0.98 \), or else \(-0.2 \); and \( S \) = empirical function of the variable \( \tau_{sh}/\tau_{sh} \) where \( \tau_{sh} = \) shrinkage-square half time, proportional to the square of the dimension of the cross section, \( S(t, t_0) = [1 + \tau_{sh}/(t-t_0)]^{-n} \) where \( \tau_{sh} = \) shrinkage-square half time = const. \( \times (k_p D)^{2/3} \) = volume-to-surface ratio, and \( k_s = \) factor taking into account the shape of cross section.

The influences of the type and composition of concrete and of environmental variables (relative humidity \( h \) and temperature \( T \)) are described by empirical or semipirical formulas which yield coefficients \( k_{sh}, \varepsilon_{shm}, \tau_{sh}, E_o, \varphi, m, n, \) and \( \alpha \) and functions \( S, C_o, C_d, \) and \( C_p \). These formulas involve the basic variables listed in Table 1. A computer program for determining all these variables and functions according to the BP formulas is available.

The mean 28 day cylinder strength is calculated by Bolomey’s formula

\[ E(f') = 27 \left( \frac{1}{W/C} - 0.5 \right) \text{MPa} \]  

\[ \text{(3)} \]

<table>
<thead>
<tr>
<th>Table 1 — Assumed coefficients of variation for influencing parameters</th>
<th>Parameter</th>
<th>Coefficient of variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h )</td>
<td>mean relative humidity of environment</td>
<td>0.2</td>
</tr>
<tr>
<td>( h_s )</td>
<td>initial relative humidity at which the specimen was in moisture equilibrium before time ( t )</td>
<td>~0.2</td>
</tr>
<tr>
<td>( T )</td>
<td>mean environmental temperature</td>
<td>0.1</td>
</tr>
<tr>
<td>( D )</td>
<td>effective thickness of specimen</td>
<td>~0.2</td>
</tr>
<tr>
<td>( 2 \times \text{area} / \text{perimeter of cross section} )</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( k_i )</td>
<td>shape factor for specimen</td>
<td>0.05</td>
</tr>
<tr>
<td>( c )</td>
<td>cement content in kg/m³</td>
<td>0.1</td>
</tr>
<tr>
<td>( w/c )</td>
<td>water-cement ratio, by weight</td>
<td>0.1</td>
</tr>
<tr>
<td>( s/c )</td>
<td>sand-cement ratio, by weight</td>
<td>0.1</td>
</tr>
<tr>
<td>( g/c )</td>
<td>gravel-cement ratio, by weight</td>
<td>0.1</td>
</tr>
<tr>
<td>( a_i )</td>
<td>parameter depending on cement type</td>
<td>0.1</td>
</tr>
<tr>
<td>( f' )</td>
<td>28 day cylinder strength</td>
<td>0.1</td>
</tr>
</tbody>
</table>
where \( E \) denotes expectation and \( 1 \text{MPa} = 145.04 \text{ psi} \).

In probabilistic analysis, each variable is represented by its expected value and the coefficient of variation. The numerical values for the coefficients of variation listed in Table 1 were not selected on the basis of precise data, which do not seem to exist, but on an intuitive judgement.

The BP formulas do not predict shrinkage and creep perfectly, and so the resulting model uncertainty will now be analyzed.

**MODEL UNCERTAINTY FACTORS**

Model uncertainty is accounted for by applying a random factor to each term: shrinkage, basic creep, and drying creep. The formula for shrinkage is then

\[
\epsilon_{sh}(t, t_o) = \Psi_s \epsilon_{shm} k_s S(t, t_o)
\]  

(4)

where \( \Psi_s \) is the model uncertainty factor. Other formats for dealing with model uncertainty could have been chosen, such as applying an additive random variable independent of time

\[
\epsilon_{sh}(t, t_o) = \Psi_s + \epsilon_{shm} k_s S(t, t_o)
\]  

(5)

or applying a multiplicative or additive random function of time

\[
\epsilon_{sh}(t, t_o) = \Psi_s(t) \epsilon_{shm} k_s S(t, t_o)
\]  

(6)

\[
\epsilon_{sh}(t, t_o) = \Psi_s(t) + \epsilon_{shm} k_s S(t, t_o)
\]  

(7)

With the formats of Eq. (4) or (5), a measured curve ought to correspond to one value of \( \Psi_s \), independent of time [see Fig. 1(a) and (b)]. With the formats of Eq. (6) and (7), measured and predicted curves can differ much more in their shape [see Fig. 1(c)]. A comparison between predicted and measured curves suggests that Eq. (4) provides a sufficiently good description. The mean value and coefficient of variation of a time averaged value of \( \Psi_s \) are estimated in Reference 6 in which the test data are represented by a hand-smoothed curve and are compared to the predicted curve at discrete times, usually one or two time points per decade in the logarithmic time scale.

The model uncertainty for the creep function is assigned in a similar way and the formula for the creep function is then

\[
J(t, t') = \Psi_c \left[ \frac{1}{E_s} + C_c(t, t') \right] + \Psi_c[C_c(t, t', t_o) - C_c(t, t', t_o)]
\]  

(8)

The mean values and coefficients of variation of the \( \Psi \)-factors [of Eq. (4) and (8)] reported in Reference 6 are

\[
\begin{align*}
E[\Psi_i] &= 1; \\
V_{\Psi_i} &= 0.17 \\
E[\Psi_j] &= 1; \\
V_{\Psi_j} &= 0.24 \\
E[\Psi_k] &= 1; \\
V_{\Psi_k} &= 0.16 \\
\end{align*}
\]

(9)

Due to the way the statistics for the \( \Psi \)-factors are determined, they reflect three sources of uncertainty and each \( \Psi \) is consequently written as

\[
\Psi_i = \Psi_i^* \psi_s \psi_x (i = 1, 2, 3)
\]  

(10)

where \( \Psi_i^* \) factor due to inadequacy of the prediction formula; \( \psi_s \) factor due to internal uncertainty; \( \psi_x \) factor due to measurement errors and uncertainty in the laboratory (or site) environment. The factors to be used in Eq. (4) and (8) are \( \Psi_i^* \) and the coefficients of variations in Eq. (9) must therefore be corrected. The factors in Eq. (10) are assumed independent, and the relation between the coefficients of variation is

\[
(1 + V_{\Psi_i}^*) = (1 + V_{\Psi_i}^* \psi_s^*) (1 + V_{\psi_s}^*) (1 + V_{\psi_x}^*)
\]  

(11)

Since the test results were hand-smoothed and the laboratory test conditions were well controlled, the coefficient of variation \( V_{\psi_s} \) is estimated as 0.05. Scant data are available for the estimation of \( V_{\psi_x} \), but the results in References 11, 12, and 13 indicate that a value between 0.06 and 0.10 is reasonable for test specimens. In the examples reported later in this paper, the following corrected values obtained from the foregoing information and from Eq. (9) are therefore used

| Shrinkage | \( E[\Psi_s] = 1; \) | \( V_{\Psi_s} = 0.14 \) (corrected) |
| Basic creep | \( E[\Psi_k] = 1; \) | \( V_{\Psi_k} = 0.23 \) (values) |
| Drying creep | \( E[\Psi_j] = 1; \) | \( V_{\Psi_j} = 0.13 \) |

(12)

The coefficients of variation \( V_\psi \) reported in Reference 6 are based on numerous test series involving a large variety of external parameters. Leaving out some test series would reduce the coefficients of variation drastically, but to do that would be a dubious matter. Instead, a weighting procedure for the individual test series, in which the weights would be assigned according to the similarity with the particular concrete and environmental parameters at hand, should be developed. It appears that this could be done within a Bayesian framework. (This approach, presently pursued by J. C. Chern at Northwestern University, is, however, beyond the scope of this paper.)

**CALCULATION OF STRUCTURAL EFFECTS**

Creep law is assumed linear. The principle of superposition is then valid and the uniaxial relation between stress \( \sigma \) and strain \( \epsilon \) is

\[
\epsilon(t) - \epsilon^*(t) = \int_0^t J(t, t') d\sigma(t')
\]  

(13)
in which \( t = \) time representing the age measured from the set of concrete; \( J(t, t') \) = compliance function (also called creep function) = strain at time \( t \) caused by a unit constant stress acting since time \( t' \); and \( \varepsilon^o = \) stress-independent strain due to shrinkage and thermal effects.

With this creep law the solution of structural analysis problems leads to Volterra integral equations. In Reference 4 various methods giving exact and approximate results are compared. To obtain accurate results, numerical step-by-step methods are most efficient. In these methods the time interval \([t_i, t_{i+1}]\) of interest is divided by discrete times \( t_1, t_2, \ldots, t_n \) into time steps \( \Delta t_i = t_i - t_{i-1} \). The integral relation in Eq. (13) may then be approximated by

\[
\varepsilon_i - \varepsilon^o_i = \sum_{j=1}^{i} \frac{1}{2} [J(t_i, t_j) + J(t_i, t_{j+1})] \Delta \sigma_j
\]

(14)

where subscripts \( i, j \) refer to times \( t_i, t_j \) or to time steps ending at these times, e.g., \( \varepsilon_i = \varepsilon(t_i) \) and \( \Delta \sigma_j = \sigma(t_j) - \sigma(t_{j-1}) \). The principles for choosing the division points are given in References 8 and 15. Eq. (14) is applicable even when some stress increments are instantaneous, in which case the corresponding time step is of zero duration. Accuracy of the method is generally very good even for a small number of time intervals. For a moderate number of time points, the results can be considered as exact solutions of the integral equations.\(^8,15,16\)

Structural analysis with the stress-strain relation in Eq. (14) may be reduced to a succession of incremental elastic analyses for the individual time steps.\(^8,15,16\) The solution from Reference 4 will be applied here in a matrix form,\(^17\) convenient for the analyst, in which all time steps are solved simultaneously. In contrast to the usual step-by-step solution,\(^8,15,16\) such an approach, admittedly, wastes computer time and storage, but this is unimportant for structures with few unknowns, which are of interest here.

Grouping \( \varepsilon_i \) and \( \Delta \sigma_i \) in column matrices \( \varepsilon \) and \( \Delta \sigma \),\(^*\) we may rewrite Eq. (14) as

\[
\varepsilon - \varepsilon^o = J \Delta \sigma
\]

(15)

in which

\[
J = \left[ \frac{1}{2} [J(t_i, t_j) + J(t_i, t_{j+1})] \right] = (i \times i) \text{ square matrix}
\]

Similarly, \( \sigma = \Sigma \Delta \sigma \), can be written as

\[
\sigma = L \Delta \sigma
\]

(16)

in which \( \sigma = \{ \sigma_i \} \) and \( L = (1, \ldots, 1) \) = a square matrix whose elements are 1 if \( i \leq j \), and otherwise are 0. Eq. (15) may then be written as

\[
\varepsilon - \varepsilon^o = E_{c}^{-1} \cdot \sigma
\]

(17)

where

\[E_{c}^{-1} = J L^{-1}\]  

(17a)

The formal similarity of Eq. (17) with Hooke's law reflects the analogy between linearly elastic and linearly viscoelastic materials.\(^4\) As demonstrated in Reference 17, the matrix formulation allows a convenient algebraic treatment of linearly viscoelastic structures (with not too many unknowns), similar to that for linearly elastic problems.

A simple application of the matrix notation may now be shown. Shrinkage of a plain concrete bar is completely restrained causing tensile concrete stresses which are modified by creep. The governing equation is

\[
\varepsilon(t) = \varepsilon_n(t) + \int_{0}^{t} J(t', t') \cdot d \sigma(t') = 0
\]

(18)

which in matrix formulation reads

\[
\varepsilon = \varepsilon_n + E_{c}^{-1} \cdot \sigma = 0
\]

(19)

and from which the concrete stress is found to be

\[
\sigma = - E_{c} \cdot \varepsilon_n
\]

(20)

Due to uncertainties in shrinkage and creep properties, the matrix \( E_{c} \) and the vector \( \varepsilon_n \) are both uncertain.

It may be noted that Eq. (17), on which all the present structural analysis is based, is also applicable for approximate solution by the age-adjusted effective modulus method. In that case, \( E_{c}^{-1} \) becomes a diagonal matrix with diagonal elements \( 1/E_{c} \) (\( t, t_{0} \)), in which \( E_{c} \) (\( t, t_{0} \)) is the age-adjusted effective modulus.\(^4\) Advantages of this method are that it allows approximate solution of the structural response at a certain time \( t \), without solving the response for the preceding times, and that the matrix inversion and multiplication indicated in Eq. (17a) need not be carried out. These advantages make hand calculations feasible; they are, however, insignificant when a computer is used.

**UNCERTAINTY ANALYSIS**

Before deciding on the type of uncertainty analysis, a number of observations can be made. Creep and shrinkage effects are mainly considered in serviceability analysis, and interest is therefore in the variations close to the mean values rather than extreme values. The relation between creep effects and basic variables is highly nonlinear. The number of basic variables is very large, and for the set of basic variables, hardly more than second moment information is available, i.e., means, variances, and covariances. It then appears that a satisfactory characterization of creep effects consists in achieving good estimates of the mean value at any time, the covariances between values at any two times, and for each basic variable a measure of the relative importance for the creep effect of variations of this parameter.

Previous work\(^9\) gives second moment analysis tools which are directly applicable in connection with the matrix method outlined here for the calculation of structural effects. Application of these tools is demonstrated

\(^*\)Editor's note — A single line under a Greek letter indicates a matrix quantity.
in Reference 18; they are even applicable when some basic variables are uncertain processes rather than uncertain variables. Although the method is conceptually very simple, serious computational difficulties arise, however, when several correlated uncertain matrices and their inverse matrices are dealt with simultaneously, as is the case of this paper’s examples. Other, simpler uncertainty analysis methods will therefore be considered.

In the creep and shrinkage models chosen, all basic variables \( \theta = (\theta_1, \ldots, \theta_k) \) are random variables. Given the value of \( \theta \), any creep effect is then represented as a function of time. Let the creep effect be denoted \( X(\theta, t) \). A linearization of \( X(\theta, t) \) around the mean value point \( E[\theta] \) gives the following mean value and covariance function

\[
E[X(\theta, t)] = X(E[\theta], t) \tag{21}
\]

\[
\text{Cov}[X(\theta, t_1), X(\theta, t_2)] = \sum_{i=1}^{k} \sum_{j=1}^{k} \frac{\partial X}{\partial \theta_i} (E[\theta], t_1) \frac{\partial X}{\partial \theta_j} (E[\theta], t_2) \text{Cov}[\theta_i, \theta_j] \tag{22}
\]

where \( \text{Cov} \) denotes the covariance and the sign \( \perp \) symbolizes the linearization of the right-hand side. Due to the complex functional relationship between the creep effects and the basic variables, the partial derivatives in Eq. (22) are, however, difficult to calculate analytically. If the partial derivatives are calculated numerically, the method becomes less attractive than the method which is proposed next and is based on point estimates for probability moments.

Let \( \theta \) be a random variable with mean value \( \mu \) and standard deviation \( s \), and let the interest be focused on the mean value of some function \( g(\theta) \). Possible observations \( \theta_{i(1)}, \theta_{i(2)}, \ldots, \theta_{i(m)} \) of \( \theta \) are chosen together with \( m \) nonnegative numbers \( p_1, p_2, \ldots, p_m \), the sum of which equals 1. The mean value of \( g(\theta) \) may then be assigned as

\[
E[g(\theta)] = p_1 g(\theta_{i(1)}) + p_2 g(\theta_{i(2)}) + \ldots + p_m g(\theta_{i(m)}) \tag{23}
\]

Since the mean value and variance of \( \theta \) are given, the \( \theta_{i(m)} \) and \( p_i \) cannot be chosen arbitrarily but must satisfy the conditions

\[
E[\theta] = p_1 \theta_{i(1)} + p_2 \theta_{i(2)} + \ldots + p_m \theta_{i(m)} = \mu \\
E[\theta^2] = p_1 \theta_{i(1)}^2 + p_2 \theta_{i(2)}^2 + \ldots + p_m \theta_{i(m)}^2 = \mu^2 + s^2 
\]

(24)

The simplest choice consists in choosing two points \( \theta_{i(m)} = \theta_{i(1)} = \mu \pm s \) with both \( p_i \) equal to \( \frac{1}{2} \) (Fig. 2). Thus, for two-point estimates, the expected value of \( g(\theta) \) is assigned as

\[
E[g(\theta)] = \frac{1}{2}[g(\mu + s) + g(\mu - s)] 
\]

(26)

Use of this procedure as an aid to assign mean values to functions of random variables with some mean values already given was suggested in Reference 14. Reference 10 describes a more formal mathematical setup of a so-called \( n \)-mass uncertainty algebra which also includes this procedure. The generalization to two and three dimensions is also shown in Fig. 2. Extensions to four or more dimensions are rather straightforward, but there are here infinitely many possible sets of \( p \)-values and it must be checked that no negative \( p \)-values are selected.

It is useful to note the validity of the following relations from linear regression analysis

\[
E[\theta_{i|j}] = \mu_j \pm s_j \] \hspace{1cm} (27)
\[
\text{Var}[\theta_{i|j}] = \mu_j \pm s_j \] \hspace{1cm} (28)
\]

where \( \text{Var} \) denotes the variance, \( s_j \) the standard deviations of random variables \( \theta_{i} \) and \( \rho_j \) their correlation coefficients. An intuitive measure of the relative importance of uncertainty of each parameter is \( \alpha_i \), defined as

\[
\alpha_i = \frac{\text{Var}[g(\theta)] - \text{Var}[g(\theta)|\theta] \text{Var}[g(\theta)]}{\text{Var}[g(\theta)|\theta]} = \frac{\text{Var}[E[g(\theta)|\theta]]}{\text{Var}[g(\theta)]} \tag{29}
\]

Image: Fig. 2 — Weights and points
where the last equality is valid for the choice of $\theta_0$ and $\phi_0$ as shown in the figures.

Applying the procedure involves a calculation of creep effects for each of the $2^k$ possibilities of $\theta$ ($k$ being the number of variables). The mean value and the covariance functions are then calculated by the appropriate weighting of results. For a large $k$, the number of calculations is thus very large, but, as will be seen, the relative importance factor $\alpha_i$ for several basic variables is so small that these variables can just be taken at their mean values.

A third procedure, which can always be used, is simulation. A formal choice of the distribution type of the set of basic variables $\theta$ must then be made. Sample $\theta$s are then simulated and the corresponding creep effects calculated. The mean value and covariance functions are obtained by giving each sample the same weight. The number of simulations necessary to give good estimates of the mean and covariance functions is not extremely high, but it is difficult to estimate any relative measure of importance for each variable.

The point estimate method seems to be the best one for this application with regard to the information on the input, computation time, and output.

**RELATIVE IMPORTANCE FACTORS**

Including the three model uncertainty factors, there are 11 basic variables in the shrinkage and creep formulas, Eq. (4) and (8). The relative importance factor defined in Eq. (29) is for the shrinkage function determined for each basic variable. For the creep function this is done only in two cases. The mean values of the basic variable and the deterministic parameters are listed in Table 2. These values are chosen somewhat arbitrarily, but very similar relative importance factors were observed for other choices of parameter values. Coefficients of variation of the basic variables are taken from Table 1 and Eq. (12).

Tables 3 to 5 show that the relative importance factors for the temperature $T$ and the shape factor $k$, are very small. These two parameters influence the time scale of shrinkage and creep development rather than the magnitude. The sand-cement ratio $s/c$ and the model uncertainty factor for drying creep $\Psi_1$ also have very small relative importance factors. In some of the following examples, these four parameters are taken as deterministic, thus reducing the number of random variables and the computer time.

**EXAMPLES**

We now combine the structural analysis method described in Eq. (13) through (20) with the present uncertainty analysis, considering a number of examples of practical importance. The examples include both non-composite and composite structures. For a noncomposite structure the age of concrete is assumed to be uniform, and so the same creep and shrinkage function is valid for all parts of the structure. A composite structure may comprise reinforcement, and the age and composition of concrete may vary from one part of the structure to another. In each example the structural analysis is briefly described and the values of the variables listed. Results of the calculations are given as the mean value function and the standard deviation function of the structural effects.

**Restrainted shrinkage in a plain concrete bar**

The concrete bar of Fig. 3 is considered. Concrete stress due to restrained shrinkage is given in Eq. (20). The parameter values are given in Table 6(a), and results of the calculations are shown in Fig. 3.

**Restrainted shrinkage in a frame**

Fig. 4 shows a plane frame structure which has a uniform cross section and is subjected to shrinkage and creep which causes development of horizontal forces at the supports. Relative displacements of the supports due to shrinkage and the reaction $R(t)$ in the primary stati-
Fig. 3 — Stresses in concrete bar subjected to shrinkage and creep

cally determinate system are denoted as \( \delta_s(t) \) and \( \delta_n(t) \), respectively. Since the supports are fixed, the governing compatibility condition is \( \delta(t) = \delta_s(t) + \delta_n(t) = 0 \). The displacement due to shrinkage is \( \delta_s(t) = 2a \varepsilon_n(t) \).

According to the ordinary bending theory, in which shear deformations are neglected, the displacement due to the reaction \( R(t) \) is

\[
\delta_s(t) = \frac{8a^3}{3I} \int_0^t J(t, t') dR(t')
\]  
(30)

In matrix notation, the governing equation is

\[
\Delta = 2a \varepsilon_n + \frac{8a^3}{3I} E_c^{-1} R = 0
\]  
(31)

From this equation, the reaction \( R \) is calculated

\[
R = -\frac{3I}{4a^2} E_c \varepsilon_n
\]  
(32)

Results for a frame with \( a = 5 \text{ m} \) (196.9 in.) and \( I = 2.13 \times 10^{-7} \text{ m}^4 \) (5117 in.) are shown in Fig. 4. Parameter values given in Table 6(b) were used in the calculations.

**Simply supported beams made continuous**

Fig. 5 shows two identical, simply supported concrete beams that are loaded at time \( t = t_i \) and coupled (without enforcing any rotations at the beam ends) at \( t = t_i \) to form a continuous beam. The change of structural system introduces a redundant bending moment \( M(t) \) governed by the compatibility condition \( \delta_s(t) + \delta_n(t) = 0, t \geq t_i \). This equation expresses the fact that the mutual rotation, due to the load \( g \) and the redundant moment \( M(t) \), between the two beam ends (in the sense of \( M \)) remains unchanged after time \( t_i \). According to the ordinary bending theory, the two rotations are

\[
\delta_s(t) = 2 \frac{gP}{24I} [J(t, t_i) - J(t_i, t_i)]
\]  
(33)

\[
\delta_n(t) = 2 \frac{1}{3I} \int_0^t J(t, t') dM(t')
\]  
(34)

In matrix formulation the governing equation is

\[
2 \frac{gP}{24I} c + 2 \frac{1}{3I} E_c^{-1} M = 0
\]  
(35)

where the elements in the column matrix \( c \) are the values \( J(t, t_i) - J(t_i, t_i) \). The solution for \( M \) is

\[
M = -\frac{1}{8} gl^2 E_i c
\]  
(36)

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<tr>
<th>Parameter</th>
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<th>( h_s )</th>
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<th>( D )</th>
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Table 6 — Mean values \( E \) and coefficients of variation used in examples
Fig. 5 — Redundant bending moment from coupling of beams

Fig. 5 shows the results for two different times $t_4$ of coupling. The parameter values used in the calculations are given in Table 6(c).

Coupling of cantilever beams of different ages

Fig. 6 shows two opposite cantilever beams of different ages, which are connected together at midspan. This causes development of shear forces and bending moments in the connection. The load history is simplified as shown in Fig. 6. The moment $M(t)$ and shear force $V(t)$ may be determined from equations expressing the fact that the mutual displacement and rotation are zero at the connection after the time of coupling. These equations are

$$
\frac{1}{8} \frac{gL^4}{I} \left[ J(t,60) - J(t,270,60) \right] + \frac{1}{3} \frac{L}{I} \int_{t_0}^{t} J(t,t') \, dt' = \frac{1}{8} \frac{gL^4}{I} \left[ J(t,60) - J(t,270,60) \right]
$$

$$
dV(t') - \frac{1}{2} \frac{L^2}{I} \int_{t_0}^{t} J(t,t') \, dM(t') = \frac{1}{2} \frac{L^2}{I} \int_{t_0}^{t} J(t,t') \, dM(t')
$$

$$
\left[ J(t - 180,60) - J(t - 90,60) \right] + \frac{1}{3} \frac{L}{I} \int_{t_0}^{t} J(t - 180,60) \, dV(t') + \frac{1}{2} \frac{L}{I} \int_{t_0}^{t} J(t - 180,60) \, dM(t') = 0 \quad (37)
$$

$$
\frac{1}{8} \frac{gL^4}{I} \left[ J(t,60) - J(t,270,60) \right] - \frac{1}{2} \frac{L^2}{I} \int_{t_0}^{t} J(t,t') \, dM(t') - \frac{1}{6} \frac{gL^3}{I}
$$

$$
\left[ J(t - 180,60) - J(t - 90,60) \right] + \frac{1}{2} \frac{L^2}{I} \int_{t_0}^{t} J(t - 180,60) \, dV(t') + \frac{1}{2} \frac{L}{I} \int_{t_0}^{t} J(t - 180,60) \, dM(t') = 0 \quad (38)
$$

In matrix formulation, these equations may be written as

$$
\frac{1}{8} \frac{gL^4}{I} (c_1 - c_2) + \frac{1}{3} \frac{L^3}{I} (E_c^1 - E_c^2) V
$$

$$
- \frac{1}{2} \frac{L^2}{I} (E_c^1 - E_c^2) M = O \quad (39)
$$

$$
- \frac{1}{6} \frac{gL^3}{I} (c_1 + c_2) - \frac{1}{2} \frac{L^2}{I} (E_c^1 - E_c^2) V
$$

$$
+ \frac{L}{I} (E_c^1 + E_c^2) M = 0 \quad (40)
$$

where $V$ and $M$ are the unknown column matrixes of $V(t)$ and $M(t)$. $E_c$ is determined from $J(t,t')$ and $E_d$ from $J(t - 180,60)$ and where the elements in the column matrix $c_i$ are $J_i(t,60) - J_i(t,270,60)$ and in the column matrix $c_j$ are $J_j(t - 180,60) - J_j(t,90,60)$. The solution is

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bending. The prestressing steel consists of one tendon with cross section area \( A_p = 3000 \text{ mm}^2 \) at distance \( y_p = 0.52 \) from the centroidal axis of the concrete cross section. The prestressing steel is assumed to be linearly elastic, with modulus of elasticity \( E_p = 200,000 \text{ MN/m}^2 \) (29 \( \times \) 10⁶ psi).

In matrix notation, stress-strain relations are

\[
\varepsilon_p = \frac{1}{E_p} \sigma_p, \quad \varepsilon_c = E_c^{-1} \sigma_c + \varepsilon_{sh} \tag{43}
\]

where the subscripts \( p \) and \( c \) distinguish between prestressing steel and concrete. The concrete strain distribution is assumed to be planar, and the difference between the steel strain \( \varepsilon_p \) and concrete strain \( \varepsilon_{cp} \) at steel level \( y = y_p \) is \( \varepsilon_{sh} \), which defines the prestress. Therefore

\[
\varepsilon_c = \varepsilon_{cp} + \varepsilon_c y_c, \quad \varepsilon_p = \varepsilon_{sh} + \varepsilon_{so} \tag{44}
\]

Equilibrium conditions relating the normal forces \( N_p \) and \( N_c \) in steel and concrete, bending moment \( M_c \) in concrete to the total normal force \( N \), and bending moment \( M \) are

\[
N = N_p + N_c, \quad M = M_p Y_p + M_c \tag{45}
\]

where the column matrices \( N, N_c, M_c, N, \text{ and } M \) group the discrete values at times \( t_i \). Since the concrete stresses must be linearly distributed

\[
\sigma_c = \frac{1}{A_c} N_c + \frac{v}{I_c} M_c, \quad \sigma_p = \frac{1}{A_p} N_p \tag{46}
\]

These equations are solved with respect to the steel force \( N_p \)

\[
N_p = A_p E_p \left[ I + \left( \frac{A_p}{A_c} + \frac{A_p y_c^2}{I_c} \right) E_c E_p^{-1} \right]^{-1} \varepsilon_{sh} + \varepsilon_{so} \left( \frac{N}{A_c} + \frac{M y_c}{I_c} \right) \tag{47}
\]

in which \( I \) is the unit matrix. For the cross section of Fig. 7 we have \( A_c = 0.60 \text{ m}^2 \) (930 in.²) and \( I_c = 63.36 \times 10^{-3} \text{ m}^4 \) (152,200 in.⁴). The prestress strain \( \varepsilon_{sh} \) was taken constant in time and equal to \( 6 \times 10^{-3} \) (bonded reinforcement), and the cross-sectional forces were taken constant in time as \( N = 0, M = 1 \text{ MNm} \) (8851 kip-in.). Shrinkage and creep parameters are listed in Table 6(e). Results for the steel force are shown in Fig. 7.

**Stress redistribution in steel-concrete composite beam**

Fig. 8 shows a symmetric cross section of a composite steel-concrete beam subjected to compound bending. The governing equations in matrix formulation are, with the notations defined in Fig. 8, as follows.

Stress strain relations
Fig. 8 — Changes of steel girder moments in steel-concrete composite beam

\[ \varepsilon_s(y) = \frac{1}{E_s} \sigma_s(y), \quad \varepsilon_c(y) = E_c^{-1} \sigma_c(y) + \varepsilon_{sh} \]  

(48)

Strain distribution

\[ \varepsilon_s(y) = \varepsilon_s(y) = \varepsilon_n + \kappa y \]  

(49)

Equilibrium conditions

\[ N = N_c + N_s, \quad M = M_s + N_s e_s + M_c - N_c e_c. \]  

(50)

Stress distributions

\[ \sigma_s(y) = \frac{N_s}{A_s} + \frac{M_s}{I_s} (y - e_s), \]

\[ \sigma_c(y) = \frac{N_c}{A_c} + \frac{M_c}{I_c} (y + e_c). \]  

(51)

These equations are solved with respect to the steel force \( N_s \) and bending moment \( M_s \).

Cross-sectional forces are taken as constant in time, with \( N = 0 \) and \( M = 0.284 \text{ MNm} \) (8851 kip-in.). The cross-sectional constants are \( A_c = 1650 \text{ cm}^2 \) (255.1 in.\(^2\)), \( A_s = 94.9 \text{ cm}^2 \) (14.71 in.\(^2\)), \( I_c = 14210 \text{ cm}^4 \) (341.4 in.\(^4\)), \( I_s = 33300 \text{ cm}^4 \) (800 in.\(^4\)), \( e = 0.279 \text{ m} \) (10.98 in.), and the modulus of steel is \( E_s = 210,000 \text{ MN/m}^2 \) (30.46 × 10\(^6\) psi). Shrinkage and creep parameters are given in Table 6(f). Fig. 8 shows the results for the steel bending moment. Deflections are proportional to the steel bending moment.

Creep buckling deformations

Fig. 9 shows the deformations at midspan of a slender concrete beam-column with square cross section. The column has a small initial sinusoidal curvature of amplitude \( y_0 \). The beam-column is subjected to an axial load \( P \) and a lateral sinusoidal load of amplitude \( q_0 \), both of which can vary in time. Two cross sections, one with and one without reinforcement, are considered. Ordinary beam-column theory is used and reinforcement is assumed linearly elastic. If the concrete is modeled as linearly elastic and the loads are constant in time, the differential equation for the deflection \( y(x) \) is then

\[
(E_c I_c + E_s I_s) [y''(x) - y_0''(x)] = -P y(x) - \frac{P}{\pi^2} q(x)
\]  

(53)

The corresponding matrix equation for linear viscoelastic concrete and for time-varying loads is
The concrete strain distribution is assumed to be planar, and the stress-strain relations are

\[ \epsilon(y) = E_{c}^{-i} \sigma(y) \quad (i = 1, 2, \ldots, 5) \]  

(57)

where \( \sigma \) is the stress in the \( i \)-th layer. The concrete stresses must be linearly distributed

\[ \sigma(y) = \frac{N_i}{A} + \frac{M_i}{I_y} \left( y - \frac{i - 3}{5} h \right) \]

(58)

From these equations the curvature \( \kappa = E_c M / I \) is calculated

\[
\kappa = \frac{1}{\int \left[ (E_{c} - 11E_{c5} + E_{c9} + 37E_{c4} + 97E_{c3}) 
- 12(2E_{c5} + E_{c4} - E_{c2} - 2E_{c1}) 
(2E_{c4} + 5E_{c5} + 3E_{c3} + E_{c1}) \right]^{-1} 
\left\{ M - \frac{h}{5} (2E_{c5} + E_{c4} - E_{c2} - 2E_{c1}) 
(E_{c4} + E_{c5} + E_{c3} + E_{c1}) \right\}
\]

(59)

The parameter values used are given in Table 6(h). For simplicity, all variables except the model uncertainty factors \( \Psi_i \) are taken as deterministic and identical for all five layers. The mean values as well as the coefficients of variation for the five \( \Psi_i \) factors are taken to be identical. The \( \Psi_i \) factors are further taken equicorrelated with correlation coefficient \( \rho_i \).

In Fig. 10 results for the curvature are shown for various combinations of the cross-sectional forces for various values of \( \rho_i \). The value of \( \rho_i \) does not affect the mean curvature but can have a large effect on the variance of the curvature. The effect is, however, opposite for the cases of pure bending and pure axial load.

**CONCLUSIONS**

1. The random variability of creep and shrinkage effects in concrete structures is often very large and should be accounted for in design. Determining the variance functions for the structural variables of interest is probably sufficient for design.

2. Any chosen deterministic model for predicting creep and shrinkage can be randomized by considering the entering parameters as random variables and by also inserting uncertainty factors that characterize inadequacy of the chosen model.

3. Uncertainty in the internal forces and deformations of the structure can be efficiently obtained by point estimates of probability moments based on a number of usual deterministic structural analyses. This can be conveniently carried out in a matrix form.

4. Numerical examples show that certain internal forces or deformations which almost vanish in a deterministic analysis can have a very large uncertainty, which
should be accounted for in design. In some other cases, a large uncertainty in the entering parameters may cause only a small uncertainty in the internal force or deformation. The coefficient of variation can substantially vary with time.

ACKNOWLEDGMENT
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CONVERSION FACTORS
1 m = 100 cm = 39.37 in.
1 MPa = 145.0 psi
1 MNm = 10 fps = 8.851 1bf x in.

NOTATION

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