Constitutive Model with Rotating Active Plane and True Stress

By Aleksander Zubelewicz[1] and Zdeněk P. Bažant,[2] F. ASCE

Abstract: A novel constitutive model for concrete, which approximately describes the basic known test data on nonlinear triaxial behavior including strain-softening, is presented. The model rests on two basic ideas: (1) The stress-strain relation is defined as the relation between the normal and shear components of stress and strain on a certain special plane, called the active plane, the orientation of which varies as a function of accumulated inelastic strains; and (2) the stress-strain relation is written in terms of microstresses or true stresses that are obtained as the macrostresses divided by the resisting area fraction of the material. Strain-softening is obtained principally due to decrease of this area fraction. Thus, an incremental plasticity law satisfying the normality rule may be introduced on the microlevel, and a symmetric stiffness matrix is obtained. The loading surface for the active plane on the microlevel is an ellipse in the normal-shear stress space, similar to the critical state theory for soils. The model involves only six empirical inelastic material parameters, for which a simple sequential identification procedure is developed.

Introduction

Despite many significant contributions (1,4–9,14,16,27), nonlinear triaxial constitutive relations applicable up to complete failure continue to pose a formidable challenge to continuum mechanics of concrete, as well as rocks and soils. The most sophisticated models developed so far describe the existing basic experimental data quite well, however, the material parameters in such models are difficult to identify from test results, and micromechanics foundations for these models are lacking (2,9,11,13,15,22,24,28), see Table 1. It is now widely agreed that further advances have to be based, at least partially, on the understanding of micromechanics of deformation.

A promising approach, which has so far been shown to work for tensile microcracking, is the microplane model (8), which represents an analog of the earlier slip theory of plasticity (3). In this approach, the inelastic deformation is described independently for planes of various orientation within the microstructure (the microplanes) and the contributions from all these planes are then superimposed according to some suitably assumed micro-macro constraint. Numerical studies with the microplane model reveal that usually the inelastic deformation originates predominantly from a single active microplane, the orientation of which varies during the deformation process. This plane represents the direction of active (i.e., growing or slipping) microcracks or slip-planes. A similar conclusion ensues from certain studies of failure (slip) criteria.

[1]Visiting Scholar, Ctr. for Concrete and Geomaterials, Northwestern Univ., Evanston, IL 60201.

[2]Prof. of Civ. Engrg. and Dir., Ctr. for Concrete and Geomaterials, Northwestern Univ., Evanston, IL 60201.

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TABLE 1.—Comparison of Various Nonlinear Triaxial Models for Concrete

Model (1)	Number of free parameters (including elastic) (2)	Properties Covered				Level of Approximation				
		Plasticity (3)	Hardening (4)	Softening (5)	Fracture (6)	Poor (7)	Fair (8)	Good (9)	Very good (10)	Excellent (11)
Mohr-Coulomb with cut-off	6	x	x			x				
Cap model	7	x	x				x			
Chen and Chen's	8	x	x					x		
Critical state model	9	x	x	x				x		
Endochronic model	15[a]	x	x	x	x					x
Plastic-fracturing model	15[a]	x	x	x	x					x
Total strain model	22[a]		x	x	x					x
Present model	6[a]	x	x	x	x				x	

[a]Some of the parameters are fixed values.
[b]Plus 2 fixed parameters.

As is well known, the von Mises or Drucker-Prager criterion may be formulated as the relation between the stresses acting on the octahedral plane. Matsuoka (18–20) showed that an improved failure criterion may be obtained as the relation between the stress components on a plane whose direction is not fixed as the octahedral plane but is allowed to rotate in a certain manner according to the values of the principal stresses. Due to the rotation of this plane, the influence of the third stress invariant is brought in and the result is a failure criterion which is quite similar to the recently proposed yield criteria of Lade (16,17) and Schreyer (26,27) using the third invariant of a stress tensor shifted in the stress space.

Aside from failure criteria, the stress-strain relations have also been formulated in terms of stresses and strains on one single plane; this plane was previously considered as fixed and identical to the octahedral plane; e.g., Gerstle, et al. (11,12) and Kotsovos (14).

Realizing the predominant single-plane source of inelastic deformation and the importance of the change of orientation and area of this plane during the deformation process, Zubelewicz (29) proposed that a simple yet effective nonlinear triaxial constitutive relation may be obtained in terms of the stress components on a certain special plane the orientation of which rotates as a function of the strain rate. This plane will be called the *active plane* because it characterizes the orientation of predominant cracking and slipping. The objective of the present study is to develop this idea and show that a reasonably good representation of the behavior of concrete can be obtained with surprisingly few material parameters which can be easily identified from given test data.

Another basic idea of the present model is the description of macroscopic strain-softening by means of a decrease of the resisting area fraction of the material that defines the ratio of the macro-stress to the micro-stress or true stress. In terms of *true stresses*, the constitutive law need not exhibit strain-softening, which makes it possible to adhere to normality and Drucker's postulate on the micro-level. The resisting area fraction concept has previously been used by Zubelewicz (29) for a similar type of model, and also by Bažant and Oh (parameter μ on p. 161 of Ref. 6) in a different type of model for strain-softening due to fracturing.

Macro- and Micro- Stresses on Rotating Active Plane

Let n_i be the cartesian components of the unit normal vector of the active plane, in cartesian coordinates x_i (i = 1, 2, 3); Fig. 1(a). The stress vector on the active plane, $\vec{S}$, has then the components $S_i = \sigma_{ij} n_j$, where σ_{ij} are the cartesian components of the macroscopic stress tensor. The normal stress and the components of the shear stress vector on the active plane are

$$\sigma = S_i n_i = \sigma_{ij} n_j n_i \quad (1)$$

$$T_i = S_i - \sigma n_i = \sigma_{ij} n_j - \sigma_{kj} n_j n_k n_i = S_k(\delta_{ik} - n_i n_k) = \sigma_{kj} n_j (\delta_{ik} - n_i n_k) \quad (2)$$

The shear stress magnitude [Fig. 1(a)] then is

$$\tau = \sqrt{T_i T_i} = [S_k S_m (\delta_{km} - n_k n_m)]^{1/2} = [\sigma_{ik} \sigma_{jm} n_i n_j (\delta_{km} - n_k n_m)]^{1/2} \quad (3)$$

The components of the unit vector **t** of the shear stress resultant, representing the direction of the maximum shear stress on the active plane, then are

$$t_i = \frac{T_i}{\tau} \quad (4)$$

The unit vectors **n** and **t**, together with the lateral vector **m** = **n** × **t** (whose components are m_i), define a local cartesian coordiante system for the active plane [Fig. 1(a)].

Since we will try to formulate the stress-strain relation using only the stress components on the active plane, we need to answer the opposite question: Given the values of n_i, t_i, σ and τ, what is the associated macroscopic stress tensor σ^N_{ij}? The answer is any tensor satisfying the condition

$$\sigma_{ij} n_j = \sigma^N_{ij} n_j \quad (5)$$

The solution of this equation is not unique; any tensor such that

$$\sigma^N_{ij} = \sigma_{ij} - g_i t_j - h_i m_j \quad (6)$$

with arbitrary constants g_i and h_i, satisfies Eq. 5. The tensors σ^N_{ij} and σ_{ij} may differ by arbitrary normal stresses in the directions parallel to the active plane, i.e., $\sigma^N_{ij} t_j$ and $\sigma^N_{ij} m_j$. Because these normal stresses are irrelevant for the stress state on the active plane, we may choose them arbitrarily, and we choose them as zero;

$$\sigma^N_{ij} t_j = 0; \quad \sigma^N_{ij} m_j = 0 \quad (7)$$

Now, multiplying Eq. 6 by m_j or t_j, and noting that $m_j t_j = 0$ (a scalar product of two orthogonal vectors), we find that $g_i = (\sigma_{ij} - \sigma^N_{ij}) t_j$, $h_i = (\sigma_{ij} - \sigma^N_{ij}) m_j$ from which, according to Eq. 7, $g_i = \sigma_{ij} t_j$, $h_i = \sigma_{ij} m_j$. Substituting this into Eq. 6, we have $\sigma^N_{ij} = \sigma_{ij} - g_i t_j - h_i m_j = \sigma_{ij} - \sigma_{ik}(t_k t_j + m_k m_j)$. This may be further simplified using the relation

$$n_i n_j + t_i t_j + m_i m_j = \delta_{ij} \quad (8)$$

expressing the fact that the vectors (n_i, t_i, m_i) and (n_j, t_j, m_j) are orthogonal if $i \neq j$ and parallel if $i = j$ (because these vectors represent the projections of the unit vector of axis x_i or x_j onto the coordinate system **n**, **t** and **m**). Consequently, we have $\sigma^N_{ij} = \sigma_{ij} - \sigma_{ik}(\delta_{jk} - n_j n_k)$, from which we finally obtain

$$\sigma^N_{ij} = \sigma_{ik} n_k n_j \quad (9)$$

or alternatively $\sigma^N_{ij} = S^N_i n_j$ in which $S^N_i = \sigma^N_{ik} n_k = \sigma_{ik} n_k$. The normal and shear components of tensor σ^N_{ij} are the same as those of tensor σ_{ik} and are

$$\sigma^N = S^N_i n_i = \sigma^N_{ij} n_j n_i = \sigma_{ik} n_k n_i$$

$$\tau_N = S^N_i t_i = \sigma^N_{ij} n_j t_i = \sigma_{ij} n_j t_i \quad (10)$$

The macrostresses σ^N_{ij} represent the stress resultants over a full unit area of the active plane. However, only a certain fraction, η, of the active plane resists stress, while the remaining area due to cracking or damage carries no stress. Denoting the resisting area fraction as η, the micro-

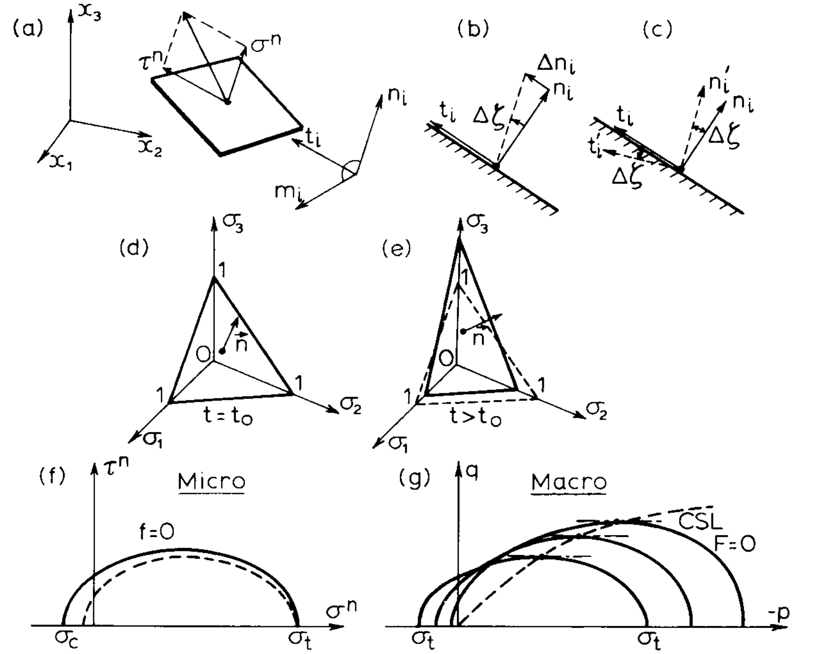

FIG. 1.—Active Plane: (a) Stress Components; (b) and (c) Rotation of Active Plane; (d) Initial Octahedral Orientation; (e) Rotated Active Plane; (f) Loading Surface on Micro-Level, $f(\sigma^n, \tau^n) = 0$; and on Macro-Level, $F(q, \eta) = 0$ (CSL = Critical State Line)

stresses, which approximately characterize the *true* stresses in the microstructure, may be expressed as

$$\sigma_{ij}^n = \frac{1}{\eta}\sigma_{ij}^N = \frac{1}{\eta}\sigma_{ik}n_k n_j = \frac{1}{\eta}S_i^N n_j = S_i^n n_j \quad (11)$$

The normal and shear microstress components on the active plane are

$$\sigma^n = \frac{1}{\eta}\sigma^N = \frac{1}{\eta}\sigma_{ij}n_j n_i = \frac{1}{\eta}S_i^N n_i = S_i^n n_i \quad (12)$$

$$\tau^n = \frac{1}{\eta}\tau^N = \frac{1}{\eta}\sqrt{T_i^N T_i^N} = \frac{1}{\eta}[\sigma_{ik}^N\sigma_{jm}^N n_i n_j(\delta_{km} - n_k n_m)]^{1/2} \quad (13)$$

As the material deforms, the active plane in general rotates. Denoting as $\dot{\zeta}$ is the angular rotation rate of the normal of the active plane, we may write:

$$\dot{n}_i = \dot{\zeta} t_i \quad (14)$$

The fact that vector **n** rotates in the plane (**n**, **t**) is merely a hypothesis, albeit a reasonable one. That **n** should turn in the **t** direction is suggested by the fact that shear causes the plane of maximum extension (the weakest plane) to rotate in the direction of ζ; see Fig. 1(*b,c*).

In view of the rotation of normal **n**, we need to distinguish the rate of stress with regard to the material, denoted by superimposed dot, from the rate of stress on the active plane with regard to the coordinate system **n**, **t**, **m** that rotates with the active plane. This rate may be called co-rotational and denoted by a superimposed triangle. It must not be confused with the co-rotational rates or objective stress rates known from the theory of finite deformation; we deal with small deformations only.

Let us now seek the co-rotational macrostress rates. In terms of the microstresses we have $\overset{\Delta}{\sigma}{}^N = (\eta\sigma_{ij}^n n_j n_i)^\Delta$, $\overset{\Delta}{\tau}{}^N = (\eta\sigma_{ij}^n n_j t_i)^\Delta$, from which we obtain

$$\overset{\Delta}{\sigma}{}^N = \eta\dot{\sigma}_{ij}^n n_j n_i + 2\eta\sigma_{ij}^n n_i \dot{n}_j + \dot{\eta}\sigma_{ij}^n n_j n_i \quad (15)$$

$$\overset{\Delta}{\tau}{}^N = \eta\dot{\sigma}_{ij}^n n_j t_i + \eta\sigma_{ij}^n n_j \dot{t}_i + \dot{\eta}\sigma_{ij}^n n_j t_i + \eta\sigma_{ij}^n \dot{n}_j t_i \quad (16)$$

The co-rotational terms are those with $\dot{n}_j$ and $\dot{t}_i$ and the remaining terms represent partial derivatives at constant **n**-direction. Eqs. 15–16 may be simplified by substituting $\dot{t}_i = -(\sigma^n/\tau^n)\, t_i\dot{\zeta}$, $\sigma_{ij}^n n_j n_i = \sigma^n$, $\dot{\sigma}_{ij}^n n_j n_i = \dot{\sigma}^n$, $\dot{\sigma}_{ij}^n n_j t_i = \dot{\tau}^n$, $\sigma_{ij}^n \dot{n}_i n_i = \sigma_{ij}^n n_i t_j \dot{\zeta} = \tau^n\dot{\zeta}$, and by introducing the simplification $\sigma_{ij}^n \dot{n}_j t_i = \eta\sigma_{ij}^n t_j t_i \dot{\zeta} = \eta\sigma_t\dot{\zeta} = 0$; this yields

$$\overset{\Delta}{\sigma}{}^N = \eta\left(\dot{\sigma}^n + 2\tau^n\dot{\zeta} + \sigma^n\frac{\dot{\eta}}{\eta}\right) \quad (17)$$

$$\overset{\Delta}{\tau}{}^N = \eta\left(\dot{\tau}_n - \sigma^n\dot{\zeta} + \tau^n\frac{\dot{\eta}}{\eta}\right) \quad (18)$$

The last simplification rests on the assumption that $\sigma_{ij}^n t_i t_j$, representing the normal stress σ_t in the direction parallel to the active plane, vanishes. In the strict mathematical sense this is not true, but physically it seems reasonable since the normal stress along the contact layer between adjacent aggregate pieces should generally be small and anyway have a negligible effect.

Since Eqs. 17–18 are basic for the present formulation, it may be helpful to rederive them in an alternative way. Consider the stress transformation due to rotating the coordinate axes through angle $\Delta\zeta$ around axis **m** [Fig. 2(*b*)];

$$\sigma' = \sigma\cos^2\zeta + \sigma_t\sin^2\zeta + \tau\sin 2\zeta$$

$$\tau' = \tau\cos 2\zeta + (\sigma_t - \sigma)\sin\zeta\cos\zeta \quad (19)$$

where the primes label the normal and shear stress after the transformation, and σ_t is the normal stress in the **t**-direction. For a small rotation angle, $\zeta = \dot{\zeta}dt$, and setting $\sigma_t = 0$, as explained above, Eq. 19 simplifies as $\sigma^{N'} = \sigma^N + 2\tau^N\dot{\zeta}dt$, $\tau^{N'} = \tau^N - \sigma^N\dot{\zeta}dt$ (now written for the macrostresses). At constant magnitudes of the stress components, the co-rotational derivatives are then obtained as $\overset{\Delta}{\sigma}{}^N = (\sigma^{N'} - \sigma^N)/dt$, $\overset{\Delta}{\tau}{}^N = (\tau^{N'} - \tau^N)/dt$, which provides $\overset{\Delta}{\sigma}{}^N = 2\tau^N\dot{\zeta}$, $\overset{\Delta}{\tau}{}^N = -\sigma^N\dot{\zeta}$. Then, adding the rate due to the changes of magnitude of σ^N and τ^N, we obtain the following co-rotational derivatives of the macrostresses

$$\overset{\Delta}{\sigma}{}^N = \dot{\sigma}^N + 2\tau^N\dot{\zeta}; \quad \overset{\Delta}{\tau}{}^N = \dot{\tau}^N - \sigma^N\dot{\zeta} \quad (20)$$

and in terms of the variable area of fraction, η, we finally have

$$\dot{\sigma}^N = \eta\left(\dot{\sigma}^n + \sigma^n\frac{\dot{\eta}}{\eta}\right), \quad \dot{\tau}^N = \eta\left(\dot{\tau}^n + \tau^n\frac{\dot{\eta}}{\eta}\right) \quad (21)$$

Substituting this into Eq. 20, we obtain again Eqs. 17–18.

Micro-Macro Relations for Stresses and Strains

The normal and shear macrostresses on the active plane may be expressed in terms of the principal stresses σ_1, σ_2 and σ_3 as follows

$$\sigma^N = \sigma_1 n_1^2 + \sigma_2 n_2^2 + \sigma_3 n_3^2 \quad (22a)$$

$$\tau^N = [(\sigma_1 - \sigma_2)^2 n_1^2 n_2^2 + (\sigma_2 - \sigma_3)^2 n_2^2 n_3^2 + (\sigma_3 - \sigma_1)^2 n_3^2 n_1^2]^{1/2} \quad (22b)$$

Note that the direction cosines appear only in their squares. Therefore, as long as the stress-strain relation for the macroplane ignores the orientation of vector **t** within the active plane, the relations are the same for all the planes for which n_1^2, n_2^2 and n_3^2 are the same. Thus, four active planes actually exist, and are characterized by the normals (n_1, n_2, n_3), $(-n_1, n_2, n_3)$, $(n_1, -n_2, n_3)$, $(n_1, n_2, -n_3)$. The negatives of these four vectors also yield the same σ^N and τ^N (Eq. 22), however, they represent the same planes. Thus, the stress-strain relations that we are going to formulate will actually apply to four distinct planes even though we will speak of only one active plane.

In plasticity, it is customary to formulate the stress-strain relations in terms of variables p and q defined as $p = \sigma_{kk}/3 = (\sigma_1 + \sigma_2 + \sigma_3)/3$ and $q = \sqrt{J_2}$, where $J_2 = s_{ij}s_{ij}/2 = [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]/6$. The principal stresses may then be obtained as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{Bmatrix} = \frac{2}{\sqrt{3}} q \begin{Bmatrix} \sin\left(\theta + \frac{2}{3}\pi\right) \\ \sin\theta \\ \sin\left(\theta + \frac{4}{2}\pi\right) \end{Bmatrix} + p \qquad (23)$$

in which $\theta = (1/3)\arcsin(-3\sqrt{3}\, I_3/2q)$, $I_3 = \sigma_1\sigma_2\sigma_3$, and $-\pi/6 \le \theta \le \pi/6$.

Using Eqs. 22–23, one can derive the following matrix relations

$$\begin{Bmatrix} \sigma^N \\ \tau^N \end{Bmatrix} = \mathbf{\Omega} \begin{Bmatrix} p \\ q \end{Bmatrix}; \quad \mathbf{\Omega} = \begin{bmatrix} 1 & 0 \\ \dfrac{-A_\sigma}{A_\tau} & \dfrac{1}{A_\tau} \end{bmatrix} \qquad (24)$$

$$\begin{Bmatrix} \sigma^N \\ \tau^N \end{Bmatrix} = \mathbf{T}^T \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{Bmatrix}; \quad \mathbf{T}^T = \begin{bmatrix} n_1^2 & n_2^2 & n_3^2 \\ \rho_1 n_1^2 & \rho_1 n_2^2 & \rho_3 n_3^2 \end{bmatrix}; \quad \begin{Bmatrix} \sigma^N \\ \tau^N \end{Bmatrix} = \mathbf{Q}^T \boldsymbol{\sigma} \qquad (25)$$

in which $\boldsymbol{\sigma} = (\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31})^T$, superscript T denotes a transpose, and

$$A_\sigma = \frac{2}{\sqrt{3}} \left[n_1^2 \sin\left(\theta + \frac{2}{3}\pi\right) + n_2^2 \sin\theta + n_3^2 \sin\left(\theta + \frac{4}{3}\pi\right) \right];$$

$$A_\tau = \left\{ \frac{4}{3} \left[n_1^2 \sin^2\left(\theta + \frac{2}{3}\pi\right) + n_2^2 \sin^2\theta + n_3^2 \sin^2\left(\theta + \frac{4}{3}\pi\right) \right] - A_\sigma^2 \right\}^{1/2};$$

$$\rho_1 = \left[\frac{2}{\sqrt{3}} \sin\left(\theta + \frac{2}{3}\pi\right) - A_\sigma \right] \frac{1}{A_\tau}; \quad \rho_2 = \left(\frac{2}{\sqrt{3}} \sin\theta - A_\sigma \right) \frac{1}{A_\tau};$$

$$\rho_3 = \left[\frac{2}{\sqrt{3}} \sin\left(\theta + \frac{4}{3}\pi\right) - A_\sigma \right] \frac{1}{A_\tau} \qquad (26)$$

$$\text{Furthermore,} \quad \mathbf{Q} = \begin{bmatrix} N_1^2 & N_1 T_1 \\ N_2^2 & N_2 T_2 \\ N_3^2 & N_3 T_3 \\ 2N_1 N_2 & (N_1 T_2 + N_2 T_1) \\ 2N_1 N_3 & (N_1 T_3 + N_3 T_1) \\ 2N_2 N_3 & (N_2 T_3 + N_3 T_2) \end{bmatrix} \qquad (27)$$

in which components N_1, N_2, N_3, and T_1, T_2, T_3 of unit vectors $\mathbf{N}$, $\mathbf{T}$ are defined in the cartesian coordinate system x_i (i = 1, 2, 3). The relations between components n_i and N_i, and between t_i and T_i, take the form $N_i = P_{ij} n_j$, $T_i = P_{ik}^T t_k$, $t_1 = n_1\rho_1$, $t_2 = n_2\rho_2$, $t_3 = n_3\rho_3$ in which matrix $\mathbf{P}$ transforms the components of the unit vectors $\mathbf{n}$, $\mathbf{t}$ from the principal directions into the cartesian coordinate system. In the loading step (r, $r+1$), the normal is updated according to the relations:

$$n_{i_{(r+1)}} = n_i^*(n_j^* n_j^*)^{-1/2}, \quad n_i^* = n_{i_{(r)}} + t_{i_{(r)}} \Delta\zeta \qquad (28)$$

Noting that $(n_j + t_j\Delta\zeta)(n_j + t_j\Delta\zeta) = 1 + \Delta\zeta^2$, we may simplify Eq. 28 as

$$n_{i_{(r+1)}} = (n_{i_{(r)}} + t_{i_{(r)}}\Delta\zeta)(1 + \Delta\zeta^2)^{-1/2} \qquad (29)$$

Furthermore, expressing t_i in terms of A_τ and A_σ, we may obtain the recurrent relation

$$n_{i_{(r+1)}} = \frac{(n_i + \Delta\zeta t_i)_{(r)}}{[(n_k + \Delta\zeta t_k)(n_k + \Delta\zeta t_k)]_{(r)}}; \quad t_{i_{(r)}} = \left(\frac{\sigma_i - \sigma^N}{\tau^N} n_i \right)_{(r)} \qquad (30)$$

in which $(n_i + \Delta\zeta t_{i_{(r)}}) = [n_i(1 + \Delta\zeta\rho_i)]_{(r)}$, $[(n_k + \Delta\zeta t_k)(n_k + \Delta\zeta t_k)]_{(r)} = n_k n_k + 2t_k n_k \Delta\zeta + t_k t_k \Delta\zeta^2 = 1 + \Delta\zeta^2$ because $n_k n_k = 1$, $t_k n_i = 0$, $t_k t_k = 1$. Consequently,

$$n_{i_{(r+1)}} = \frac{1 + \rho_1 \Delta\zeta}{\sqrt{1 + \Delta\zeta^2}} n_{i_{(r)}} \qquad (30a)$$

As the initial condition (stress-free state, isotropic properties), no particular direction may be favored, and so it is necessary that $n_1 = n_2 = n_3 = 1/\sqrt{3}$. Thus the initial orientation of the active plane is the same as for the von Mises yield criterion.

The normal and shear macrostresses and microstresses are related as

$$\sigma^N = \eta\sigma^n; \quad \tau^n = \eta\sigma^n \qquad (31)$$

According to the hypothesis of a single active plane, the energy dissipation written in terms of the stress and strain components on the active plane must be equal to that written in terms of the macroscopic stresses and strains, as well as that written on the active plane in terms of the microstresses and microstrains;

$$\sigma^N \dot{\epsilon}^N + \tau^N \dot{\gamma}^N = p\dot{\epsilon}_p + q\dot{\epsilon}_q; \quad \sigma^N \dot{\epsilon}^N + \tau^N \dot{\gamma}^N = \sigma_1\dot{\epsilon}_1 + \sigma_2\dot{\epsilon}_2 + \sigma_3\dot{\epsilon}_3$$

$$\sigma^N \dot{\epsilon}^N + \tau^N \dot{\gamma}^N = \sigma_{11}\dot{\epsilon}_{11} + \sigma_{22}\dot{\epsilon}_{22} + \sigma_{33}\dot{\epsilon}_{33} + \sigma_{12}\dot{\epsilon}_{12} + \sigma_{23}\dot{\epsilon}_{23} + \sigma_{31}\dot{\epsilon}_{31}$$

$$\sigma^N \dot{\epsilon}^N + \tau^N \dot{\gamma}^N = \sigma^n \dot{\epsilon}^n + \tau^n \dot{\gamma}^n \qquad (32)$$

Although there are normally four active planes, we rewrite Eq. 32 for only one active plane. This can be done because the left-hand sides of Eq. 32 are the same for all the four planes, which means that merely a factor of 4 is omitted; this is irrelevant because only the relative values matter.

Now substituting Eqs. 24–26 and 31, and noting that Eqs. 32 must hold for any possible values of stresses, we obtain the following relations

$$\mathbf{\Omega}^T \begin{Bmatrix} \dot{\epsilon}_p \\ \dot{\epsilon}_q \end{Bmatrix} = \begin{Bmatrix} \dot{\epsilon}^N \\ \dot{\gamma}^N \end{Bmatrix} \qquad (33)$$

$$\mathbf{R}^T \begin{Bmatrix} \dot{\epsilon}_1 \\ \dot{\epsilon}_2 \\ \dot{\epsilon}_3 \end{Bmatrix} = \begin{Bmatrix} \dot{\epsilon}^N \\ \dot{\gamma}^N \end{Bmatrix} \qquad (34)$$

$$\mathbf{Q}^T \dot{\boldsymbol{\epsilon}} = \begin{Bmatrix} \dot{\epsilon}^N \\ \dot{\gamma}^N \end{Bmatrix} \qquad (35)$$

$$\dot{\epsilon}^N = \eta \dot{\epsilon}^n; \quad \dot{\gamma}^N = \eta \dot{\gamma}^n \qquad (36)$$

in which $\dot{\boldsymbol{\epsilon}} = (\dot{\epsilon}_{11}, \dot{\epsilon}_{22}, \dot{\epsilon}_{33}, \dot{\epsilon}_{12}, \dot{\epsilon}_{23}, \dot{\epsilon}_{31})^T$.

Constitutive Relation for Active Plane

First we need the elastic parts of the normal and shear strain rates on the active plane. They may be expressed as

$$\begin{Bmatrix} \dot{\epsilon}^e \\ \dot{\gamma}^e \end{Bmatrix} = \mathbf{C} \begin{Bmatrix} \dot{\sigma}^n \\ \dot{\tau}^n \end{Bmatrix}, \quad \mathbf{C} = \begin{bmatrix} k_\sigma & 0 \\ 0 & k_\tau \end{bmatrix} \qquad (37)$$

Note that one can introduce an elastic potential function $\Psi = 1/2\ (\epsilon^n, \gamma^n)\ \mathbf{C}^{-1} (\epsilon^n, \gamma^n)^T$ from which the stress rates may be obtained as $\dot{\sigma}^n = (\partial \Psi / \partial \epsilon^n)^{\cdot}$, $\dot{\tau}^n = (\partial \Psi / \partial \gamma^n)^{\cdot}$. In writing Eq. 37, we lump all the elastic deformation into the active plane. Although this might not be quite true, it is convenient to do so, especially since the resulting elastic part of the macroscopic stress-strain relation is the same as usual.

As for the formulation of the inelastic part of deformation, note that it should be written in terms of the microscopic rather than macroscopic stress and strain rates on active planes. This is because these rates describe the changes with regard to the material coordinates. The use of microstresses is convenient because they relate to the true stresses that act in the resisting area fraction (i.e., unfractured, undamaged fraction) of the material, which makes it possible to relegate the softening of the material entirely to the growth of the area fraction η. Furthermore, since strain softening is in this manner eliminated from the constitutive relation for the microstresses, it is possible to use the mathematical framework of plasticity, which represents the simplest, most consistent, and numerically trouble-free approach to constitutive relations. The use of plasticity for the microstresses instead of the macrostresses is better justified because strain-softening, the chief objection to the use of normality rule (or Drucker's postulate), is modeled principally by the reduction of the resisting area fraction η. Nevertheless, we should be aware that another objection to the normality rule, namely that of friction, is not eliminated by the introduction of the resisting area fraction η and the corresponding microstresses.

The simplest loading surface which exhibits the principal features of the behavior of geomaterials, including concrete, is the loading surface used in the so-called critical state theory for soil plasticity. This surface has the form of an ellipse in the shear stress-normal stress space. The ellipse changes its size and shape according to neither isotropic nor kinematic hardening, but in the simple manner depicted in Fig. 1(f). The elliptical loading surface may be written as

$$f(\sigma^n, \tau^n) = \tau^{n^2} - \mu^2 (\sigma^n - \sigma_t)(\sigma_c + \sigma^n) = 0 \qquad (38)$$

in which μ is a material parameter characterizing internal friction, and σ_t and σ_c are hardening parameters which may be regarded as the current values of the tensile and compressive strength limits; Fig. 1(g).

According to the normality rule, the inelastic rates of the normal and shear microstrains on the active planes are

$$\dot{\epsilon}^{n''} = \dot{\lambda} \frac{\partial f}{\partial \sigma^n}; \quad \dot{\gamma}^{n''} = \dot{\lambda} \frac{\partial f}{\partial \tau^n} \qquad (39)$$

in which $\dot{\lambda}$ is a certain common multiplier. The total rates of normal and shear microstrains on the active plane then are the sums of the elastic and inelastic components, i.e., $\dot{\epsilon}^n = \dot{\epsilon}^{n^e} + \dot{\epsilon}^{n''}$, $\dot{\gamma}^n = \dot{\gamma}^{n^e} + \dot{\gamma}^{n''}$, which then yields

$$\dot{\epsilon}^n = k_\sigma^{-1} \dot{\sigma}^n + \dot{\lambda} \frac{\partial f}{\partial \sigma^n}; \quad \dot{\gamma}^n = k_\tau^{-1} \dot{\tau}^n + \dot{\lambda} \frac{\partial f}{\partial \tau^n} \qquad (40)$$

To determine $\dot{\lambda}$ we use the usual procedure in plasticity. Because $f(\sigma^n, \tau^n, \sigma_t) = 0$ at all times, we must have $\dot{f}(\sigma^n, \tau^n, \sigma_t) = 0$, i.e.,

$$\dot{f}(\sigma^n, \tau^n, \sigma_t) = \frac{\partial f}{\partial \sigma^n} \dot{\sigma}^n + \frac{\partial f}{\partial \tau^n} \dot{\tau}^n + \frac{\partial f}{\partial \sigma_t} \dot{\sigma}_t = 0 \qquad (41)$$

which is in fact Prager's consistency condition. Substituting Eq. 40 into Eq. 41, we have

$$f_\sigma k_\sigma (\dot{\epsilon}^n - \dot{\lambda} f_\sigma) + f_\tau k_\tau (\dot{\gamma}^n - \dot{\lambda} f_\tau) + \frac{\partial f}{\partial \sigma_t} \frac{\partial \sigma_t}{\partial \epsilon^{n''}} \dot{\lambda} f_\sigma = 0 \qquad (42)$$

in which $f_\sigma = \partial f / \partial \sigma_n$, $f_\tau = \partial f / \partial \tau_n$. Now we may solve this equation for $\dot{\lambda}$ and obtain

$$\dot{\lambda} = \frac{(f_\sigma k_\sigma \dot{\epsilon}^n + f_\tau k_\tau \dot{\gamma}^n)}{H} \qquad (43)$$

in which $H = f_\sigma^2 k_\sigma + f_\tau^2 k_\tau - (\partial f / \partial \sigma_t)(\partial \sigma_f / \partial \epsilon^{n''})\, f_\sigma$. Later we will introduce an expression for $(\partial f / \partial \sigma_t)(\partial \sigma_t / \partial \epsilon^{n''}) f_\sigma = \alpha \sigma_t f_{\sigma_t} |f_\sigma|$, which then permits solving $f_\sigma (\partial f / \partial \epsilon^{n''}) = \alpha f_{\sigma_t} |f_\sigma|$, with f_{σ_t} given later by Eq. 45. Then we substitute Eq. 43 along with the expression for H into Eq. 40, and solve Eq. 40 for $\dot{\sigma}^n$ and $\dot{\tau}^n$; this yields the tangential relation for the normal and shear components of the microstresses and microstrains on the active plane;

$$\begin{Bmatrix} \dot{\sigma}^n \\ \dot{\tau}^n \end{Bmatrix} = \begin{bmatrix} k_\sigma \left(1 - \dfrac{f_\sigma^2 k_\sigma}{H}\right); & -k_\sigma k_\tau \dfrac{f_\sigma f_\tau}{H} \\ -k_\sigma k_\tau \dfrac{f_\sigma f_\tau}{H}; & k_\tau \left(1 - \dfrac{f_t^2 k_\tau}{H}\right) \end{bmatrix} \begin{Bmatrix} \dot{\epsilon}^n \\ \dot{\gamma}^n \end{Bmatrix} \qquad (44)$$

Note that the matrix of this equation, representing the stiffness matrix for the active plane, is symmetric. This is a necessary consequence of the normality rule.

Further, it is necessary to specify the hardening-softening rule for the evolution of the loading surface as well as parameters η and ζ. Similar to many previous works on plasticity, it appears possible to describe the decay of tensile strength in terms of the path length of the inelastic normal strain;

$$\sigma_t = f_t' e^{-\alpha_1 \int |\epsilon^{n''}| dt} \qquad (45)$$

in which α_1 and f_t' are constants, f_t' being the initial tensile strength value. Without the decay of tensile strength, it would be impossible to fit the test data. Note that Eq. 45 indicates tensile strength decay even for compressive loading with regard to the active plane, and for cyclic loading. The parameters η and ζ appear to be approximately functions of the total accumulated inelastic normal and shear strains on the active plane, and the following particular forms have been found to work reasonably well

$$\eta = e^{\alpha_0 \epsilon^{n''}}; \quad \zeta = -\frac{\omega - \omega_0}{\omega_1}; \quad \omega = \omega_0 e^{-\omega_1 \gamma^{n''}} \tag{46}$$

in which α_0, ω_0 and ω_1 are empirical constants. Eq. 46 implies that the initial conditions are $\eta = 1$ and $\zeta = 0$.

Note that ζ is a cumulative angular rotation, which represents an actual angle only if vector $\mathbf{n}$ has constant direction during the loading process. If the direction of $\mathbf{n}$ is rotating, then ζ represents no actual angle, but only a sum of all the previous rotation angle increments.

Macroscopic Constitutive Relation

The microscopic constitutive relation given by Eq. 42 (for loading) must now be translated into a macroscopic constitutive relation. This can be done by using the previously determined micro-macro relations for the stresses (Eqs. 24–26). From these relations we obtain

$$\begin{Bmatrix} \overset{\Delta}{\sigma}{}^N \\ \overset{\Delta}{\tau}{}^N \end{Bmatrix} = \eta^2 \mathbf{K} \begin{Bmatrix} \dot{\epsilon}^N \\ \dot{\gamma}^N \end{Bmatrix}; \quad \mathbf{K} = \begin{bmatrix} k_{\sigma\sigma} & k_{\sigma\tau} \\ k_{\tau\sigma} & k_{\tau\tau} \end{bmatrix} \tag{47}$$

in which $$K_{\sigma\sigma} = k_\sigma + \frac{f_\sigma^2 k_\sigma}{H}\left(\frac{1}{\eta}\sigma^n \alpha_0 - k_\sigma\right) + \frac{1}{\eta} 2\tau^n \omega \frac{f_\sigma f_\tau k_\sigma}{H} \tag{48a}$$

$$K_{\sigma\tau} = \frac{f_\sigma f_\tau k_\tau}{H}\left(\frac{1}{\eta}\sigma^n \alpha_0 - k_\sigma\right) + \frac{1}{\eta} 2\tau^n \omega \frac{f_\tau^2 k_\tau}{H} \tag{48b}$$

$$K_{\tau\sigma} = \frac{f_\sigma f_\tau k_\sigma}{H}\left(\frac{1}{\eta}\sigma^n \omega - k_\tau\right) + \frac{1}{\eta}\tau^n \alpha_0 \frac{f_\sigma^2 k_\tau}{H} \tag{48c}$$

$$K_{\tau\tau} = k_\tau - \frac{f_\tau^2 k_\tau}{H}\left(\frac{1}{\eta}\sigma^n \omega + k_\tau\right) + \frac{-1}{\eta}\tau^n \alpha_0 \frac{f_\sigma f_\tau k_\tau}{H} \tag{48d}$$

This relation holds only for loading on the active plane. For unloading on the active plane it must be replaced by

$$\begin{Bmatrix} \overset{\Delta}{\sigma}{}^N \\ \overset{\Delta}{\tau}{}^N \end{Bmatrix} = \eta^2 \begin{bmatrix} k_\sigma & 0 \\ 0 & k_\tau \end{bmatrix} \begin{Bmatrix} \dot{\epsilon}^N \\ \dot{\gamma}^N \end{Bmatrix} \tag{49}$$

Furthermore, using the previously derived matrices $\mathbf{\Omega}$, $\mathbf{R}$ and $\mathbf{Q}$, we may obtain the following form of the stress-strain relation

$$\begin{Bmatrix} \dot{p} \\ \dot{q} \end{Bmatrix} = (\mathbf{\Omega}^T \mathbf{K}^{-1} \mathbf{\Omega})^{-1} \begin{Bmatrix} \dot{\epsilon}_p \\ \dot{\epsilon}_q \end{Bmatrix} \tag{50}$$

$$\begin{Bmatrix} \dot{\sigma}_1 \\ \dot{\sigma}_2 \\ \dot{\sigma}_3 \end{Bmatrix} = (\mathbf{R}^T \mathbf{K}^{-1} \mathbf{R})^{-1} \begin{Bmatrix} \dot{\epsilon}_1 \\ \dot{\epsilon}_2 \\ \dot{\epsilon}_3 \end{Bmatrix} \tag{51}$$

$$\dot{\boldsymbol{\sigma}} = (\mathbf{Q}^T \mathbf{K}^{-1} \mathbf{Q})^{-1} \dot{\boldsymbol{\epsilon}} \tag{52}$$

The last form, with the full 6×6 stiffness matrix, is the form generally required for finite element programs. However, for fitting test data it is more convenient to use Eq. 51 with the 3×3 stiffness matrix; this form of the stress-strain relation is admissible only if the principal stress directions do not rotate and if they coincide with the principal strain directions. In general, of course, the principal stress directions do rotate and do not coincide with the principal strain directions.

As in plasticity, the loading criterion is

$$df(\sigma^n, \tau^n) > 0 \tag{53}$$

If this criterion is violated, the stiffness matrix must be changed to the elastic stiffness matrix. After that, the elastic stiffness matrix is used throughout unloading and reloading as long as $f(\sigma^n, \tau^n) \le 0$. During unloading and reloading the orientation of the active plane (which is inactive) is kept fixed. Plastic deformation starts again when $f(\sigma^n, \tau^n)$ attains a zero value.

As an alternative, the constitutive relation could be also set up entirely on the macrolevel. For that purpose, the plastic loading surface would have to be expressed in terms of the macrostresses, i.e., σ_1, σ_2, σ_3, or $p_{,q}$ or σ_{11}, σ_{22}, ... σ_{31}. Such a loading surface is illustrated for the present model in Fig. 1(f,g) (right). The expansion or contraction of the macroscopic loading surface in the (p,q) space is governed by parameter η, while the change of shape of the surface is governed primarily by the rotation of the active plane [Fig. 1(d,e)].

Identification of Material Parameters

One principal advantage of the present model is that the material parameters are few and can be easily identified from test data. Material parameter identification begins with a uniaxial compression test. For the sake of simplicity, we assume the stress-strain diagram to be linear up to the elastic limit at which a sudden decrease of slope occurs. The present model could no doubt be adjusted to also represent the curved shapes of the rising branches of the uniaxial and biaxial stress-strain diagrams and obtain correct initial elastic moduli. However, this would mean more material parameters and a less simple identification procedure. The principal intention of this paper is to show what can be achieved with the simplest model.

The elastic limit is considered as $\sigma_1 = f_c'$, and the correesponding strains are denoted as $\epsilon_1 = \epsilon_1'$ and $\epsilon_2 = \epsilon_2'$. Using the elastic stress-strain relation (Eq. 37), we obtain the following expressions for the elastic stiffnesses;

$$k_\sigma = \frac{f_c'}{3(\epsilon_1' + 2\epsilon_2')}; \quad k_\tau = \frac{f_c'}{3(\epsilon_1' - \epsilon_2')} \tag{54}$$

Furthermore, using the known value of the uniaxial tensile strength f_t', we obtain according to our loading function (Eq. 38) the expressions

$$\sigma_t^0 = \frac{1}{2} a \left(\sqrt{1 + \frac{4}{b-2}} - 1 \right); \quad \sigma_c = \sigma_t^0 + a \quad (55)$$

in which $$a = \frac{6 + 3\mu^2}{\mu^2} f_t'; \quad b = 10.1 + \frac{16.2}{\mu^2} \quad (56)$$

Further we use the equi-biaxial compression strength f_c'', i.e., the limit of elasticity in biaxial compression with $\sigma_1 = \sigma_2$. Using the micro-macro relations for stresses in Eqs. 24, 25 and 31, we obtain $\sigma'' = 2f_c''/3$, $\tau'' = f_c'' \sqrt{2}/3$ and substituting this into the initial loading surface (Eq. 38) we acquire the following condition from which the internal friction parameter μ can be solved;

$$f_c''^2(1 - 2\mu^2) - 3f_c''\mu^2(\sigma_c - \sigma_t^0) + \frac{9}{2}\mu^2\sigma_c\sigma_t = 0 \quad (57)$$

Next we need to use the hydrostatic compression test ($\sigma_1 = \sigma_2 = \sigma_3$) to identify parameter α_0. For this test we have $\dot{p} = k_\sigma \dot{\epsilon}_o$ until the limit surface (Eq. 38) is reached. The subsequent inelastic behavior is defined by Eq. 48 and Eq. 50, which simplify because $f_\tau = 0$, $\tau = 0$, $\dot{\gamma}^{n''} = 0$. Thus we find

$$\dot{p} = \mu^2 k_{\sigma\sigma} \dot{\epsilon}_p \quad (58)$$

in which $$k_{\sigma\sigma} = k_\sigma + \frac{f_\sigma^2 k_\sigma}{H} \left(\frac{1}{\eta} \sigma^n \alpha_0 - k_\sigma \right) \quad (59)$$

$$H = f_\sigma^2 k_\sigma + \alpha_1 \sigma_t f_{,\sigma_t} |f_\sigma|; \quad f_{,\sigma_t} = -\mu^2(\sigma_c - \sigma) \quad (60)$$

Here, for hydrostatic compression, we have $f_{,\sigma_t} = 0$, and thus Eq. 58 is reduced to $\dot{p} = p\alpha_0 \dot{\epsilon}_p$. Integration then yields

$$p = \sigma_c e^{\alpha_0(\epsilon_p - \epsilon_p^0)} \quad (61)$$

in which $\epsilon_p^0 = \sigma_c/k_\sigma$. By matching the exponential curves in Eq. 61 to the measured hydrostatic compression diagram, one can easily identify parameter α_0.

Parameters α_0 and α_1 govern the softening properties in uniaxial compression. Knowing already the values of parameters k_σ, k_τ, σ_c, σ_t^0, μ and α_0, it suffices to assign only the value of α_1 in order to be able to plot the strain-softening diagram for uniaxial compression. By choosing various α_1-values, the proper value can be found in a trial-and-error manner.

Eq. 46 for η and ζ, and particularly parameters ω_0 and ω_1 from Eq. 46, control principally the shape of the failure surface (the maximum load surface) in the octahedral cross section. This shape is rather sensitive to the evolution of the orientation of the active plane, characterized by the relations $\dot{n}_i = \zeta t_i$ and $\dot{\zeta} = \omega \dot{\gamma}''$. The shape of the failure surface in the octahedral cross section is not known very accurately; however, it is generally agreed that the surface is noncircular at low hydrostatic compression, having the shape of a rounded triangle, and expands to a nearly circular shape at a very high hydrostatic compression (the circular shape

corresponds to the von Mises or Drucker-Prager failure criteria). The present formulation with Eq. 46 provides this type of failure surfaces in the octahedral cross section. Moreover, it has been found empirically that parameters ω_0 and ω_1 may be considered to be about the same for all concretes and can be fixed once for all as

$$\omega_0 = 50 \text{ psi}; \quad \omega_1 = 5\omega_0 \quad (62)$$

Since parameters ω_0 and ω_1 can apparently be fixed for all concretes, there are only six free inelastic material parameters to be found from test data: f_c', ϵ_1', ϵ_2', α_0, α_1, and μ. This is indeed a relatively small number of parameters. Moreover, these parameters need not be solved simultaneously from the given test data but may be identified in sequence.

Comparison with Test Data

Important test data from the literature (2,9,11,12,13,15,22,24,28) have been fitted with the present model; see Figs. 2–4. In these figures the

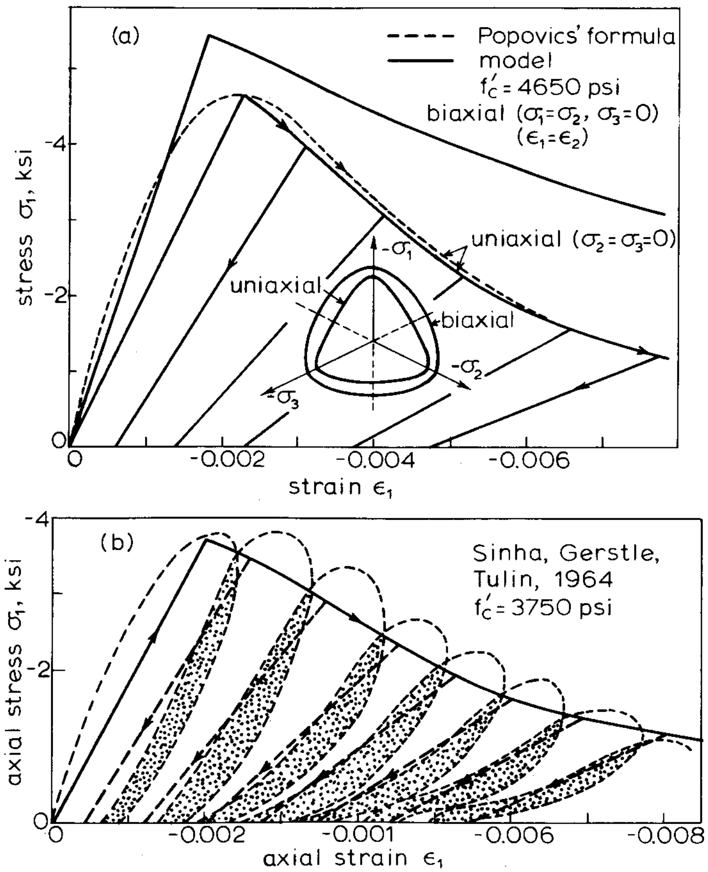

FIG. 2.—(*a*) Fit of Uniaxial Test Data by Popovics (19), and (*b*) Fit of Uniaxial Cyclic Compression Data of Sinha, Gerstle, Tulin (25)

data are represented by the data points, and the present constitutive model is plotted as the solid curves. Unfortunately, no sufficiently large data set exists for one and the same concrete, and therefore, test data for different concretes with different strengths have to be used at present. The material parameter values obtained by the foregoing material

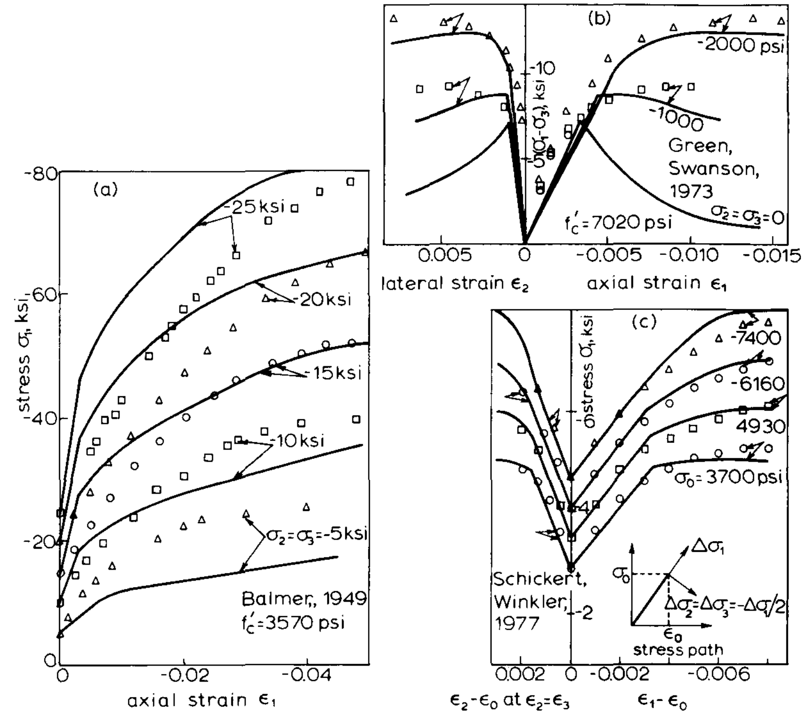

FIG. 3.—(*a*) Comparison with Balmer's Triaxial Test Data (2); (*b*) Fit of Triaxial Test Data of Green and Swanson (11); and (*c*) Fit of Experimental Nonproportional Triaxial Stress-Strain Data of Schickert and Winkler

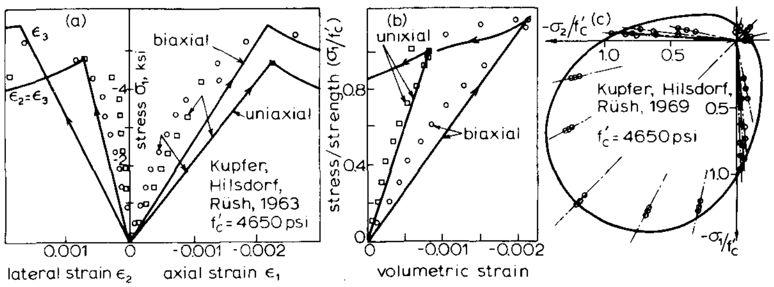

FIG. 4.—(*a*) Fit of Axial and Lateral Strain Diagrams; (*b*) Volume Change Diagrams; and (*c*) Biaxial Failure Envelope of Concrete for Uniaxial and Biaxial Tests of Kupfer, Hilsdorf and Rüsch (1963)

identification procedure are listed for all these data in Table 2. Note that the material parameter values for the lowest strength concrete ($f_c' = 3{,}570$ psi) differ appreciably from those for the other concretes, which are of a considerably higher strength. Especially, it may be noted that for this concrete at $\sigma_1 = f_c'$ the axial strain is very large ($\epsilon_1' = 0.05$); obviously, this low strength concrete is quite soft and therefore not very well comparable with the other concretes.

The fit of Popovics' empirical formula (22) [Fig. 2(*a*)] and Balmer's test data (28) [Fig. 2(*b*)] demonstrated a good representation of strain-soft-

TABLE 2.—Material Parameter Values Identified by Data Fitting

f_c' (psi) (1)	ϵ_1' (2)	ϵ_2' (3)	α_0 (4)	α_1 (5)	μ^2 (6)	ω_0 (7)	ω_1 (8)
3,750	−0.0025	0.00080	70	300	0.4	50	300
4,650	−0.0023	0.00075	100	500	0.5	50	300
4,930	−0.0023	0.00075	100	500	0.4	50	300
7,020	−0.0033	0.00100	100	500	0.6	50	300

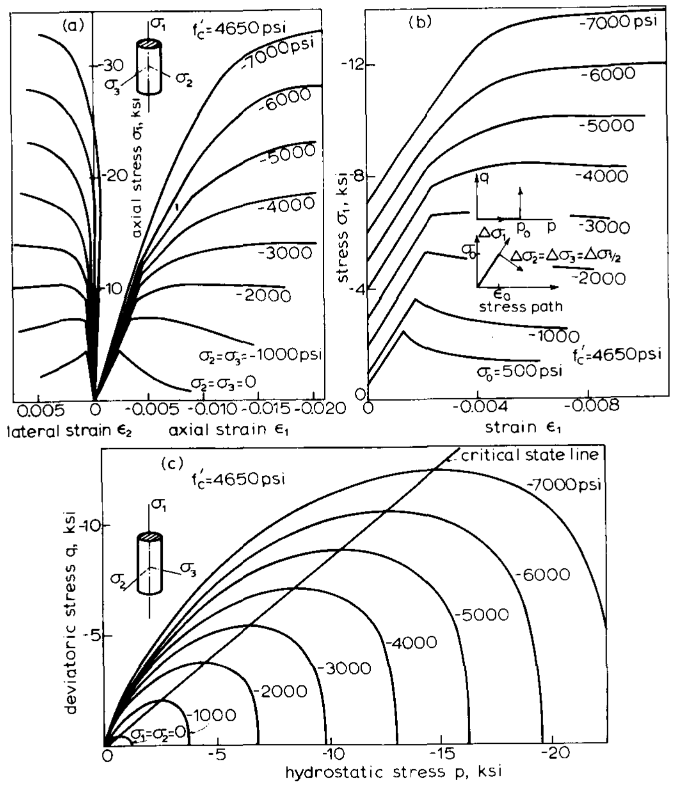

FIG. 5.—(*a*) Standard Triaxial Stress-Strain Diagram for Concrete; (*b*) Nonstandard Triaxial Stress-Strain Diagram for Concrete; (*c*) Failure Surface for Triaxial Compression Test, Concrete (f_c' = 4,640 psi)

ening and unloading after strain-softening. Due to the use of secant moduli, the rising branch does not start with the correct initial elastic modulus. However, the present model could doubtless be adapted to describe the correct initial elastic modulus if further parameters were introduced. In particular, this could be done by introducing a dependence of the rotation of the active surface on the total strains. The material identification, however, would then become more involved.

Note also in Fig. 2(*a*) that the failure surfaces on octahedral (π plane) projection have the well-known non-circular shapes (24) even though the third stress invariant is not used in the plastic loading function; this shape is a consequence of the rotation of the active plane.

The biaxial failure envelope (15) for plane stress, representing a collection of the peak stress points for proportional loading at various constant stress ratios, is fitted in Fig. 4(*c*). Its shape is reasonable, although it gives much too high strength values in the compression-tension quadrant. This could be avoided by using, instead of Eq. 38, the loading surface $f = \tau^{n^2} + (\xi - \sigma_t')(\sigma_c' + \xi) = 0$ with $\xi = (c\sigma_t - \sigma)^n/k^n$, $\sigma_t' = [(c - 1)\sigma_t/k]^n$, $\sigma_c' = [(c - 1)/\sigma_c/k]^n$. Three additional parameters c, k, n would then have to be identified.

In Fig. 2(*b*), representing cyclic uniaxial compression data for large strains (28), no attempt was made to describe the hysteretic loops. However, the overall representation is satisfactory. The fits of the data by Green and Swanson (13) and by Schickert and Winkler (24) in Figs. 3(*b,c*) and 5(*a,b*) are also acceptable, with errors in the range of typical scatter of measurements.

In judging the goodness of the fits in Figs. 2–5, one should realize that only six free (and two fixed) inelastic material parameters are used (Table 1). Closer fits would no doubt be possible with more elaborate formulas involving a greater number of parameters.

Summary and Conclusions

The present constitutive model has the following basic characteristics

1. A certain active plane is assumed to exist such that the constitutive relation may be written as a relation between the normal and shear stresses and strains on this plane.
2. The orientation of the active plane, initially coinciding with the octahedral plane, is not fixed but is assumed to vary during the deformation process.
3. The constitutive law on the active plane is written in terms of microstresses (or true stresses) and microstrains which refer to the resisting area fraction of the material.
4. The resisting area of fraction on the active plane and the orientation of the active plane are variable and depend on the previously accumulated inelastic normal and shear strains on the active plane.
5. The instantaneous axis of rotation of the active plane is considered to lie within the active plane and be normal to the resolved shear stress component on the active plane.

Since the constitutive law is written in terms of the microstresses (true stresses) and the microstrains, it need not exhibit strain softening, which

can be obtained on the macrolevel by means of a variation of the resisting area fraction η which relates the microstresses (true stresses) to the macrostresses. Consequently, the constitutive law on the microlevel may be assumed to follow the classical theory of incremental plasticity, including the normality rule. An incremental (tangential) stiffness matrix, which is independent of the rates of stress and strain (except for the choice between loading and unloading), exists and is always symmetric. For practical reasons, though, it has been convenient to include tensile strain-softening due to fracturing on the micro-level.

The loading surface on the active plane is chosen to be of the same type as that in the critical state theory for soils (Cam clay), i.e., an ellipse (in the plot of normal stress versus shear stress) which expands (neither isotropically nor kinematically) according to an evolution law for the tensile strength and compression strength limits.

The important test data from the literature can be relatively closely approximated with the present constitutive model (Fig. 2–4).

The present model involves only six free inelastic material parameters to be identified from test data, plus two further parameters which are fixed for all concretes and need not be identified from test data.

The material parameter values need not be identified simultaneously but their sequential identification is possible, and is relatively simple.

To identify the inelastic material parameters, the following data are needed: the compression strength, the tensile strength, the axial and lateral strains at the peak stress point, the equi-biaxial strength in compression, the approximate shape of the hydrostatic compression curve beyond the elastic limit, and the approximate shape of the strain-softening diagram in uniaxial compression.

The model gives the correct non-circular shape of the failure surface in the octahedral cross section; however, this shape is achieved automatically, without fitting any test data on this shape. The noncircular shape is caused by rotation of the active plane.

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