# Microplane Model for Brittle-Plastic Material: I. Theory

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**Abstract**: A generalized microplane model for brittle-plastic heterogeneous materials such as concrete, which describes not only tensile cracking but also nonlinear triaxial response in compression and shear, is presented. The constitutive properties are characterized separately on planes of various orientations within the material, called the microplanes, on which there are only few stress and strain components and no tensorial requirements need to be observed. These requirements are satisfied automatically by integration over all spatial directions. The state of each microplane is characterized by normal deviatoric and volumetric strains and shear strain, which makes it possible to match any Poisson ratio. The microplane strains are assumed to be the resolved components of the macroscopic strain tensor. The central assumption is that on the microplane level the stress-strain diagrams for monotonic loading are path-independent and that all the path dependence on the macrolevel is due to unloading, which happens selectively on microplanes of some orientations. The response on the microplane is assumed to depend on the lateral normal strain, which does no work. In consequence, the incremental elastic moduli tensor is nonsymmetric, which is necessary to model friction and dilatancy. This tensor is also generally anisotropic and fully populated (i.e., none of its elements can be prescribed as zero). The model involves many fewer free material parameters than the existing comprehensive macroscopic phenomenologic constitutive models for concrete.

## Introduction

During the last decade we have witnessed a tremendous progress in the development of constitutive models for brittle-plastic heterogeneous materials such as concrete, motivated chiefly by the needs of finite element analysis. The existing models are basically of two kinds: (1) Macroscopic phenomenologic models; and (2) micromechanics models. The macroscopic phenomenologic models include: (1) The classical plastic models (Chen and Chen 1975; Takahashi and Marchertas 1985), which include Drucker-Prager plasticity, the cap model, the critical state theory (originally developed for soils), models with rounded-triangle deviatoric section (Willam and Warnke 1974) and with slanted ellipse volumetric section (Lin et al. 1987), etc.; (2) the bounding surface model (Yang et al. 1985) which is really plasticity with a refined hardening rule; (3) the rotating active plane model (Zubelewicz and Bažant 1987); (4) the hypoelastic models

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(these include the orthotropic models, which, however, violate tensorial form invariance); (5) the total strain models (deformation theory of plasticity), also called nonlinear elastic models (Bažant and Tsubaki 1980; Cedolin et al. 1977; Gerstle et al. 1980; Gerstle 1981; Kotsovos and Newman 1978); (6) the continuum damage mechanics models (Janson and Hult 1977; Kachanov 1958; Krajcinovic and Fonseka 1981; Mazars 1981), which can be seen as a refinement of the total strain models; (7) the fracturing material of Dougill (1976); (8) the plastic-fracturing theory (Bažant and Kim 1979), which is a combination of the fracturing and plastic models; (9) the viscoplastic and endochronic models (Bažant 1974; Bažant and Bhat 1976; Bažant and Shieh 1980); and (10) the damage model of Ortiz (1985), in which all 21 anisotropic elastic moduli are treated as damage variables.

Despite significant initial success during the 1970s, however, the macroscopic phenomenologic approach seems to have run into a dead-end street. Probably only relatively minor further improvements can be expected. As the experimental data base in the literature is expanding, many aspects of the material response apparently cannot be modeled by a unified general theory in which the material parameters are the same for all types of loading.

A greater promise is offered by the second kind of models (i.e., those based on micromechanics of the inelastic phenomena in the material microstructure). These models include: (1) the classical slip theory of plasticity and its recent variants and extensions called the microplane models; and (2) particle simulation, which started as the distinct element method of Cundall for granular solids, and was recently developed for the fracturing of concrete as the interface element model (Zubelewicz and Bažant 1987). The latter models, in which a random simulation of the aggregate framework in the microstructure of concrete is used, pose enormous demands for computer time and storage when large structures or three-dimensional analysis is considered.

This study will focus on the microplane model. The original idea underlying this approach is due to G. I. Taylor (1938), who proposed in 1938 that the stress-strain relation be specified independently on various planes in the material, assuming that either the stresses on such a plane (presently called the microplane) are the resolved components of the macroscopic stress tensor, which is called the static constraint, or that the strain components on such plane are the resolved components of the macroscopic strain tensor, which is called the kinematic constraint. In practice, however, only the static constraint had been used until the recent development of the microplane model for tensile fracturing of concrete (Bažant and Gambarova 1984; Bažant 1984; Bažant and Oh 1985). The responses on the planes of various orientation were related to the macrolevel simply by superposition, while recently (Bažant and Gambarova 1984; Bažant 1984; Bažant and Oh 1985) the principle of virtual work has been used for that purpose, for both static and kinematic constraints. The static constraint formulation was developed initially for polycrystalline metals by Batdorf and Budianski in 1949 and is known as the slip theory of plasticity. This theory has long been viewed as the best available description of the plastic behavior of metals, especially for highly nonproportional loading paths with sharp corners. The static constraint approach was



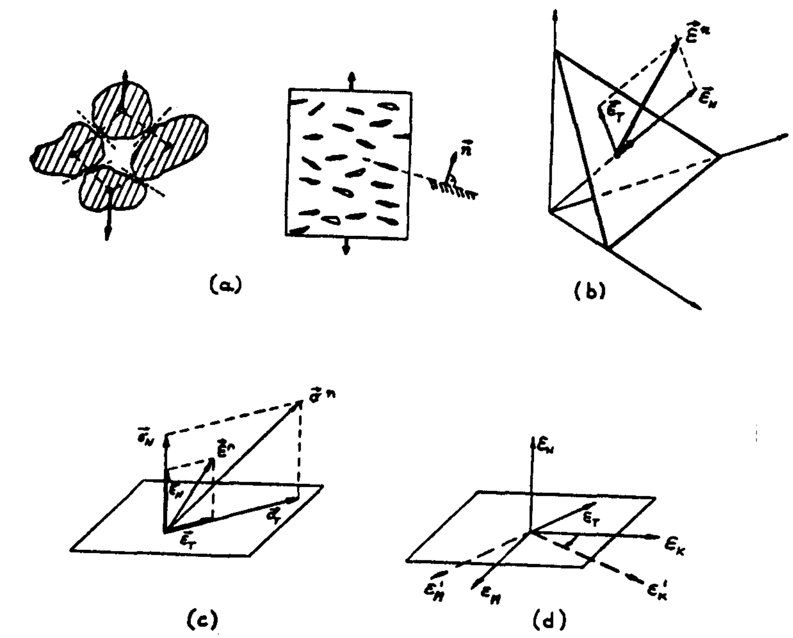


**FIG. 1. (*a*) Assumed Microstructure; (*b*) Strain Components on Microplane; (*c*) Stress and Strain on Microplane; and (*d*) Lateral Strains**

further developed for soils and rocks by Zienkiewicz and Pande (1977) and Xiong (1982), and more recently for creep of clay by Bažant and Kim (1986) and Bažant and Prat (1987).

In an effort to pursue this approach for tensile fracturing of concrete it was found (Bažant and Gambarova 1984; Bažant 1984; Bažant and Oh 1985) that, due to strain softening, the static constraint is impossible since it always leads to unstable response of snapback type. This problem has been overcome by adopting the kinematic constraint. Such a constraint seems to better describe the relation between the microscopic deformations and the macroscopic strain in an aggregate framework (Bažant and Gambarova 1984; Bažant 1984; Bažant and Oh 1985). The new term *microplane* had to be introduced (Bažant and Gambarova 1984; Bažant 1984; Bažant and Oh 1985) because the classical term *slip theory* gives the connotation of plasticity, which makes no sense as a description of concrete cracking. The prefix *micro* refers to the fact that the behavior is characterized on the weak planes of various orientations that are found in the microstructure, (e.g., the interaggregate contact planes) [Fig. 1(*a*)]. Keep in mind, though, that the microplane model does not represent a complete micromechanics formulation, which would have to reflect also spatial interactions between the weak planes, cracks and various defects on the microstructural level. The microplane model takes into account the microscopic interactions among various orientations but not those at distance. The latter ones are perhaps best described by nonlocal concepts.

From the mathematical viewpoint, the basic advantage of the microplane model is its conceptual simplicity. In contrast to the phenomenologic macroscopic models, one does not have to enforce any tensorial invariance requirements for the stress-strain relations, since the stress-strain relations refer to a single plane which cannot be rotated, by definition. The tensorial invariance requirements are automatically fulfilled postfacto (i.e., after the stress-strain relation is stated); this is done simply by integrating over the microplanes of all spatial orientations.

Due to this conceptual simplicity, one might hope to describe the material with fewer material parameters. This will be borne out by our analysis, which shows that only about four *independent material parameters* are needed to describe the general nonlinear triaxial behavior of any concrete with practically sufficient accuracy. By contrast, the existing phenomenologic macroscopic models for concrete, which are intended to describe a broad range of responses, all involve at least 15 unknown material parameters. When the endochronic theory, the first comprehensive nonlinear model for concrete, was presented in 1974 (Bažant 1974), the fact that it involved so many material parameters was unappealing and was justly criticized. Nevertheless, despite extensive efforts, neither this theory nor other general phenomenologic macroscopic models developed subsequently were able to succeed with fewer than about 15 material parameters. This attests to the presence of a fundamental limitation in the phenomenologic macroscopic approach.

The microplane model for concrete initially covered only tensile cracking (Bažant and Gambarova 1984; Bažant 1984; Bažant and Oh 1985), although it could also describe, by means of inclined tensile microcracks, the shear response of cracks or crack bands (Bažant and Gambarova 1984). The main objective of Part I of the present study is to develop a general microplane model that is applicable both to tension and compression, and both to brittle cracking and plastic response that characterize the damage in concrete. A further objective is to use as few material parameters as possible, and to cover a broader range of responses than the previous macroscopic phenomenologic models. Part II of this study will calibrate and verify this model by experimental data.

## Basic Hypotheses

**Hypothesis I**

The strains on any microplane are the resolved components of the macroscopic strain tensor $\varepsilon_{ij}$, which represents a tensorial kinematic constraint [Fig. 1(*b*)]. (However, we will also introduce an additional static constraint of the nontensorial, scalar type.)

**Hypothesis II**

The microplane resists not only normal strains $\varepsilon_N$, but also shear strains, $\varepsilon_{Ti}$, and their vector has the same direction as the vector of shear stresses $\sigma_{Ti}$ (i.e., $\sigma_{Ti} \sim \varepsilon_{Ti}$) [Fig. 1(*c*)].

**Hypothesis III**

The response on each microplane depends on the mean lateral strain $\varepsilon_L$, which is equivalent to a dependence on the volumetric strain $\varepsilon_V$ ($\varepsilon_V = \varepsilon_{kk}/3$).



**Hypothesis IV**

The stress-strain curves of each microplane are path independent as long as this microplane suffers no unloading. During unloading and reloading, the curves of the stress and strain differences from the state at the start of unloading are also path independent.

**Hypothesis V**

The volumetric, deviatoric and shear responses on each microplane are mutually independent.

As already mentioned, hypothesis I makes it possible to obtain stable response during strain softening while the classical hypothesis of static constraint would not. At the same time one should be aware of the fact that the kinematic constraint in general yields the stiffest possible response for the given stiffness properties of the material components. This is suggested by the fact, known from the theory of elastic composites, that the Voigt estimate for the elastic modulus, which corresponds to a kinematic constraint, represents an upper bound, while the Reuss estimate, which corresponds to a static constraint, represents a lower bound. The reality would no doubt be better described by a mixed kinematic-static constraint. In fact, we will need to introduce an additional static constraint, but we will be able to keep it nontensorial (scalar), which preserves simplicity. Due to the mixed constraint, our model will not be the stiffest possible.

According to hypothesis I, the strain vector on a microplane whose direction cosines are $n_i$ is $\varepsilon_j^n = \varepsilon_{jk}n_k$. The normal strain component and its vector then are

$$\varepsilon_N = n_j\varepsilon_j^n = n_jn_k\varepsilon_{jk} \qquad (1a)$$

$$\varepsilon_{N_i} = n_in_jn_k\varepsilon_{jk} \qquad (1b)$$

The Latin lowercase subscripts refer to cartesian coordinates $x_i$ ($i = 1, 2, 3$). Repeated Latin lower case subscripts indicate summation over 1, 2, 3. The magnitude of the strain vector on the microplane is $\|\varepsilon^n\| = \sqrt{\varepsilon_j^n\varepsilon_j^n} = \sqrt{n_i\varepsilon_{ji}n_k\varepsilon_{jk}}$. The vector of the shear (tangential) strain component is (Fig. 1*b*) $\varepsilon_T = \varepsilon^n - \varepsilon_N$. So the shear strain components and the shear strain magnitudes are:

$$\varepsilon_{T_i} = (\delta_{ij} - n_in_j)n_k\varepsilon_{jk} = \frac{1}{2}(n_j\delta_{ik} + n_k\delta_{ij} - 2n_in_jn_k)\varepsilon_{jk} \qquad (2a)$$

$$\varepsilon_T = \sqrt{\varepsilon_{T_i}\varepsilon_{T_i}} = \sqrt{n_k\varepsilon_{jm}n_m(\varepsilon_{jk} - n_in_j\varepsilon_{ik})} \qquad (2b)$$

in which $\delta_{ij}$ is the Kronecker's unit delta tensor. The purpose of the longer expression for $\varepsilon_{Ti}$ is to make the tensor that multiplies $\varepsilon_{jk}$ symmetric with respect to $j$ and $k$. The reason for doing this is that the antisymmetric part of this tensor has no effect on the value of $\varepsilon_{Ti}$ since $\varepsilon_{jk}$ is symmetric.

In the original microplane model for tensile cracking of concrete, simplicity was achieved by neglecting the shear stiffness of the microplanes and dealing only with normal strains $\varepsilon_N$. On the macroscopic level, shear stiffness was provided by resistance against compression in inclined directions, which was exploited in the crack band microplane model (Bažant and Gambarova 1984). However, this approach proved insufficient for the modeling of the response in compression, for various reasons. For example, simulation of the uniaxial compression tests with the original microplane model does yield a peak stress value; however, shortly into the strain-softening regime the system becomes unstable and exhibits snapback, which does not correspond to reality. After exploring a number of alternatives it has been concluded that the microplane model for general behavior must include not only the normal strain but also the shear strain on each microplane [see also Bažant (1984)]. It may be noted that the classical slip theory of plasticity involves only plastic shear response on each slip plane and ignores the normal response, which is the opposite of the original microplane model for tensile cracking.

Even with the inclusion of shear strains, however, a good description of the response in compression cannot be achieved, as learned in extensive numerical simulations. It appears that the response on a microplane is also affected by the mean normal strain $\varepsilon_L$, called the lateral strain, which influences the microplane from the planes that are normal to the microplane. This strain is defined as $\varepsilon_L = (\varepsilon_K + \varepsilon_M)/2$, where $\varepsilon_K$ and $\varepsilon_M$ are the normal strains in two orthogonal directions normal to that of $\varepsilon_N$; see Fig. 1(*d*). The value of $\varepsilon_L$ is invariant with regard to rotations of the directions K and M around the normal to the microplane. It appears that the lateral strain $\varepsilon_L$ considerably affects the brittleness of the response, although it does no work on the normal and shear stresses acting on the given microplane.

Instead of using the lateral strain $\varepsilon_L$, one may equivalently and more conveniently use the volumetric strain $\varepsilon_V$, which is expressed as $\varepsilon_V = (\varepsilon_N + \varepsilon_K + \varepsilon_M)/3$, and is a three-dimensional invariant. Defining the deviatoric normal strain as $\varepsilon_D = \varepsilon_N - \varepsilon_V$, one has $\varepsilon_D = \frac{2}{3}(\varepsilon_N - \varepsilon_L)$. Thus, hypothesis III is equivalent to the requirement that the volumetric and deviatoric normal strain components $\varepsilon_V$ and $\varepsilon_D$ should be used separately in the stress-strain relation for each microplane.

There is also another physical argument for introducing the shear strains on each microplane and for separating the volumetric and deviatoric normal strains. As is well known (Bažant and Oh 1985), the microplane model which exhibits only normal stiffness yields always the elastic Poisson's ratio $\nu = 0.25$ (in three dimensions; in two dimensions it is 1/3). Therefore, the artifice of a series coupling with a fictitious additional elastic element had to be used previously to adjust the Poisson's ratio to a physically correct value. If both the shear and normal strains are considered in a kinematically constrained microplane model but the deviatori and volumetric components are not separated, the Poisson's ratio i restricted to the limits $-1 \leq \nu \leq 0.25$. This range could describe the Poisson's ratio observed for concrete, but the fact that Poisson's ratio does not exceed 0.25 is suspicious on physical grounds, because the microplane model should in principle also work for other materials for which the Poisson's ratio might exceed 0.25. It has been discovered by Bažant that an arbitrary Poisson's ratio, satisfying the thermodynamic requirements $-1 \leq \nu \leq 0.5$, can be obtained if, in addition to the consideration of both the shear and normal strains, the normal strains are further split into the volumetric and deviatoric components. It may be also noted that the additional scalar static constraint that we introduce later will have no effect on the elastic response, and therefore none on the Poisson's ratio.



As a reasoning behind hypothesis IV, one may recall that the nonlinear triaxial response of concrete in the hardening regime (prepeak states) is nearly path independent for monotonic loading paths. This is evident from the success of the total strain theory (deformation theory) of Cedolin et al. (1977, 1983) [later adapted and restated by Kotsovos and Newman (1978)], as well as the model of Gerstle et al. (1980) and Gerstle (1981) and the comprehensive model of Bažant and Tsubaki (1980). From these studies it transpired that the path dependence of response becomes significant only if there is unloading or strain softening. Strain softening, however, always implies unloading of the material on planes of some orientations at the microstructural level. This suggests that the main reason for the path dependence of concrete, as well as other similar materials, should be the unloading.

The typical response of the microplane model combines loading on the microplanes of some orientations and unloading on the microplanes of other orientations. There are numerous possible combinations of such loading and unloading on planes of various orientations, which cannot be captured by the existing phenomenologic macroscopic models. Thus it becomes clear that even if the microplane model assumes path independence on each microplane, the overall material response can and does exhibit significant path dependence. This has been the reason for introducing hypothesis IV.

The consequence of this hypothesis is that for monotonic loading, the secant moduli for each microplane can be considered to be functions of the current strains and stresses only. We will assume the stress-strain relation for monotonic loading to be of the form

$$\sigma_V = C_V(\varepsilon_V, \varepsilon_D, \varepsilon_T)\varepsilon_V \quad (3a)$$

$$\sigma_D = C_D(\varepsilon_V, \varepsilon_D, \varepsilon_T)\varepsilon_D \quad (3b)$$

$$\sigma_T = C_T(\varepsilon_V, \varepsilon_D, \varepsilon_T, \boldsymbol{\sigma})\varepsilon_T \quad (3c)$$

in which $C_V$, $C_D$, and $C_T$ are the secant moduli, which depend on $\varepsilon_V$, $\varepsilon_D$, and $\varepsilon_T$, and $C_T$ also on an invariant of stress tensor $\boldsymbol{\sigma}$. Modeling experience, however, indicated that the volumetric, deviatoric and shear responses on each microplane can be assumed to be mutually independent, as stated in hypothesis V. Thus, the stress-strain relations for monotonic loading are assumed to be decoupled as follows:

$$\sigma_V = C_V(\varepsilon_V)\varepsilon_V \quad (4a)$$

$$\sigma_D = C_D(\varepsilon_D)\varepsilon_D \quad (4b)$$

$$\sigma_T = C_T(\varepsilon_T, \sigma_C)\varepsilon_T \quad (4c)$$

in which $C_V$ depends only on $\varepsilon_V$ and $C_D$ only on $\varepsilon_D$ [Fig. 2(*c*)]. Among strains, $C_T$ depends only on $\varepsilon_T$, but empirically it is found to also depend on the stress $\sigma_C = (\sigma_{II} + \sigma_{III})/2$ where $\sigma_{II}$ and $\sigma_{III}$ are the medium and minimum principal stresses. $\sigma_C$ is a macroscopic stress invariant, which we will call the confining stress. Test data indicate that $\sigma_C$ has a significant effect on the shear stiffness. The use of $\sigma_C$ in Eq. 4 means that the

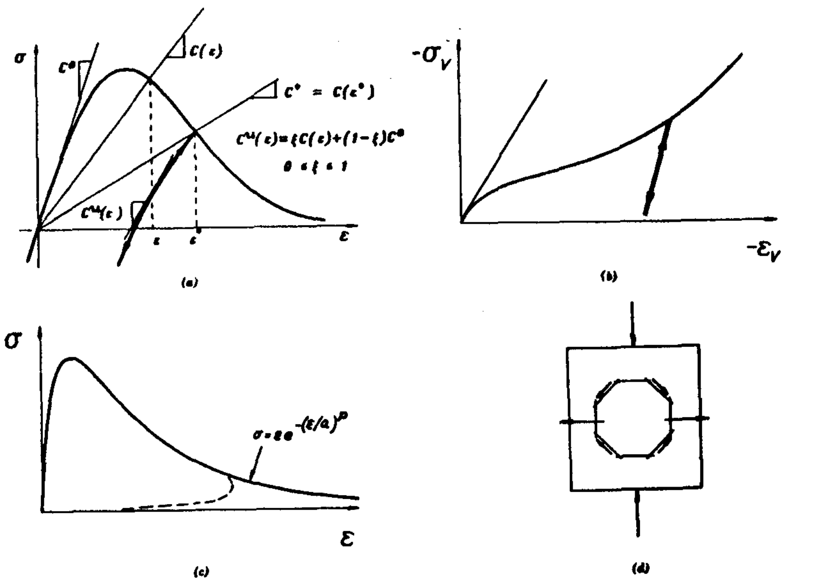


**FIG. 2. (*a*) Unloading criterion; (*b*) Volumetric Stress-Strain and Unloading; (*c*) Strain Softening on Microplane and Snapback; and (*d*) Lateral Strains Produced by Uniaxial Compression**

constraint of the microplanes is not solely kinematic but also static. The static constraint, however, is not tensorial but merely scalar. $\sigma_C$ appears only as a parameter, not as a basic variable, and does not affect the elastic part of response.

The stress-strain relation for shear on the microplane, $\sigma_T = C_T\,\varepsilon_T$, neglects any possible effect of the direction of the shear strain vector $\varepsilon_T$ on the components $\varepsilon_{Ti}$ within the microplane. Whether the direction of this vector remains constant or rotates during loading, the secant modulus depends only on the final state. Although one might suspect this to be an oversimplification, good agreement with test data has been obtained. Note also that we do not need to specify the shear stiffness separately for each component $\varepsilon_{Ti}$, but do so only for the magnitude $\varepsilon_T = \|\varepsilon_T\|$.

To handle unloading, one records in a step-by-step computation the maximum and minimum values of every strain component achieved so far (i.e., $\varepsilon_V^{max}$, $\varepsilon_V^{min}$, $\varepsilon_D^{max}$, $\varepsilon_D^{min}$, and $\varepsilon_T^{max}$ ($\varepsilon_T^{min} = 0$ always). These values are updated after each loading step. Hypothesis IV means that we need to specify for unloading or reloading a curve of a certain fixed shape to be followed by each strain component. After experimenting with various rules, the following simple rule, which is a slight modification of the rule introduced by Bažant and Chern (1985) and involves only one additional parameter, $\xi$, has been adopted:

$$d\sigma_V = C_V^u\,d\varepsilon_V \quad \text{where} \quad C_V^u = \xi C_V(\varepsilon_V) + (1-\xi)C_V^0 \qquad (5a)$$

$$d\sigma_D = C_D^u\,d\varepsilon_D \quad \text{where} \quad C_D^u = \xi C_D(\varepsilon_D) + (1-\xi)C_D^0 \qquad (5b)$$



$$d\sigma_T = C_T^u\,d\xi_T \quad \text{where} \quad C_T^u = \xi C_T(\varepsilon_T, \boldsymbol{\sigma}) + (1-\xi)C_T^0 \qquad (5c)$$

$C_V^u$, $C_D^u$, and $C_T^u$ represent the tangential moduli during unloading or reloading (i.e., for unloaded states). Eq. 5*a* is valid for unloading of the volumetric component at $\varepsilon_V > 0$ (volumetric tension); if $\varepsilon_V < 0$ (volumetric compression), then the unloading modulus is $C_V^u = kC_V^0$ where $k$ is the ratio of the unloading modulus to the initial elastic modulus of the macroscopic hydrostatic compression response [Fig. 2(*b*)]. In the data studies that follow, $k$ is not obtained from curve fitting (and thus it is not a parameter of the model), but from direct measurement of the slope change at unloading. If such test results are unavailable, one may use $k = 1$, which has been found acceptable for all data sets used in this study.

The expressions in Eq. 5 graphically mean that the tangential modulus for unloading or reloading is a weighted average of the initial elastic modulus $C_V^0$, $C_D^0$ or $C_T^0$ and the secant modulus for loading at the same strain [Fig. 2(*a*)]; the case of $\varepsilon_T$ is an exception to this rule if $\sigma_C$ varies. Parameter $\xi$ is assumed to be the same for all the strain components, although it could have different values for each of them.

The loading criterion is in a step-by-step computer program formulated as follows: If $\varepsilon_V\,\Delta\varepsilon_V \ge 0$ and $(\varepsilon_V - \varepsilon_V^{max})(\varepsilon_V - \varepsilon_V^{min}) \ge 0$: virgin loading (Eq. 4) is appropriate; otherwise, unloaded states occur (Eq. 5). The same criterion is introduced for $\varepsilon_D$ and $\varepsilon_T$. The increment of shear strain $\Delta\varepsilon_T$ is defined as $\Delta\varepsilon_T = \|\boldsymbol{\varepsilon}_T + \Delta\boldsymbol{\varepsilon}_T\| - \|\boldsymbol{\varepsilon}_T\|$, because otherwise $\varepsilon_T\Delta\varepsilon_T \ge 0$ always (e.g., if $\Delta\varepsilon_T = \|\Delta\,\boldsymbol{\varepsilon}_T\|$).

The foregoing simple rule means that if unloading after previous compression continues into tensile states or if unloading after previous tension continues into compressive states, the tensile or compression strength limits for the microplanes, $f_t^M$ and $f_c^M$, could be greatly exceeded. This could be grossly unrealistic. To prevent it from happening, one needs to impose for unloading of $\varepsilon_v$ after previous compression the condition that, approximately, $\sigma_N \le 0.75 f_t^M$, and for unloading after previous tension of $\varepsilon_D$ and $\varepsilon_V$ the condition that $\sigma_N \ge -0.75 f_c^M$. If this condition is violated, a more sophisticated unloading rule should be specified. If this is not done the computation must be aborted.

Lest it be thought that hypothesis V prevents the modeling of friction, several facts should be noted:

1. Even though the model has no shear friction on the microplane per se, there is a dependence of the normal strain $\sigma_N$ on the lateral strain $\varepsilon_L$. This represents a frictional phenomenon as explained later. The dependence of $C_T$ on $\boldsymbol{\sigma}$ is of course a frictional phenomenon too.

2. On the macroscopic level, the model exhibits friction due to the fact that a shear stress is manifested at inclinations ±45° as compressive and tensile stresses that are not equal in magnitude and have unequal responses [Fig. 2(*d*)]. This was amply demonstrated for the crack band microplane model (Bažant and Gambarova 1984), in which the microplanes were assumed to possess no shear stiffness, yet the model was shown to describe friction on the macrolevel very well (e.g., upon applying a pure shear stress on the crack band, the inclined compressive stresses produce shear and normal components which are not canceled by the effect of the tensile stresses of opposite inclination since these tensile stresses are

smaller or vanish). The inclined normal strains caused by these inclined compressive and tensile stresses also produce dilatancy of the crack band subjected to shear.

3. Despite the neglect of shear stiffness on the microplanes, the crack band microplane model (Bažant and Gambarova 1984) was shown capable to describe all the available test results for friction and dilatancy on cracks in concrete. The reason is that a rough crack in shear behaves in the same manner as a crack band in shear. The contacts of the asperities are inclined and transmit inclined compressive stresses across the crack, which are the source of friction.

4. If any friction interaction were included for the microplane, it would have to model only the remaining part of friction which is in excess of that already exhibited by interactions among the microplanes of various orientations.

5. At a sufficient resolution of the microstructure, friction generally disappears. For example, there is no friction between atoms, only central forces. Although this analogy cannot be taken literally, it does illustrate that friction is a macroscopic phenomenon, caused by interaction of forces and deformations of various orientations in the material. There is no compelling requirement for attributing macroscopically observed friction to a coupling between normal and shear components on a single plane in the material.

## Incremental Macroscopic Stress-Strain Relation

For incremental solutions with step-by-step loading, we need to differentiate Eq. 4, which yields (together with Eq. 5) the following incremental stress-strain relations for each microplane:

$$d\sigma_V = C_V^t \, d\varepsilon_V - d\sigma_V'' \tag{6a}$$

$$d\sigma_D = C_D^t \, d\varepsilon_D - d\sigma_D'' \tag{6b}$$

$$d\sigma_{T_r} = C_T^t \, d\varepsilon_{T_r} - d\sigma_{T_r}'' \tag{6c}$$

in which for virgin loading

$$d\sigma_V'' = -\varepsilon_V \, dC_V, \qquad C_V^t = C_V(\varepsilon_V) \tag{7a}$$

$$d\sigma_D'' = -\varepsilon_D \, dC_D, \qquad C_D^t = C_D(\varepsilon_D) \tag{7b}$$

$$d\sigma_{T_r}'' = -\varepsilon_{T_r} \, dC_T, \qquad C_T^t = C_T(\varepsilon_T, \boldsymbol{\sigma}) \tag{7c}$$

and for unloaded states:

$$d\sigma_V'' = 0 \qquad C_V^t = C_V^u \tag{8a}$$

$$d\sigma_D'' = 0 \qquad C_D^t = C_D^u \tag{8b}$$

$$d\sigma_{T_r}'' = 0 \qquad C_T^t = C_T^u \tag{8c}$$

$C_V^t$, $C_D^t$, and $C_T^t$ represent the incremental elastic moduli for the current loading step for the microplane, and $d\sigma_V''$, $d\sigma_D''$, and $d\sigma_{Ti}''$ represent the inelastic stress increments for the microplane, which occur only for virgin loading.

Following the same procedure as used in the original derivation of the microplane model (Bažant and Gambrova 1984; Bažant 1984; Bažant and Oh 1985; Bažant 1985), we use the principle of virtual work to approximately enforce the equivalence of forces on the microscale and macroscale:

$$\frac{4\pi}{3} d\sigma_{ij}\delta\varepsilon_{ij} = 2 \int_S (d\sigma_N \delta\varepsilon_N + d\sigma_{T_r}\delta\varepsilon_{T_r}) f(\mathbf{n}) \, dS \tag{9}$$

in which $\delta\varepsilon_{ij}$, $\delta\varepsilon_N$, and $\delta\varepsilon_{T_r}$ are small variations of the macroscopic strain and of the strain components on a microplane. The macroscopic work on the left-hand side is taken over the volume of a unit sphere, and the integral extends over the surface of a unit hemisphere, $S$. The factor 2 is used because the integrand values on diametrically opposite points of the sphere are equal. Function $f(\mathbf{n})$ is a weighting function of the normal direction $\mathbf{n}$ which in general can introduce anisotropy of the material in its initial state. For concrete, one can approximately assume isotropy, in which case one sets $f(\mathbf{n}) = 1$, but for some concretes (such as rolled dam concrete) consideration of initial anisotropy would be appropriate and would not add much to the complexity of analysis. (It is one advantage of the microplane model that analysis of inelastic anisotropic materials is not appreciably more difficult than isotropic ones.)

Substituting now $\delta\varepsilon_N = n_i n_j \delta\varepsilon_{ij}$, $\delta\varepsilon_{Tr} = \frac{1}{2}(n_i\delta_{rj} + n_j\delta_{ri} - 2n_i n_j n_r)\delta\varepsilon_{ij}$, according to Eqs. 1 and 2, and setting $d\sigma_N = d\sigma_V + d\sigma_D$, we get

$$\frac{2\pi}{3} d\sigma_{ij}\delta\varepsilon_{ij} = \int_S \left[ n_i n_j (d\sigma_V + d\sigma_D) + \frac{1}{2}(n_i\delta_{rj} + n_j\delta_{ri} - 2n_i n_j n_r) \, d\sigma_{T_r} \right] f(\mathbf{n}) \, dS\delta\varepsilon_{ij} \tag{10}$$

This variational equation must hold for any variations $\delta\varepsilon_{ij}$; consequently

$$d\sigma_{ij} = \frac{3}{2\pi} \int_S \left[ n_i n_j (d\sigma_V + d\sigma_D) + \frac{1}{2}(n_i\delta_{rj} + n_j\delta_{ri} - 2n_i n_j n_r) \, d\sigma_{T_r} \right] f(\mathbf{n}) \, dS$$

$$= \frac{3}{2\pi} \int_S \left[ n_i n_j (C_V^t \, d\varepsilon_V - d\sigma_V'' + C_D^t \, d\varepsilon_D - d\sigma_D'') + \frac{1}{2}(n_i\delta_{rj} + n_j\delta_{ri} \right.$$

$$\left. - 2n_i n_j n_r)(C_T^t \, d\varepsilon_{T_r} - d\sigma_{T_r}'') \right] f(\mathbf{n}) \, dS \tag{11}$$

in which we have substituted Eq. 6. According to Eqs. 1 and 2, we may further substitute $d\varepsilon_V = \delta_{km} d\varepsilon_{km}/3$, $d\varepsilon_D = n_k n_m d\varepsilon_{km} - d\varepsilon_V$, $d\varepsilon_{Tr} = \frac{1}{2}(n_m\delta_{rk} + n_k\delta_{rm} - 2n_r n_k n_m) d\varepsilon_{km}$. Thus, we finally obtain a macroscopic stress-strain relation of the form

$$d\sigma_{ij} = C_{ijkm} \, d\varepsilon_{km} - d\sigma_{ij}'' \tag{12}$$

in which $C_{ijkm}$ denotes the incremental stiffness tensor (elastic moduli tensor):

$$C_{ijkm} = \frac{3}{2\pi}\int_S \left[(C'_D - C'_T)n_in_jn_kn_m + \frac{1}{3}(C'_V - C'_D)n_in_j\delta_{km}\right.$$

$$\left. + \frac{1}{4}C'_T(n_in_k\delta_{jm} + n_in_m\delta_{jk} + n_jn_k\delta_{im} + n_jn_m\delta_{ik})\right] f(\mathbf{n})\ dS \qquad (13)$$

and $d\sigma''_{ij}$ denotes the inelastic stress increments:

$$d\sigma''_{ij} = \frac{3}{2\pi}\int_S \left[n_in_j\ d\sigma''_N + \frac{1}{2}(n_i\delta_{rj} + n_j\delta_{ri} - 2n_in_jn_r)\ d\sigma''_{T_r}\right] f(\mathbf{n})\ dS \qquad (14)$$

## Nonsymmetry of Incremental Elastic Moduli Tensor and Isotropy

Upon carrying an interchange of indices $i$, $j$ and $k$, $m$ we notice a surprising fact: the incremental elastic moduli tensor is nonsymmetric unless $C'_V = C'_D$, which, however, cannot be expected to occur in general. This nonsymmetry, which represents a major difference from the previous constitutive models, can be understood from the following relations:

$$\sigma_N = \sigma_V + \sigma_D = C'_V\varepsilon_V + C'_D\varepsilon_D = C'_V\varepsilon_V + C'_D(\varepsilon_N - \varepsilon_V)$$

$$= C'_D\varepsilon_N + (C'_V - C'_D)\varepsilon_V = C'_V(\varepsilon_N - \varepsilon_D) + C'_D\varepsilon_D$$

$$= C'_V\varepsilon_N + (C'_D - C'_V)\varepsilon_D = \frac{1}{3}(C'_V + 2C'_D)\varepsilon_N + \frac{2}{3}(C'_V - C'_D)\varepsilon_L \qquad (15)$$

Since $C'_V = C'_D$ is required for symmetry, we conclude that symmetry is obtained if $\sigma_N$ depends only on the normal strain $\varepsilon_N$ and does not separately depend on the volumetric strain $\varepsilon_V$ and the deviatoric strain $\varepsilon_D$, and thus also the lateral strain $\varepsilon_L$. The lateral strain $\varepsilon_L$ has the special property that none of the microplane stresses (neither $\sigma_N$ nor $\sigma_T$) does any work on this strain. The phenomenon that a nonworking variable in this case $\varepsilon_L$, influences the response is in the generalized sense called friction (Bažant 1980). Frictional phenomena are undeniably real, they do occur. So the nonsymmetry of $C_{ijkm}$ cannot be objected on physical grounds, although it violates Drucker's postulate and may cause instabilities and programming inconvenience.

Friction is of course impossible on physical grounds when the response is purely elastic, and in that case $C_{ijkm}$ must be fully symmetric even if $C'_V \neq C'_D$. Moreover, $C_{ijkm}$ must exhibit isotropy (if $f(n) = 1$). That these two properties are indeed true we demonstrate now. Using spherical coordinates $\theta$ and $\varphi$, we may substitute in Eq. 13:

$$n_1 = \cos\theta \qquad (16a)$$

$$n_2 = \sin\theta\cos\varphi \qquad (16b)$$

$$n_3 = \sin\theta\sin\varphi \qquad (16c)$$

$$dS = \sin\theta\ d\theta\ d\varphi \qquad (16d)$$

We also substitute the initial elastic moduli $C'_V = C^0_V$, $C'_D = C^0_D$, and $C'_T = C^0_T$. Since $C^0_V$, $C^0_D$ and $C^0_T$ are independent of $\theta$ and $\varphi$, we may integrate explicitly for all the possible combinations of subscripts $i$, $j$, $k$, $m$. For isotropy, we have by definition $f(\mathbf{n}) = 1$. In this manner we obtain the following elastic relation:

$$\begin{Bmatrix} d\sigma_{11} \\ d\sigma_{22} \\ d\sigma_{33} \\ d\sigma_{12} \\ d\sigma_{23} \\ d\sigma_{31} \end{Bmatrix} = \begin{bmatrix} A & B & B & 0 & 0 & 0 \\ B & A & B & 0 & 0 & 0 \\ B & B & A & 0 & 0 & 0 \\ 0 & 0 & 0 & C & 0 & 0 \\ 0 & 0 & 0 & 0 & C & 0 \\ 0 & 0 & 0 & 0 & 0 & C \end{bmatrix} \begin{Bmatrix} d\varepsilon_{11} \\ d\varepsilon_{22} \\ d\varepsilon_{33} \\ d\varepsilon_{12} \\ d\varepsilon_{23} \\ d\varepsilon_{31} \end{Bmatrix} \qquad (17)$$

in which

$$A = \frac{1}{15}(5C^0_V + 4C^0_D + 6C^0_T) \qquad (18a)$$

$$B = \frac{1}{15}(5C^0_V - 2C^0_D - 3C^0_T) \qquad (18b)$$

$$C = \frac{1}{5}(2C^0_D + 3C^0_T) \qquad (18c$$

The matrix in Eq. 17 must coincide with the well-known elastic moduli matrix for isotropic elastic materials, and so we must have

$$A = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \qquad (19a)$$

$$B = \frac{E\nu}{(1+\nu)(1-2\nu)} \qquad (19b)$$

$$C = 2G = \frac{E}{1+\nu} \qquad (19c)$$

in which $E$ = elastic Young's modulus; $G$ = elastic shear modulus, and $\nu$ = Poisson ratio. Equating the expressions for $A$ and $B$ in Eqs. 18 and 19, and solving for $\nu$ and $E$, we obtain the relations

$$\nu = \frac{5C^0_V - 2C^0_D - 3C^0_T}{10C^0_V + 2C^0_D + 3C^0_T} \qquad (20a)$$

$$E = (1 - 2\nu)C^0_V \qquad (20b)$$

It may be verified that if one equates the expressions for $A$ and $C$ in Eqs. 18 and 19 and solves again for $\nu$ and $E$, one gets the same result. Note also that even if $C^0_V \neq C^0_D$, the integration over $\theta$ and $\varphi$ yields all the zeros in Eq. 17 that are required for material isotropy and symmetry.

The only thermodynamic restriction we have on $C^0_V$ and $C^0_D$ is that they must not be negative. From Eq. 20 one can then obtain any Poisson ratio in the range

$$-1 \le \nu \le 0.5 \qquad (21)$$

that coincides with the well-known thermodynamic restriction on $\nu$.

In fitting of the test data, it is convenient to consider $E$ and $\nu$ as the basic parameters and choose the ratio $\eta_0 = C_D^0/C_V^0$. Then one obtains $C_V^0 = E/(1 - 2\nu)$ and, by solving Eq. 20,

$$C_T^0 = \frac{1}{3}\left[\frac{5(1-2\nu)}{1+\nu} - 2\eta_0\right]C_V^0 \quad (22a)$$

$$C_D^0 = \eta_0 C_V^0 \quad (22b)$$

Choosing various values of $\eta_0$, it was found that good fits are obtained approximately for $0.25 \leq \eta_0 \leq 1$.

For the extreme case, $\eta_0 = 1$ (i.e., $C_V^0 = C_D^0$). This does not imply that $C_V^t = C_D^t$ for further loading, but does imply that $C_V^t$ is rather close to $C_D^t$ for the initial inelastic response in the hardening range. Consequently, tensor $C_{ijkm}$ is nearly symmetric for this initial range. Beyond the initial range, however, $C_V^t$ and $C_D^t$ may become significantly different (and tensor $C_{ijkm}$ highly nonsymmetric) for two reasons: (1) $C_V^t$ and $C_D^t$ depend on $\varepsilon_V$ and $\varepsilon_D$, respectively, whose values can be rather different; and (2) they change greatly in the case of unloading (either $\varepsilon_V$ or $\varepsilon_D$ may be loading and the other unloading on the same microplane). Note also that unloading occurs on some microplanes even during monotonic loading (e.g., for the uniaxial compression test).

It has been tried to determine whether replacement of tensor $C_{ijkm}$ with its symmetric part could yield good fits of test data. It cannot (especially not the observed volume changes and lateral strains).

## Numerical Implementation

To make computations efficient, the tensorial expressions $n_i n_j n_k n_m$, $\frac{1}{4}C_T^t(n_i n_k \delta_{jm} + n_i n_m \delta_{jk} + n_j n_k \delta_{im} + n_j n_m \delta_{ik})$, $n_i n_j \delta_{km}$, $n_i n_j$, and $\frac{1}{2}(n_i \delta_{rj} + n_j \delta_{ri} - 2n_i n_j n_r)$ should be calculated in advance of finite element analysis and stored in arrays in the computer's memory. The integrals over the unit hemisphere in Eqs. 13 and 14 need to be evaluated in finite element analysis in every loading step, every iteration of the step, and every integration point of every finite element. Therefore, evaluation of these integrals must be as efficient as possible. This evaluation is done by numerical integration using some suitable Gaussian-type integration formula for the surface of the hemisphere. Many such formulas are listed in Stroud's book (Stroud 1971), and some further formulas which are more efficient under certain circumstances have been derived by Bažant and Oh (1985, 1986). These formulas generally approximate the integrals in the form

$$\int_S F\, dS \approx 4\pi \sum_{\alpha=1}^{N} w_\alpha F_\alpha \quad (23)$$

in which subscript $\alpha$ refers to a certain discrete set of directions in space, and $w_\alpha$ are the weights (numerical integration coefficients) for these directions (i.e., for points on the unit hemisphere). In the present calculations, Stroud's formula with 28 points per hemisphere (Bažant and Gambarova 1984; Bažant and Oh 1985; Stroud 1971) has been used. For symmetric stress states the number of integration points can be reduced



because the values of $w_\alpha F_\alpha$ are equal for some points (e.g., for axisymmetric stress states, the summation in Eq. 23 involves only five terms). Stroud's 28-point formula is of the eleventh degree (i.e., it integrates exactly on the surface of a sphere all polynomials up to the eleventh degree). A lesser but practically still sufficient accuracy is obtained by Bažant and Oh's 21-point formula (Bažant and Gambarova 1984; Bažant and Oh 1985), and for relatively crude analysis one could use Collatz's formula with only 16 points. A smaller number of points gives unacceptable results, especially in the strain-softening range (Bažant and Gambarova; Bažant and Oh 1985).

## Conclusions

1. The microplane model that was previously formulated for tensile cracking of materials such as concrete is extended to general nonlinear triaxial behavior including compression and shear loadings. This model represents a counterpart of the slip theory of plasticity in which the structure is constrained kinematically rather than statically and the normal inelastic strains are taken into account. As confirmed in Part II, which follows, the model is capable of a good description of a broad range of the existing test data for nonlinear triaxial behavior of concrete. It seems to describe realistically the dilatancy and friction; the brittle-ductile transition at increasing hydrostatic pressure; various degrees of path dependence; the stiffening response in hydrostatic pressure tests; the extended strain softening in tension as well as in compression; and other typical features.
2. The model achieves conceptual simplicity by specifying the constitutive properties independently on planes of various orientations in the material, which are called the microplanes and characterize principally the behavior of the weak planes within the microstructural framework. On each microplane there are only few stress and strain components and there are no tensorial invariance requirements to satisfy. The tensorial invariance requirements are satisfied automatically on the macroscopic level by integrating over all the spatial directions.
3. For monotonic loading, the stress-strain relation for each microplane is assumed to be path independent and all the path dependence that is observed macroscopically is a consequence of unloading which happens selectively on some microplanes.
4. On each microplane one distinguishes the volumetric and deviatoric normal strains and the shear strains, with different elastic constants for each. This makes it possible to match with the microplane system any Poisson ratio. The previous microplane models, by contrast, had been characterized by only a certain value of the Poisson ratio, and matching of the experimentally observed Poisson ratio required the artifice of coupling in series or in parallel with the microplane system an additional fictitious elastic element.
5. The response for each microplane is assumed to depend on the volumetric strain of the material, which implies dependence on the normal strain in the lateral direction. Since this strain does no work, a frictional aspect is introduced into the response, causing the incremental elastic moduli tensor in the nonlinear range to be nonsymmetric. This tensor is fully populated, (i.e., none of its components can in general be prescribed

as zero). The nonsymmetry of this tensor appears to be important for the modeling of friction, dilatancy, and pressure sensitivity.

6. To reflect internal friction, the inelastic part of shear stiffness on the microplane is considered to depend also on the macroscopic confining stress (defined as the average of the minimum and medium principal stresses). In consequence, the micro-macro constraint is not purely kinematic but mixed (kinematic-static). This causes the response to be less stiff than for a purely kinematic constraint. While the basic kinematic constraint is tensorial, the additional static constraint is scalar.

7. The model permits extensions to anisotropic materials without any additional complexity.

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## Appendix. References

Batdorf, S. B., and Budianski, B. (1949). "A mathematical theory of plasticity based on the concept of slip." *Technical Note No. 1871*, Nat. Advisory Committee for Aeronautics, Washington, D.C.

Bažant, Z. P. (1974). "A new approach to inelasticity and failure of concrete, sand and rock: Endochronic theory." *Proc. Society of Engineering Science 11th Annual Meeting*, Duke Univ., Durham, N.C., 158–159.

Bažant, Z. P. (1980). "Work inequalities for plastic fracturing materials." *Int. J. Solids Struct.*, 16, 873–901.

Bažant, Z. P. (1984). "Microplane model for strain-controlled inelastic behavior." *Mechanics of engineering materials*, C. S. Desai and R. H. Gallagher, eds., John Wiley and Sons, Inc., New York, N.Y., 45–59.

Bažant, Z. P. (1985). *Mechanics of geomaterials: Rock, concrete, soils*. John Wiley and Sons, Inc., New York, N.Y.

Bažant, Z. P., and Bhat, P. D. (1976). "Endochronic theory for concrete." *J. Engrg. Mech.*, ASCE, 105(3), 407–428.

Bažant, Z. P., and Chern, J.-C. (1985). "Strain-softening with creep and exponential algorithm." *J. Engrg. Mech.*, ASCE, 111(3), 381–390.

Bažant, Z. P., and Gambarova, P. G. (1984). "Crack shear in concrete: Crack band microplane model." *J. Struct. Engrg.*, ASCE, 110(9), 2015–2035.

Bažant, Z. P., and Kim, S. (1979). "Plastic-fracturing theory for concrete." *J. Engrg. Mech.*, ASCE, 105(3), 407–428.

Bažant, Z. P., and Kim, J.-K. (1986). "Creep of anisotropic clay: Microplane model." *J. Geotech. Engrg.*, ASCE, 112(4), 458–475.

Bažant, Z. P., and Oh, B. H. (1983). "Crack band theory for fracture of concrete." *Matériaux et Constructions*, RILEM, Paris, 16(93), 155–177.

Bažant, Z. P., and Oh, B. H. (1985). "Microplane model for progressive fracture of concrete and rock." *J. Engrg. Mech.*, ASCE, 111(4), 559–582.

Bažant, Z. P., and Oh, B. H. (1986). "Efficient numerical integration on the surface of a sphere." *Zeitschrift für Angewandte Mathematik und Mechanik* 66(1), 37–49.

Bažant, Z. P., and Prat, P. C. (1987). "Creep of anisotropic clay: New microplane model." *J. Engrg. Mech.*, ASCE, 113(7), 1050–1064.

Bažant, Z. P., and Shieh, C.-L. (1980). "Hysteretic fracturing endochronic theory for concrete." *J. Engrg. Mech.*, ASCE, 106, 929–950.



Bažant, Z. P., and Tsubaki, T. (1980). "Total strain theory and path dependence of concrete." *J. Engrg. Mech.*, ASCE, 106(6), 1151–1173.

Cedolin, L., Crutzen, Y. R. J., and dei Poli, S. (1977). "Triaxial stress-strain relationship for concrete." *J. Engrg. Mech.*, ASCE, 103(4), 423–439.

Cedolin, L., dei Poli, S., and Iori, L. (1983). "Experimental determination of the stress-strain curve and fracture of concrete in tension." *Proc. Int. Conference on Constitutive Laws for Engrg. Materials*, Univ. of Arizona.

Chen, C. T., and Chen, W. F. (1975). "Concrete in biaxial cyclic compression." *J. Struct. Engrg.*, ASCE, 101(4), 461–476.

Dougill, J. W. (1976). "On stable progressively fracturing solids." *J. Appl. Math. Phys.* (ZAMP) 27, 423–436.

Gerstle, K. H. (1981). "Simple formulation of biaxial concrete behavior." *J. Am. Concr. Inst.*, 78(1), 62–68.

Gerstle, K. H., et al. (1980). "Behavior of concrete under multiaxial stress states." *J. Engrg. Mech.*, ASCE, 106(6), 1383–1403.

Gerstle, K. H. (1981). "Simple formulation of triaxial concrete behavior." *J. Am. Concr. Inst.* 78(9), 382–387.

Janson, J., and Hult, J. (1977). "Fracture mechanics and damage mechanics, a combined approach." *J. Mec. Theor. Appl.* 1(1), 69–84.

Kachanov, L. M. (1958). "Time of rupture process under creep conditions." Izv. Akad. Nauk SSR, Otd. Tkh. Nauk 8, 26–31.

Kotsovos, M. D., and Newman, J. B. (1978). "Generalized stress-strain relations for concrete." *J. Engrg. Mech.*, ASCE, 104(4), 845–856.

Krajcinovic, D., and Fonseka, G. U. (1981). "The continuous damage theory of brittle materials, part 1: General theory." *J. Appl. Mech. Trans.*, 48, 809–815.

Lin, F. B. et. al. (1987). "Concrete model with normality and sequential identification." *Comput. Struct.* 26(6), 1011–1026.

Mazars, J. (1981). "Mechanical damage and fracture of concrete structures." *Advances in Fracture Research, Proc. 5th International Conference on Fracture*, 4, 1499–1506.

Ortiz, M. A. (1985). "A constitutive theory for the inelastic behavior of concrete." *Mech. Mater.* 4, 67–93.

Pande, G. H., and Xiong, W. (1982). "An improved multi-laminate model of jointed rock masses." *Proc. 1st Int. Symposium on Numerical Models in Geomechanics* 218–226.

Stroud, A. H. (1971). Approximate calculation of multiple integrals. Prentice-Hall, Inc., Englewood Cliffs, N.J.

Sture, S., and Ko, H. Y. (1978). "Strain-softening of brittle geologic materials." *Int. J. Numer. Anal. Methods Geomech.*, 2, 237–253.

Takahashi, Y., and Marchertas, A. H. (1985). "A simple elasto-plastic constitutive model of concrete." *Proc. 4th Int. Conference on Struct. Safety and Reliability*,

Taylor, G. I. (1938). "Plastic strain in metals." *J. Inst. Metals* 62, 307–324.

Willam, K. J., and Warnke, E. P. (1974). "Constitutive model for the triaxial behavior of concrete." *IABSE Seminar on Concrete Structures Subjected to Triaxial Stresses*, ISMES, Bergamo, Italy, 1–30.

Yang, B.-L., Dafalias, Y. F., and Herrmann, L. R. (1985). "A bounding surface plasticity model for concrete." *J. Engrg. Mech.*, ASCE, 111(3), 359–380.

Zienkiewicz, O. C., and Pande, G. N. (1977). "Time-dependent multilaminate model of rocks—A numerical study of deformation and failure of rock masses." *Int. J. Numer. Anal. Methods Geomech.*, 1, 219–247.

Zubelewicz, A., and Bažant, Z. P. (1987). "Constitutive model with rotating active plane and true stress." *J. Eng. Mech.*, ASCE, 113(3), 398–416.

Zubelewicz, A., and Bažant, Z. P. (1987). "Interface element modeling of fracture in aggregate composites." *J. Engrg. Mech.*, ASCE, 113(11), 1619–1630.