Mathematical Modeling of Creep and Shrinkage of Concrete
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Chapter 3

Creep Analysis of Structures†

3.1 INTRODUCTION

Creep and shrinkage have important effects on the behaviour of concrete structures. The deflections due to these long-term effects can be large, normally significantly larger than the short-time or elastic deflections, and the designer must make sure that the deflections are tolerable. In redundant structures creep and shrinkage cause redistributions of internal forces. These redistributions are sometimes favourable since they tend to relax the maximum stresses produced by enforced displacements or shrinkage and temperature changes, but sometimes harmful. The stresses produced by differences in creep or shrinkage among various parts of the structure, or due to a change of the structural system during construction, can cause deleterious cracking, accompanied by degradation of structural stiffness. This may further facilitate ingress of water and promote corrosion of reinforcement, which may cause spalling of concrete and ultimately a loss of serviceability of the structure.

By altering the long-time stress state, creep and shrinkage indirectly causes a change in the stress maxima for superimposed live loads. Due to the lack of plastic response (limited ductility), creep and shrinkage may exert in this manner a significant influence on the brittle failures of concrete structures. Thus, creep and shrinkage may alter the safety margin against the collapse of a structure under short-time overloads. In slender or thin structures, creep also causes a slow long-time growth of buckling deflections. Consequently, the critical loads for long-time instability may be much less than the elastic critical loads.

The purpose of this chapter is to review the methods of linear as well as non-linear analysis of structures for the effects of creep and shrinkage and to describe the typical effects of various types of structures. The finite element analysis will be omitted from the present chapter except in so far as it directly relates to the subjects to be dealt with. This field has become so broad that it requires a separate treatment which is relegated to the next chapter.

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3.2 LINEAR METHODS

3.2.1 Basic assumptions and code recommendations

Although many strides have recently been made towards non-linear creep analysis of concrete structures, nearly all practical applications still rely on the linearity assumption, which is sanctioned by the current design code recommendations. For stresses that are constant in time, the linearity means that the strain $\varepsilon(t)$ at any time t is proportional to the stress σ and that the creep response is totally characterized by the compliance function $J(t,t')$ (see Eq. 2.2). For time-variable stresses, the linearity assumption implies the principle of superposition, which may be expressed by the history integrals in Eqs (2.4)–(2.7), and for multi-axial stress Eq. (2.9). The current codes define the creep properties usually in terms of the creep coefficient $\phi(t,t')$ (Eq. 2.3). This type of definition of creep properties, however, is prone to confusion, since it tempts the analyst to incorrectly combine $\phi(t,t')$ with an elastic modulus of concrete defined in an inconsistent manner (see Section 2.5.1).

The conditions of acceptability of the linearity assumption (see Section 2.2.2) fortunately include much of the structural behaviour under service loads, except for the errors which are due to simultaneous moisture migration and cracking and are often not insignificant.

The mathematical formulae for the creep coefficient and the method for structural analysis are given in the current code recommendations such as those of the American Concrete Institute (ACI) (1983) or the CEB–FIB Model Code (1978). These formulations, which rely on traditional concepts, have the advantage of simplicity but involve a large degree of uncertainty. The 95 per cent confidence limits for the deviations from the code predictions are on the order of $\pm$ 80 per cent of the mean prediction of the creep strain. This magnitude of error is not acceptable for design, and it may be one of the reasons why, in so many instances, concrete structures suffer excessive deflections or harmful cracking and spalling well before the end of their design lifetimes.

Presently, the most comprehensive and accurate prediction model for creep and shrinkage is the BP model (Bažant and Panula, 1978) which is to a much larger extent based on physical considerations than the ACI and CEB–FIP models. This model is, admittedly, considerably more tedious to implement, but the 95 per cent confidence limits are under 30 per cent. Furthermore, by using formulae that are better justified physically, this model offers the possibility of a significant further reduction of the prediction errors by means of calibration with short-time measurements.

Even though the time that the analyst needs to implement the BP model is considerably longer than that for the current code recommendations (several hours instead of a fraction of an hour for the ACI model), the effort spent in implementing this model is still far smaller than the effort needed to carry out the structural analysis. Ideally, more time should be devoted to the determination of the creep and shrinkage properties of the material than to structural analysis, because the former task involves a larger error. This error is in fact much larger than the accuracy gain due to the use of a computer.

3.2.2 Numerical methods based on hereditary integrals

The reason for assuming linearity is the applicability of the principle of superposition—in mathematics due to Boltzmann (1876) and Volterra (1909, 1913, 1959 see Chapter 2), and in concrete creep theory due to Maslov (1941) and McHenry (1943). According to this principle, the current strain $\varepsilon(t)$ is obtained as a hereditary integral over the previous stress history $\sigma(t)$ (Eq. 2.4), and conversely for $\sigma(t)$ expressed in terms of $\varepsilon(t)$ (Eq. 2.7). The inverse relation involves the relaxation function $R(t,t')$, which may be obtained for the purposes of design from a simple one-line formula (Eq. 2.8) due to Bažant and Kim (1979a). The multiaxial generalization of these integral-type stress–strain relations is readily obtained by assuming material isotropy; see Eqs (2.9–2.10).

Numerical analysis of structural response can be carried out in time steps $\Delta t = t_i - t_{i+1}$ $(i = 1, 2, 3, \ldots)$. The integral in the stress–strain relation is replaced by a constitutive law which yields a system of linear algebraic equations to be solved at each time step. The unknowns in these equations may be the redundant internal forces, or the displacements of structural members, or the nodal displacements of a finite element subdivision, depending on which method of spatial analysis is used. The integral in the stress–strain relation may be approximated by a sum with various degrees of accuracy. The first applications came with the advent of computers during the 1960s and were motivated chiefly, but not exclusively, by the design of nuclear reactor vessels; see, e.g. the works of Rashid (1972), Bresler and Selna (1964), Selna (1967, 1969), Cederberg and David (1969), Prokopovich (1963), Ghali *et al.* (1967) and others, in which algorithms of first-order accuracy were used. Ghali *et al.*, for example, approximated the integral in Eq. (2.4) by the sum $\sum_j J(t_{i-1/2}, t_{j-1/2})\Delta\sigma_j$ in which, the subscript $i - \frac{1}{2}$ refers to the middle of the time step and $\Delta\sigma_j = \sigma_j - \sigma_{j-1}$. The first-order algorithm was also used by Tadros *et al.* (1977a, b).

The algorithm which is now in prevalent use is that of Bažant (1972) (also used by Bažant and Najjar, 1973). It is based on approximating the integral in Eq. (2.5) or Eq. (2.9) by the trapezoidal rule, which leads to the incremental quasi-elastic stress–strain relations in Eqs (2.30)–(2.32). This algorithm is of second-order accuracy. The same accuracy may be achieved using the relaxation function, in which case the integral in Eq. (2.7) is approximated according to the trapezoidal rule. A general-purpose computer program based on Bažant's second-order algorithm was developed by Huet (1980). A third-order method, in which the error is proportional to $(\Delta\sigma)^3$, was formulated and used by Schade (1977); in his work the integral in Eq. (2.4) or Eq. (2.9) was approximated according to Simpson's rule.

Instead of the integral based on the compliance function $J(t,t')$, an integral based on the impulse memory function $L(t,t')$ (Eq. 2.5) is also possible. However, this approach is no longer used since it has been recognized to be numerically inefficient and, more importantly, to preclude an increase of the time step beyond a certain limit no matter how slow is the variation of stress and strain.

In the early work, the linear algebraic equations obtained by replacing the integral with a finite sum were formulated in terms of the structural unknowns, such as the values of the redundants or the nodal displacements at subsequent discrete times. However, as is now the general practice, it is simpler if the approximation by a sum is used to obtain an incremental quasi-elastic stress–strain relation with inelastic strain, and the structural analysis is then carried out in the same manner as for elastic materials, reducing the problem to a sequence of elasticity problems with prescribed inelastic strains (Bažant, 1966a, b, and Section 2.2.4). For the second-order or higher-order methods, the incremental elastic stiffness matrix of the structure is then different in each time step. However, certain first-order formulations have the advantage that the incremental elastic stiffness matrix changes from one time step to the next in proportion to the elastic modulus $E(t)$, making it unnecessary to generate and assemble a new structural stiffness matrix at each time step. This advantage is lost, however, if the structure is non-homogeneous, e.g. due to differences in age, thickness, humidity conditions, etc. For large structural systems, the advantage of considerably higher accuracy of the second-order method is important because the creep analysis based on the integral-type creep law is very demanding for computer time and storage. One can greatly reduce the number of time steps by using the second-order instead of the first-order approximation.

A matrix version of Bažant's second-order algorithm, which is however useful only for simple structures with few unknowns, was introduced by Madsen (1979) (for application in probabilistic analysis see Madsen and Bažant, 1982). In this version, instead of a step-by-step procedure the values of all unknowns at all time steps are solved simultaneously. The incremental quasi-elastic relations in Eq. (2.30) are written as one matrix relation $\boldsymbol{\varepsilon} = \mathbf{J}\,\Delta\boldsymbol{\sigma} + \boldsymbol{\varepsilon}^0$ in which $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)^{\mathrm{T}}$ = a column matrix grouping the values of strains ε at all discrete times $t_1, t_2, \ldots, t_n$; $\Delta\boldsymbol{\sigma}$, $\boldsymbol{\varepsilon}^0$ = similar column matrices grouping the stress increments and the inelastic strains at all discrete times, and $\mathbf{J} = n \times n$ square matrix. Introducing the relation $\boldsymbol{\sigma} = \mathbf{L}\,\Delta\boldsymbol{\sigma}$ one obtains the combined stress–strain relation for the stress and strain values at all times in the form (Madsen, 1979):

$$\boldsymbol{\varepsilon} = \mathbf{E}^{-1}\boldsymbol{\sigma} + \boldsymbol{\varepsilon}^0 \tag{3.1}$$

in which $\mathbf{E}^{-1} = \mathbf{J}\mathbf{L}^{-1}$. The advantage of this formulation is that it is easy to program. One can take an algebraic expression for an elastic solution, and by replacing $1/E$ with the matrix $\mathbf{E}^{-1}$ one obtains a matrix equation for the values of structural unknowns, e.g. the redundant forces, at all discrete times. In terms of computer time and storage, however, this method is much less efficient than the step-by-step procedure based on the incremental quasi-elastic stress–strain relation such as Eq. (2.31). The lesser efficiency does not matter when there are only a few unknowns, but for larger structures this method would be very expensive.

The solution based on even the second-order step-by-step version (e.g. Eq. 2.30) becomes nevertheless prohibitively costly and exceeds the capacity of all but the largest computers if the structure has many unknowns (over 50), except when a supercomputer is used. For such structures it is vastly more efficient to use numerical step-by-step algorithms based on rate-type constitutive relations, which we discuss next.

3.2.3 Numerical methods based on degenerate kernel

As shown in Chapter 2, the integral-type stress–strain relation can be reduced to a rate-type stress–strain relation if the compliance function (or, more generally, the kernel of the hereditary integral) is approximated by a degenerate kernel. For the discrete time-step approach, one obtains the incremental elastic stress–strain relation which was already given in Chapter 2; see Eqs (2.33–2.37), which represent the rate-type creep law corresponding to an ageing Maxwell chain model, associated with a degenerate form of the relaxation function $R(t, t')$. A similar formulation based on the Kelvin chain model and associated with the degenerate approximation of the compliance function $J(t, t')$ also exists (Bažant and Wu, 1974). The latter approach has been extended for two- and three-dimensional analysis, and implemented in a finite element program RECON (Wium and Buyukozturk, 1984, 1985). The determination of the age-dependent elastic moduli of the Maxwell or Kelvin chain, and the choice of the associated relaxation or retardation times, was also discussed in Chapter 2, and comments were made on the numerical efficiency which is far superior to the methods based directly on hereditary integrals if large structural systems are considered. The works in which the principal contributions were presented were also discussed in Chapter 2. (*Remark.* A more efficient method, in which a Kelvin chain with age-independent moduli and viscosities is used and aging is introduced by means of transformation of variables, is presented in the Addendum to Chapter 2.)

3.2.4 Simplified algebraic methods permitting any $J(t, t')$

Effective modulus method

This is the oldest and simplest method. Introduced perhaps first by McMillan (1916), it has been employed extensively in practice. This method, which admits any form of the compliance function $J(t, t')$, is usually implemented in the sense of

the total stress–strain relation

$$\boldsymbol{\varepsilon}(t) = \mathbf{D}\frac{1}{E_{\text{eff}}(t, t')}\boldsymbol{\sigma}(t) + \boldsymbol{\varepsilon}^0(t) \tag{3.2}$$

in which t_0 = age at first loading,

$$E_{\text{eff}}(t, t') = 1/J(t, t') = E(t')/[1 + \phi(t, t')]$$

is the effective modulus of elasticity (also called the sustained modulus), and $\mathbf{D} = \mathbf{B}$ is defined by Eq. (2.10) for the case of isotropy. When the structural system is changed right after the time of loading, t_0, this method must be applied incrementally for the time interval from t_0 to t, in which case

$$\left[\frac{1}{{}_{\text{eff}}(t, t')}\Delta\boldsymbol{\sigma}(t) + \frac{\sigma(t_0)}{E(t_0)}\phi(t, t_0) \right] + \Delta\boldsymbol{\varepsilon}^0(t) \tag{3.3}$$

The effective modulus method is exact only if the stress is constant from time t_0 at first loading up to the current time t. Otherwise, this method overestimates the effect of creep when the stress is increasing, and underestimates it when the stress is decreasing. Thus, for example, the final stress in a relaxation problem is obtained much too high, the reduction of the elastic shrinkage stress due to creep is obtained much too low, and the long-time creep buckling deflection is obtained much too high. It should also be noted that the error of the creep coefficient values obtained from the current code formulations (ACI, CEB) is often much larger than the error caused by the use of the effective modulus. More accurate methods are therefore unnecessary if these are used to determine the material properties.

All the solutions according to the effective modulus method are special cases of those for the age-adjusted effective modulus method to be discussed later.

Bažant's theorem

If the mechanical strain $\varepsilon - \varepsilon^0$ is zero up to time t_0, then jumps to the value a, and subsequently varies from time t_0 to t linearly with the creep coefficient, i.e.

$$\varepsilon(t) - \varepsilon^0(t) = a + c\phi(t, t_0) \quad \text{or} \quad \boldsymbol{\varepsilon}(t) - \boldsymbol{\varepsilon}^0(t) = \mathbf{a} + \mathbf{c}\phi(t, t_0) \tag{3.4}$$

then the stress for $t > t_0$ varies linearly with the relaxation function and is expressed as

$$\sigma(t) = (a - c)R(t, t_0') + cE(t_0) \quad \text{or} \quad \mathbf{D}\boldsymbol{\sigma}(t) = (\mathbf{a} - \mathbf{c})R(t, t_0) + \mathbf{c}E(t_0) \tag{3.5}$$

Here a and c are arbitrary constants, and Eqs (3.4) and (3.5) are written both for uniaxial deformation and triaxial deformation. In the latter case, matrix $\mathbf{D}$ is given by Eq. (2.10) for the case of isotropy, and $\mathbf{a}$, $\mathbf{c}$ are arbitrary real constant (6×6) matrices.

From the theorem, discovered and proven by Bažant (1972), one readily obtains by algebraic manipulation the basic incremental stress–strain relation of the age-adjusted effective modulus method given in Eqs (2.38) and (2.39). The originally presented proof may be simplified by operator formalism (Bažant, 1982b, p. 242).

Age-adjusted effective modulus method

Formulating the incremental stress–strain relation on the basis of the foregoing theorem, one finds that it is formally identical to the stress–strain relation proposed in 1967 by Trost on the basis of approximate considerations and under the restriction to a time-constant elastic modulus. Trost (1967) proposed for the time interval (t_0, t) the quasi-elastic incremental stress–strain relation given in Eq. (2.38), in which modulus $E''(t, t_0)$ is expressed similarly to the effective modulus, but with a correction of the creep coefficient value:

$$E''(t, t_0) = \frac{E(t_0)}{1 + \chi\phi(t, t_0)} \tag{3.6}$$

Coefficient χ was called by Trost the relaxation coefficient but was later renamed by Bažant the aging coefficient because this coefficient is almost exactly 1 when aging is assumed to be absent, whether or not stress relaxation occurs. Accordingly, modulus E'' was named the age-adjusted effective modulus (Bažant, 1972). The values of the aging coefficient χ are usually between 0.5 and 1.0, with 0.80–0.85 as the mean estimate.

On the basis of the foregoing theorem it was shown (Bažant, 1972) that Trost's relation, originally proposed as approximate, is in fact exact in a certain sense—namely for the cases where the strain varies linearly with the creep coefficient or where the stress varies linearly with the relaxation function. It so happens that such a time variation closely approximates many typical stress histories obtained by exact solutions of the aging viscoelasticity problems. While the effective modulus method is exact only for the condition of constant stress, the method of age-adjusted effective modulus is exact for infinitely many stress histories. These stress histories need not include, as initially thought (Trost, 1967), only decreasing stress regimes typical of the stress relaxation problems, but they also include increasing stress histories such as those in long-time buckling (Bažant, 1972). Thus, coefficient χ does not really introduce a correction for the stress relaxation. Rather, it introduces a correction for aging.

By a simple algebraic rearrangement of Eqs (3.4) and (3.5), it is shown that the age-adjusted effective modulus and the aging coefficient are exactly expressed as (Bažant, 1972)

$$E''(t, t_0) = \frac{E(t_0) - R(t, t_0)}{\phi(t, t_0)}, \qquad \chi(t, t_0) = \frac{E(t_0)}{E(t_0) - R(t, t_0)} - \frac{1}{\phi(t, t_0)} \tag{3.7}$$

provided the assumption that ε is linear with $\phi(t, t_0)$, or that σ is linear with $R(t, t_0)$, is acceptable for the given problem. Thus we see that the aging

coefficient may be calculated from the relaxation function, and vice versa.

The relaxation function is easily obtained with a high accuracy by a simple computer program, which has been used to construct tables and graphs of the aging coefficient for various typical definitions of the compliance function specified by code recommendations (Bažant, 1972; Neville, *et al.*, 1983; Chiorino *et al.*, 1984). However, the tables or graphs are not really indispensable, since the relaxation function to be used in Eq. (3.7) can be expressed with adequate accuracy from the approximate formula in Eq. (2.8), which is applicable to any conceivable shapes of the compliance curves of concrete. This is important for the use of the more realistic compliance functions, such as that in the BP Model, which depend on many parameters (humidity, size, temperature, etc.) and give for each of them compliance curves of different shapes. The variation of the aging coefficient is illustrated in Fig. 3.1 for the CEB–FIP (1970) compliance function (Bažant, 1972). The plot is given both for constant and age-dependent elastic

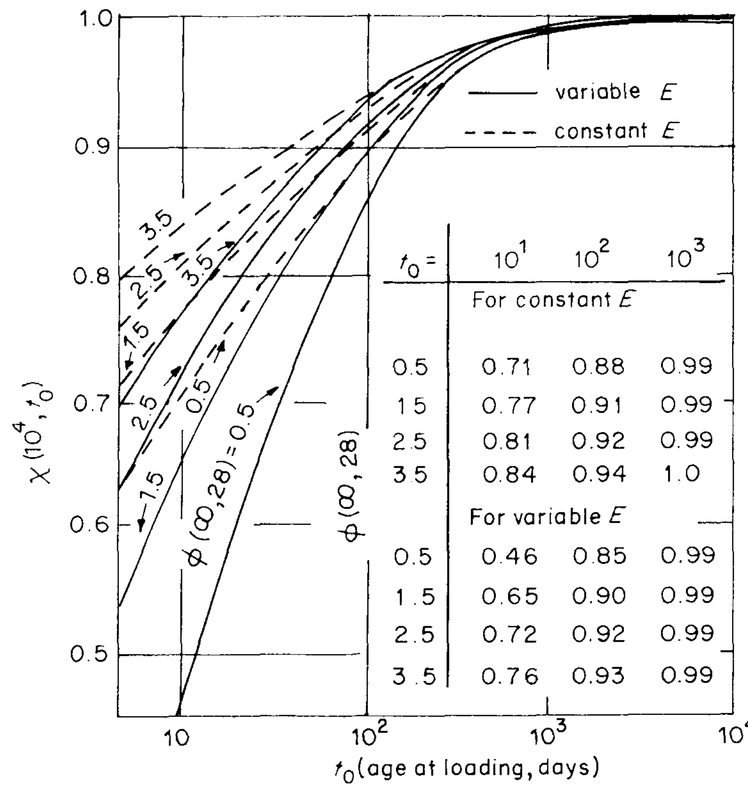


Figure 3.1 Diagram of aging coefficient $\chi(t, t_0)$ at $t - t_0 = 10^4$ days and various times of loading t_0 for CEB creep function (after Bažant, 1972)

modulus, revealing the influence of the variation of E to be quite significant, as pointed out by Bažant (1972).

The three-dimensional form of the quasi-elastic incremental stress–strain relation of the age-adjusted effective modulus method is obtained by generalizing Eq. (2.38):

$$\Delta\boldsymbol{\varepsilon}(t) = \frac{1}{E''(t,t_0)}\mathbf{D}\Delta\boldsymbol{\sigma}(t) + \Delta\boldsymbol{\varepsilon}''(t), \qquad \Delta\boldsymbol{\varepsilon}''(t) = \frac{\phi(t,t_0)}{E(t_0)}\mathbf{D}\boldsymbol{\sigma}(t_0) + \Delta\boldsymbol{\varepsilon}^0(t) \quad (3.8)$$

provided the material is assumed to be isotropic with a constant Poisson's ratio, which is approximately true for concrete.

It has been attempted in the past to verify certain values of the aging coefficient χ by comparisons with test results; see, e.g. the comparison made by Neville, Dilger and Brooks (1983-pp. 257) on the basis of the test results by Bastgen (1979). These tests, however, were of rather limited duration and the range of ages at loading did not include ages t' at loading beyond one month, so that the values of $J(t,t')$ for higher t' were left experimentally undetermined. This fact renders the interpretation of these tests questionable because χ depends on such values. Furthermore, the humidity conditions in these tests put also the linearity of creep into question so that the deviations found might not be just errors of χ but of the principle of superposition itself.

The idea to correct the creep coefficient in Eq. (3.6) with some correction factor χ is an old one and predates Trost and Bažant. For instance, the value $\chi = 0.5$ was used, which corresponded to the assumption that the stress varies between t_0 and t linearly. While this is a good assumption for very short time intervals, it is poor for the very long time intervals (t_0, t) used in the age-adjusted effective modulus method. This may be illustrated with the help of Fig. 3.2 (Bažant, 1982); the typical stress variation in curve 3 is usually much closer to a constant stress (curve 2),

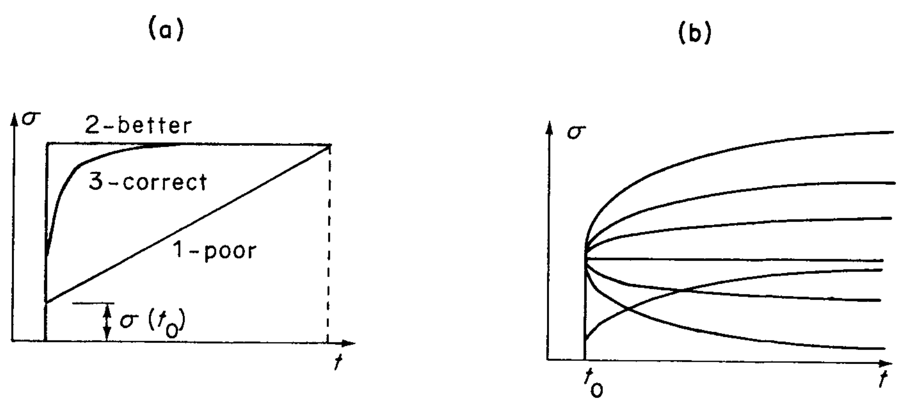
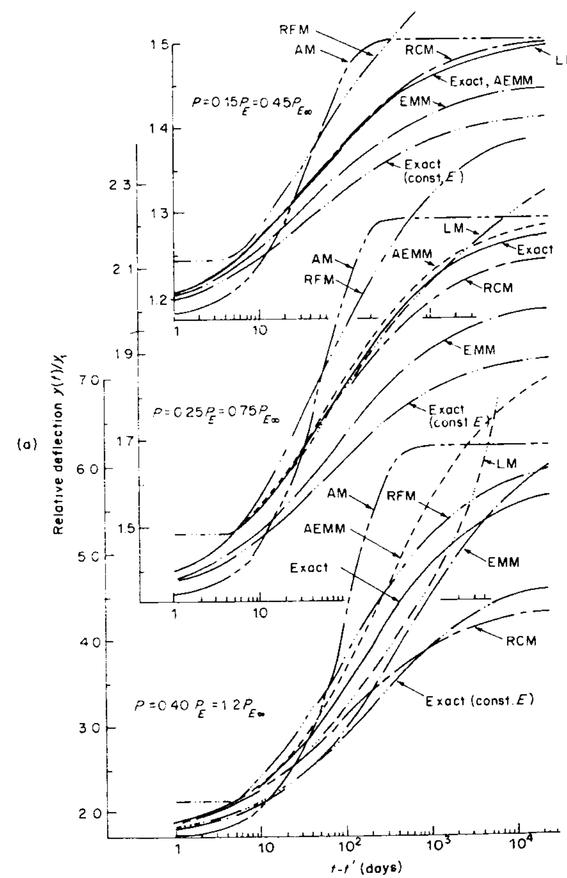


Figure 3.2 (a) Stress histories implied by various calculation methods; (b) stress histories expressed as a linear combination of the relaxation function

implied in the effective modulus method, than to the linear stress variation (curve 1). Moreover, the stress values which matter most are those at later times, because the influence of stress at time t' upon the strain at time t increases as t' approaches t, as indicated by the memory function $L(t,t')$ (see Eq. 2.5, the curve of which is also sketched in Fig. 3.2). For a discussion of these aspects see also Chiorino *et al.* (1984, pp. 104–15).

Extensive comparisons of the predictions of the age-adjusted effective modulus method for various structural creep problems with the exact solutions, as well as with the solutions by other approximate methods, were made by Bažant and Najjar (1973). Several of these comparisons are reproduced in Fig. 3.3.

The age-adjusted effective modulus method is at present widely considered to be the most efficient method for linear creep analysis of ageing structures (see for example Dilger, 1982; Neville, *et al.*, 1983; CEB–FIP Manual by Chiorino *et al.*, 1984; ACI Committee 209 Recommendation, 1982).



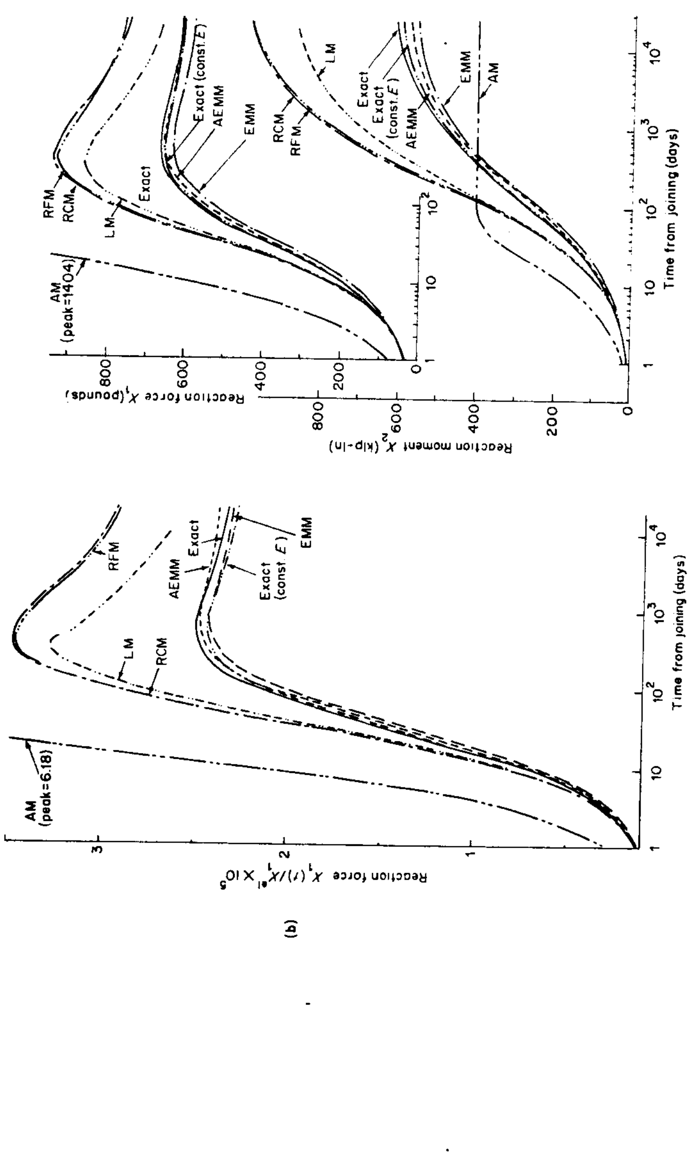


Figure 3.3 (a) Creep buckling deflections of slender column under working loads (after Bažant and Najjar, 1973). (b) Forces in midspan connection of two cantilevers of different age, due to differential creep (1 kip-in = 112.5 N × m) (after Bažant and Najjar, 1973). (EMM—effective modulus method, AEMM—age-adjusted effective modulus method, RCM—rate-of-creep method, RFM—rate-of-flow method, LM—Levi's method.)

Lazić's Theorems

Lazić and Lazić (1984a, 1981, 1977, 1982a) discovered and proved the following theorems for composite beams which are analysed under the usual assumption that plane cross-sections remain plane and normal and consist of aging linear viscoelastic parts of different properties. If the axial force N and the bending moment M depend linearly on the relaxation function R, i.e. if

$$N = N_1 R + N_2 \qquad M = M_1 R + M_2 \tag{3.9}$$

where $N_1, \ldots M_2$ are arbitrary constants, then:

1. The normal strain λ and the curvature change κ depend linearly on certain functions $B_h (h = 1, 2)$, i.e.

$$\lambda = \lambda_1 B_1 + \lambda_2 B_2 + \lambda_3, \qquad \kappa = \kappa_1 B_1 + \kappa_2 B_2 + \kappa_3 \tag{3.10}$$

where $\lambda_1, \ldots \kappa_3$ are constants;

2. The stresses in part k of the composite cross-section depend linearly on the same functions B_h and the relaxation function R, i.e.

$$\sigma_k = v_k \left(U_k + V_k R + \sum_{h=1}^{2} W_{hk} B_h \right) \tag{3.11}$$

where U_k, V_k, W_{hk}, and v_k are constants expressed according to the formulae found in Lazić and Lazić (1985) and Lazić (1985a);

3. The generalized displacement also depends linearly on the same functions B_h, i.e.

$$\Delta = \sum_{(a)} \left[P^{(a)} + \sum_{h=1}^{2} Q_h^{(a)} B_h^{(a)} \right] \tag{3.12}$$

where $P^{(a)}$ and $Q_h^{(a)}$ are constants expressed according to the formulae found in Lazić and Lazić (1982a) and Lazić (1985a).

The basic functions B_h $(h = 1, 2)$ depend not only on the properties of the material of the various parts of the cross-section, but also on the cross-section geometry. They can be obtained by introducing the function $K(\gamma, t, t') = \gamma + (1 - \gamma) J(t, t')$, in which γ is a parameter and γ $(0 \leqslant \gamma < 1)$ appears. The functions $K(\gamma, t, t')$ and $B(\gamma, t, t')$ are related in the same manner as $J(t, t')$ and $R(t, t')$, and so the known methods for determining the relaxation function can be applied—either a solution of a Volterra integral equation (Bažant, 1972; Lazić and Lazić, 1982b) or the approximate Bažant–Kim formula (2.8). Two coefficients γ_h $(0 < \gamma_h < 1)$ $(h = 1, 2)$ describe the cross-section geometry, and these particular values of γ yield the corresponding basic functions $B_h(\gamma_h, t, t')$ $(h = 1, 2)$, while $\gamma = 0$ yields the relaxation function. Function $B(\gamma, t, t')$ as well as $R(t, t')$ can be tabulated or described by graphs for any given compliance function. Equations (3.11) and (3.12) represent then the expressions for stresses and a generalized displacement in an algebraic form.

Under the assumptions stated in Eq. (3.9), they are accurate when the values of B_h ($h = 1, 2$) and R are calculated through a Volterra integral equation. Lazić and Lazić (1982a) have presented such valus of B and R for the CEB–FIP (1978) compliance function. For limitations where Eq. (3.9) is valid, see Lazić and Lazić (1985). In the special case of a homogeneous cross-section, Lazić's theorem given by Eq. (3.11) reduces to Bažant's theorem.

Improved aging coefficient

In analogy to the aging coefficient χ, Lazić and Lazić (1981, 1985) defined a new aging coefficient χ_F for the given composite cross-section. The value of this improved aging coefficient can be determined approximately so that the creep effects at most points of the cross-section are either overestimated or predicted exactly, the exact prediction being imposed for a certain particular cross-section shape often used in practice.

Coefficient χ_F, in contrast to the aging coefficient χ, introduces the influence of the cross-section geometry. It always has a smaller value than the corresponding aging coefficient χ. Thus, the use of χ in the analysis of composite cross-sections always leads to some underestimation of the creep effect. In most cases, however, the approximate results obtained by applying the usual age-adjusted effective modulus method with the aging coefficient χ to a composite cross-section are very close to the exact linear viscoelastic solution. Nevertheless, for certain forms of the compliance function the deviation from the predictions based on χ from the exact solution is appreciable, although still not large. In that case, the use of the coefficient χ_F offers an improvement, as shown by Lazić to be the case for the CEB–FIP (1978) compliance function. The gain of accuracy from the use of χ_F instead of χ becomes more pronounced if the creep coefficient is larger and if the stress relaxation in the prestressing steel is greater (Lazić and Lazić, 1985). For the ACI compliance function as well as for the BP model, the predictions based on χ were shown to be accurate enough (e.g. Bažant and Najjar, 1973). The prevailing opinion seems to be that the use of the usual χ is sufficient in most cases, especially in comparison to the errors and uncertainties of $J(t, t')$-values and of structural creep analysis.

Matrix generalization

The theorem underlying the age-adjusted effective modulus method can be generalized to any structure characterized by a linear relation between column matrix $\mathbf{u}$ of generalized displacements and their column matrix $\mathbf{f}$ of generalized forces. If the material behaviour is linear, obeying the principle of superposition, this relationship may always be cast in the form

$$\mathbf{u}(t) = \int_{t_0}^{t} \mathbf{J}(t, t')\,\mathrm{d}\mathbf{f}(t') = \underset{\sim}{\mathbf{J}}\mathbf{f}(t) \tag{3.13}$$

in which $\mathbf{J}(t, t')$ is a square compliance matrix characterizing the components of displacement $\mathbf{u}$ at age t caused by unit applied forces (components of $\mathbf{f}$) applied at age t'. This matrix characterizes, in the most general form, the creep properties of the structure. Here $\underset{\sim}{J}$ is the square matrix creep operator associated with $\mathbf{f}$. The inverse relation is

$$\mathbf{f}(t) = \int_{t_0}^{t} \mathbf{R}(t, t')\,\mathrm{d}\mathbf{u}(t') = \mathbf{R}\mathbf{u}(t) \tag{3.14}$$

in which $\mathbf{R}(t, t')$ is a square relaxation matrix representing the components of forces $\mathbf{f}$ at age t caused by unit displacements (components of $\mathbf{u}$) enforced at age t'; and $\underset{\sim}{R}$ is the square matrix relaxation operator, characterizing the material creep properties of the structure; $\underset{\sim}{R} = \underset{\sim}{J}^{-1}$. Note that the underlying operator equations of Lazić's theorem are the special case of these relations.

In similarity to Bažant's theorem, assume that the displacement histories $\mathbf{u}(t)$ are linearly dependent on the compliance matrix $\mathbf{J}(t, t_0)$, which is written as

$$\mathbf{u}(t) = \mathbf{a} + \mathbf{J}(t, t_0)\mathbf{b} \tag{3.15}$$

in which $\mathbf{a}$ and $\mathbf{b}$ are square constant matrices.

Theorem (Bažant, 1987). If Eq. (3.15) holds true, then

$$\mathbf{f}(t) = \mathbf{b} + \mathbf{R}(t, t_0)\mathbf{a} \tag{3.16}$$

Proof: Equation (3.15) may be rewritten as $\mathbf{u}(t) = H(t - t')\mathbf{a} + \underset{\sim}{J}H(t - t_0)\mathbf{b}$. Multiplying all terms of this equation by operator $\underset{\sim}{R}$ from the left, we have

$$\underset{\sim}{R}\mathbf{u}(t) = \mathrm{R}H(t - t_0)\mathbf{a} + \underset{\sim}{R}\underset{\sim}{J}H(t - t_0)\mathbf{b}.$$

Now we may note that

$$\underset{\sim}{R}H(t - t_0) = R(t, t_0), \qquad \underset{\sim}{R}\underset{\sim}{J} = \underset{\sim}{I},\ \underset{\sim}{I}\,\mathbf{b} = \mathbf{b},$$

which proves Eq. (3.16) (I is the identity operator).

Similarly to the age-adjusted effective modulus method, Eq. (3.16) can be reformulated in terms of the increments of $\mathbf{u}$ and $\mathbf{f}$ from the instant of first loading t_0, to the current time t. Denoting the initial elastic compliances and stiffnesses of the structure as $\mathbf{J}(t_0, t_0) = \mathbf{J}_0$, $\mathbf{R}(t_0, t_0) = \mathbf{R}_0$, we have

$$\Delta\mathbf{u} = [\mathbf{J}(t, t_0) - \mathbf{J}_0]\mathbf{b} \quad \text{and} \quad \Delta\mathbf{f} = [\mathbf{R}(t, t_0) - \mathbf{R}_0]\mathbf{a}$$

Expressing the column matrices $\mathbf{a}$ and $\mathbf{b}$ by matrix inversions from the last two equations, and substituting them into Eqs (3.15) and (3.16) and then into the expressions for $\Delta\mathbf{u}$ and $\Delta\mathbf{f}$, we obtain the following algebraic matrix relations between the column matrices of incremental forces and displacements (Bažant, 1987):

$$\Delta\mathbf{f} = (\mathbf{R}_0 - \mathbf{R})[\mathbf{J}_0(\mathbf{J} - \mathbf{J}_0)^{-1}\Delta\mathbf{u} - \mathbf{u}_0] \tag{3.17}$$

$$\Delta\mathbf{u} = (\mathbf{J} - \mathbf{J}_0)[\mathbf{R}_0(\mathbf{R}_0 - \mathbf{R})^{-1}\Delta\mathbf{f} + \mathbf{f}_0] \tag{3.18}$$

These equations represent generalizations of the stress–strain relation of the age-adjusted effective modulus method. Of course, they yield these stress–strain relations as a special case in which the column matrices **f** and **u** are replaced by σ and ε.

As a special case, one can also obtain an algebraic stress–strain relation for the case of general aging linearly viscoelastic material behaviour, for which the Poisson ratio ν is not a constant, but variable; $\nu(t, t')$. In that case the stress–strain relations may be written as $\boldsymbol{\varepsilon}(t) = \underline{\mathbf{J}}\boldsymbol{\sigma}(t)$ and their inverse as $\boldsymbol{\sigma}(t) = \underline{\mathbf{R}}\boldsymbol{\varepsilon}(t)$ where $\underline{\mathbf{J}}$ and $\underline{\mathbf{R}}$ are matrix operators which are inverse to each other and are expressed on the basis of the matrix compliance function $\mathbf{J}(t, t')$ and the matrix relaxation function $\underline{\mathbf{R}}(t, t')$. The incremental algebraic matrix stress–strain relations are then obtained from Eqs (3.17) and (3.18) by replacing the column matrices $\Delta\mathbf{f}$ and $\Delta\mathbf{u}$ by $\Delta\boldsymbol{\varepsilon}$ and $\Delta\boldsymbol{\sigma}$. The result is (Bažant, 1987):

$$\Delta\boldsymbol{\varepsilon} = (\mathbf{J} - \mathbf{J}_0)[\mathbf{R}_0(\mathbf{R}_0 - \mathbf{R})^{-1}\Delta\boldsymbol{\sigma} + \Delta\boldsymbol{\sigma}_0] \tag{3.19}$$

$$\Delta\boldsymbol{\sigma} = (\mathbf{R}_0 - \mathbf{R})[\mathbf{J}_0(\mathbf{J}_0 - \mathbf{J}_0)^{-1}\Delta\boldsymbol{\varepsilon} - \Delta\boldsymbol{\varepsilon}_0] \tag{3.20}$$

3.2.5 Methods based on simplified $J(t, t')$ and differential equations

While the algebraic methods just described admit any given compliance function, and make their approximation in the structural analysis phase of the solution, another group of methods solves the structure exactly but simplifies the compliance function to a form that permits an exact structural solution without too much difficulty. So, these methods are based on a certain simplification of the stress–strain relation; they were already discussed in considerable detail in Section 2.2.5. Therefore, we give now only a brief overview with some additional viewpoints.

The rate-of-creep method, also called the Dischinger method, which corresponds to a differential stress–strain relation based on an aging Maxwell model (Eqs 2.84, 2.85), played a historically important role. Next to the effective modulus method, it was the first method to be applied extensively to a multitude of complex structural problems, especially in Europe (while American and British practice favoured the simpler effective modulus method which is certainly not more inaccurate). After the introduction of the improvements consisting of the rate-of-flow method (England and Illston, 1965) or the improved Dischinger method (Nielsen 1970; Rüsch *et al.*, 1973), as well as the age-adjusted effective modulus method, the classical rate-of-creep method became obsolete as a prediction of mean response. Nevertheless, aside from providing various instructive simple solutions, the method is still useful in that it provides an approximate bound on the exact solution. When the stress varies monotonically in time, which is true of a majority of problems, the rate-of-creep method generally gives a bound on the exact solution which is opposite to the bound obtained from the classical effective modulus method. Thus, it is still acceptable

for the designer if he makes his calculations twice, once with the effective modulus method, and once with the rate-of-creep method, and then makes sure that his design is admissible for both solutions. However, using only one of these two methods is inadequate, except when the creep effects are known to be very small.

The improvement of the rate-of-creep method in the form of the rate-of-flow method and its later variant known as the improved Dischinger method (Eq. 2.86) (England and Illston, 1965; Nielsen, 1970; Rüsch *et al.*, 1973) was an important development. At the time of its introduction in 1965, the rate-of-flow method was clearly the best practical method available. However, the age-adjusted effective modulus method discovered subsequently is even more accurate in comparison to the exact solutions according to aging viscoelasticity, and at the same time it is simpler to apply since it consists of an elastic analysis, which is algebraic in time, rather than a solution of differential equations in time. The age-adjusted effective modulus method is the only method used in the ACI 209 Recommendations (1982) (next to the effective modulus) is emphasized as the main approach in the latest CEB Design Manual (Chiorino *et al.* 1982), and is recommended in Annexe e (para, e.2) of the CEB–FIP (1978) Model Code, along with the improved Dischinger method as an alternative. For extensive comparisons of various approximate methods with the exact viscoelastic solutions, see Bažant and Najjar (1973) (cf. Section 2.5.5).

Other simplified forms of the compliance function were used in the past for the purpose of facilitating structural analysis. Levi and Pizzetti (1951) proposed the first-order differential equation

$$E\frac{\mathrm{d}\varepsilon}{\mathrm{d}\psi} + E_u\varepsilon = \frac{\mathrm{d}\sigma}{\mathrm{d}\psi} + \psi \tag{3.21}$$

which may be integrated to yield

$$J(t, t') = \frac{1}{E} + \left(\frac{1}{E_u} - \frac{1}{E}\right)\{1 - \gamma\exp[\psi(t') - \psi(t)]\} \tag{3.22}$$

in which E, E_u are constants, $\gamma = E_u/(E - E_u)$, and $\psi(t) = -\ln[1 - \phi(t, t_0)\gamma]/\gamma$.

A similar method which was popular in eastern Europe was proposed by Arutyunian (1952). By using a compliance function of the form

$$J(t, t') = \frac{1}{E(t')}\left\{1 + \psi(t')\left[1 - \exp\left(-\frac{t - t'}{\tau_1}\right)\right]\right\} \tag{3.23}$$

in which τ_1 = constant and $\psi(t') = A + C/t'$ or $\psi(t') = A + C\exp(t'/\tau_1)$ with constants E and C, the integral-type constitutive law can be reduced to a first-order differential equation with age-dependent coefficients, which was shown by Bažant (1966a, b) to correspond to an aging standard-solid rheologic model. On this basis, various structural creep problems can be integrated in time with the help of the incomplete gamma function (Arutyunian, 1952; Bažant, 1965, 1968b). Later, however, it was shown (Bažant and Najjar, 1973) that these solutions deviate greatly from the exact linear viscoelastic solutions when long-time response is considered. It is now clear that, generally, this method is both more complicated and less accurate than the age-adjusted effective modulus method or the improved Dischinger method. The same conclusion applies to Levi's method.

An extension of Arutyunian's method was proposed by Aleksandrovskii (1966). However, his creep function was later shown incapable of a good representation of long-time creep data (Bažant and Thonguthai, 1978).

More recently Jordaan and England (1977) proposed the use of the Burgers model, which consists of a series coupling of a spring, a dashpot, and a Kelvin unit. He assumed that the effect of aging may be represented by replacing time t with a certain reduced time θ which is a monotonic continuous function of t. Thus, he expressed the compliance function in a non-aging form

$$J(\theta-\theta') = \frac{1}{E} + f(\theta-\theta') + A\{1-\exp[-\lambda(\theta-\theta')]\} \tag{3.24}$$

in which E, A, and λ are constants. In terms of variable θ, the aging property is removed and structural problems can be solved by Laplace transforms, just as in classical linear viscoelasticity without aging. The comparisons of Eq. (3.24) with test data were limited, and it is doubtful that a close representation can be achieved when test data of very long duration, involving rather different ages at loading, are considered.

Extensions to variable temperature and moisture content were proposed on the basis of replacing time with some reduced time (Sackmann, 1963). In detail, this approach was developed by Mukaddam and Bresler (1972), who also used a certain reduced time to transform the constitutive equation to a non-aging form.

3.2.6 Elastic–viscoelastic analogy for aging materials

The most general aging linearly viscoelastic stress–strain relation can be written as

$$\boldsymbol{\varepsilon} = \underset{\sim}{\mathbf{C}}^{-1}\boldsymbol{\sigma} + \boldsymbol{\varepsilon}^0 \quad \text{or} \quad \boldsymbol{\sigma} = \underset{\sim}{\mathbf{C}}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^0) \tag{3.25}$$

in which $\underset{\sim}{\mathbf{C}}$ and $\underset{\sim}{\mathbf{C}}^{-1}$ are Volterra integral operators. For isotropic properties and a constant Poisson ratio, $\underset{\sim}{\mathbf{C}}^{-1} = \mathbf{D}\underset{\sim}{E}^{-1}$, as stated in Eq. (2.12). These operators obey the same rules as linear algebra, except that a product is not commutative. Thus, any of the equations of elasticity in which only linear combinations of elastic constants appear may be generalized to creep by replacing them with the corresponding operators. This correspondence is called the elastic–viscoelastic analogy. It was stated in the operator form for aging materials by Mandel (1958) (see also Huet, 1970), although for the special case of homogeneous structures an equivalent analogy (due to McHenry, see the next section) was presented earlier. (Caution is needed when a product of elastic constants appears; it is then

necessary to go over the derivation of the elasticity equation to be converted to creep in order to determine what is the proper order of the operators, since they are not commutative.)

For some illustrations of the use of this analogy, see e.g. Bažant (1975).

3.2.7 Homogeneous structures and McHenry's analogy

Consider a structure that has the following properties: (1) homogeneity, i.e. the creep properties are the same at all points (this excludes, e.g., differences in age or water content); (2) constancy of creep Poisson's ratio ν; (3) absence of deformable supports; and (4) linearity of the associated elasticity problem. Further, denote by $\boldsymbol{\sigma}^{el}(t)$, $\mathbf{u}^{el}(t)$ the stresses and displacements at time t for an elastic structure of time-constant modulus E, caused by distributed surface loads $\mathbf{p}(t)$, distributed body forces $\mathbf{f}(t)$, prescribed boundary displacements $\mathbf{u}^0(t)$ and prescribed inelastic strains $\boldsymbol{\varepsilon}^0(t)$ as given functions of time. Then: (a) if $\boldsymbol{\varepsilon}^0 = \mathbf{0}$ and $\mathbf{u}^0 = \mathbf{0}$, then

$$\boldsymbol{\sigma}(t) = \boldsymbol{\sigma}^{el}(t), \qquad \mathbf{u}(t) = E\underset{\sim}{E}^{-1}\mathbf{u}^{el}(t) \tag{3.26}$$

(b) if, instead, $\mathbf{p} = \mathbf{0}$, and $\mathbf{f} = 0$, then

$$\boldsymbol{\sigma}(t) = \frac{1}{E}\underset{\sim}{\mathbf{E}}\boldsymbol{\sigma}(t), \qquad \mathbf{u}(t) = \mathbf{u}^{el}(t) \tag{3.27}$$

For a concise proof, see e.g. Bažant (1975).

As an equivalent statement of this analogy, $\boldsymbol{\sigma}(t)$ and $\mathbf{u}(t)$ are equal to the elastic solution due to the fictitious load $E\underset{\sim}{E}^{-1}\mathbf{p}(t)$, $E\underset{\sim}{E}^{-1}\mathbf{f}(t)$, plus the elastic solution due to the prescribed displacements $(1/E)\underset{\sim}{\mathbf{E}}\mathbf{u}(t)$, and inelastic strains $(1/E)\underset{\sim}{\mathbf{E}}\boldsymbol{\varepsilon}^0(t)$. In this latter form, the analogy was discovered by McHenry (1943) although for the special case of rate-of-creep method and redundant frames (for which the constancy of ν need not be required due to the neglect of shear strains), the essence of this analogy was previously implied in Dischinger's work (1973). A rigorous mathematical proof was given by Arutyunian (1952).

From the foregoing analogy we know that in a homogeneous structure, under the conditions specified above, all the displacements and stresses vary in the same proportion as in a homogeneously stressed specimen.

Applying the principle of superposition, McHenry's analogy can also be used to obtain solutions for homogeneous structures in which the structural system was changed while the structure was under load; see Bažant (1975), in which an illustrative example is also given.

3.2.8 Conversion of inelastic strains to applied loads

The quasi-elastic stress–strain relations obtained by various types of numerical methods or by the age-adjusted effective modulus method involve inelastic strains known or approximately estimated in advance. In elastic finite element analysis, the effect of such inelastic strains is replaced, according to the principle of virtual work, by equivalent inelastic nodal forces. The use of the following theorem is, however, more general, and is applicable for any method of structural analysis.

Consider a general (anisotropic) elastic stress–strain relation $\sigma_{ij} = C_{ijkm}(\varepsilon_{km} - \varepsilon''_{km})$ in which C_{ijkm} are the elastic moduli and ε''_{km} is the prescribed inelastic strain tensor. Further define

$$\sigma''_{ij} = C_{ijkm}\varepsilon''_{km}, \qquad \tilde{f}_i = -\sigma_{ij,j} \tag{3.28}$$

$$\tilde{p}_i = n_i\sigma''_{ij} \text{ on } \Gamma, \qquad \tilde{p}^*_i = n^*_j(\sigma''^{+}_{ij} - \sigma''^{-}_{ij}) \text{ on } \Gamma^* \tag{3.29}$$

in which σ''_{ij} is the inelastic stress tensor; $\tilde{f}_i$ and $\tilde{p}_i$, $\tilde{p}^*_i$ are fictitious volume and surface loads equilibrating σ''_{ij}; Γ^* is an interface with unit normal n^*_i across which σ''_{ij} changes discontinuously from the value σ''^{-}_{ij} to the value σ''^{+}_{ij}. Then the stresses, strains, and displacements caused by ε''_{ij} are

$$\sigma_{ij} = \tilde{\sigma}_{ij} - \sigma''_{ij}, \qquad \varepsilon_{ij} = \tilde{\varepsilon}_{ij}, \qquad u_i = \tilde{u}_i \tag{3.30}$$

in which $\tilde{\sigma}_{ij}\tilde{\varepsilon}_{ij}$ and $\tilde{u}_i$ are the solutions corresponding to loads $\tilde{f}_i$, $\tilde{p}_i$, and $\tilde{p}^*_i$, with no inelastic strains, and to the given boundary displacements, u^*_i, if any.

The special case of this theorem, the proof of which is given in Bažant (1975), is known in thermoelasticity as the body force analogy (Lin, 1968), which is due to Duhamel (1838) and Neumann (1841) for the case of the isotropic tensor ε_{ij}. For deviatoric plastic strains, this analogy was derived in 1931 by Reissner and, in different contexts, by Eshelby and others (Bažant, 1966a, b, 1967, 1964a). First applications to concrete creep were in the analysis of frames (Bažant, 1964a, 1966a, b), and later in linear and non-linear creep of concrete plates (Bažant, 1967, 1971a, b, 1970, and Lin, 1968).

3.2.9 Variational principles

For aging linearly viscoelastic structures it is possible to formulate variational principles analogous to the principle of minimum potential energy or the minimum complementary energy in elasticity. Such principles may be best obtained according to the elastic–viscoelastic analogy, replacing E or E^{-1} with the corresponding operators. For example, by replacing E^{-1} in the expression for the elastic complementary energy Π^* with the operator $(E^{-1}\mathrm{d}/\mathrm{d}\psi + E_c^{-1})(\mathrm{d}/\mathrm{d}\psi)^{-1}$ which results from Eq. (2.88) for the rate-of-creep method, it was shown (Bažant, 1961, 1964a, b, 1966a) that for redundant (i.e. statically indeterminate) structures:

$$\Phi = \int_V \left(\frac{\sigma}{E(t)}\frac{\mathrm{d}\sigma}{\mathrm{d}\psi} + \frac{\sigma^2}{2E_c}\right)\mathrm{d}V - \sum_k P_k\frac{\mathrm{d}w_k}{\mathrm{d}\psi} = \min \tag{3.31}$$

This was called the principle of minimum deformation resistance; P_k = concentrated loads ($k = 1, 2, \ldots n$), w_k = load-point displacements, and V = volume of the

structure. This principle, which implies the compatibility conditions, was shown to yield the differential equations for the redundants X_k in a statically indeterminate system. The deformation rates are obtained as $\mathrm{d}w_k/\mathrm{d}\psi = \partial\Phi/\partial P_k$ and the compatibility conditions are $\partial\Phi/\partial X_k = 0$ (stresses σ depend linearly on X_k according to equilibrium conditions). The principle of minimum deformation resistance in Eq. (3.31) is also applicable for the CEB–FIP improved Dischinger method if $1/E(t)$ is replaced with $1/E(t') + 1/E^{\mathrm{d}}_{\mathrm{ef}}$, according to Eq. (2.86). A similar principle for the rate-of-creep method was later expounded by England and co-workers (e.g. Zeitouni and England, 1987).

Although the principles such as Eq. (3.31) are theoretically interesting, they do not seem to provide new essential information since solutions can be obtained without them in other ways, e.g. directly from the elastic–viscoelastic analogy on the basis of the variational principles of elasticity. In incremental numerical solutions, which reduce the creep problem to a sequence of elasticity problems, again only elastic variational principles are needed.

It may be pointed out that for non-linear creep of concrete at high stress, there exist no variational principles analogous to the lower and upper bound theorems of plasticity.

3.3 NON-LINEAR METHODS

In contrast to metals and clays, and in similarity to many polymers, concrete creep may be considered to be approximately linear for many types of response in the service stress range. Significant non-linearities nevertheless exist. In the high stress range, concrete creep is highly non-linear. Although the understanding of non-linear methods for concrete creep is much more limited than it is for the linear methods, some noteworthy advances have been made and will now be briefly reviewed.

3.3.1 Non-linearity due to humidity and temperature variation

This is the main source of non-linearity under constant loads in the service stress range. Its source is twofold:

1. The cross coupling of creep with shrinkage or thermal expansion, which causes that these strains are not additive and need to be taken into account in the form of, for example, stress-induced shrinkage (or swelling) and stress-induced thermal expansion (or contraction).
2. Cracking or tensile strain softening, which inevitably accompanies the changes of environmental humidity and temperature and is caused by the delay due to diffusion (see Sections 2.3.3–2.3.5).

These effects may be analysed numerically in a step-by-step fashion, with iterations in each time step. This technique was employed, e.g. by Bažant and Wu

(1974), Becker and Bresler (1977), Iding and Bresler (1982), Wittmann and Roelfstra (1981), and Jonasson (1977). A rather simple analysis of these effects for nuclear reactor containments was made by Bažant *et al.* (1975), who employed the age-adjusted effective modulus method, introducing corrections to the effective modulus on the basis of approximate drying histories of the layers throughout the shell thickness.

3.3.2 Non-linearity due to cracking or strain-softening

These effects are normally taken into account in step-by-step time integration by adjusting the incremental stiffness according to the stress and strain values in the previous iteration of the same step or, for the first iteration, in the last iteration of the previous step. There are, nevertheless, serious problems with convergence of the iterations, as well as with convergence when the structural discretization is refined. These problems are due to strain-localization instabilities and border on fracture mechanics. A vast amount of research is being done in this field (for a review see, e.g. Bažant, 1986), most of it not motivated by concrete creep and shrinkage but still applicable to it.

A particular difficulty arises with strain-softening when the time steps are decreased substantially beyond the shortest relaxation time, as is necessary when long-time creep is analysed. It was found (Bažant and Chern, 1985) that an arbitrary increase of the time step is made possible by an exponential algorithm which is analogous to the exponential algorithm for the Maxwell chain model and is based on an exact solution of the incremental stress–strain relation for a strain-softening element under the assumption that the coefficients of the associated differential equation are constant.

3.3.3 Non-linearity due to cyclic loading

Even in the linear service stress range, a cyclic stress component superimposed on a constant stress produces after many repetitions a significant increase of deformation, a phenomenon often referred to as the cyclic creep and sometimes considered an aspect of fatigue. The increase of the deformation due to stress cycling depends on the stress non-linearly, and so the phenomenon is not covered by linear constitutive laws (see Section 2.4.2). This type of deformation is probably caused by progressive formation and irreversible openings of microcracks.

In cyclic creep we are usually interested in a large number of repetitions, at least 1000 and even over 10^6. Thus it is impossible to follow the stress history in a step-by-step calculation. A simple way to calculate the effect of cyclic creep is again to adjust the effective modulus E_{ef} (or E'') on the basis of the cyclic creep deformation predicted for the stress history which would take place if there were no redistributions of stress in the structure due to cyclic creep. After solving the structural problem with such modified effective modulus values, which are generally different for each point of the structure, the calculation can be iteratively repeated updating the magnitude of the cyclic creep on the basis of the stress history with stress redistributions obtained in the previous iteration. As an approximation, only one long step from the time of first loading to the final time after many cycles is considered. This approach was used by Bažant (1968a) in the analysis of cyclic creep effect in long-span prestressed concrete bridges.

3.3.4 Non-linearity at unloading and adaptation

In concrete, significant deviations from linearity (i.e. from the principle of superposition) are observed at decreasing strain, especially at large sudden unloading (see Section 2.4). Generally, the unloading deformation behaviour is stiffer than that predicted according to linearity. The deviations from linearity are not caused by the decrease of stress, since stress relaxation follows the predictions of linear analysis, but by a decrease of strain. There are some stress–strain relations (e.g. the rate of flow and improved Dischinger methods) which are calibrated according to the behaviour at unloading, but then they sacrifice the possibility of good representation of test data at constant stress for a wide range of ages at loading and for long durations. In these approaches, one cannot of course speak of non-linearity at unloading.

The non-linearity which is responsible for the reduced recoverability of strain at unloading has been called the adaptation. Described by an adaptation parameter in the kernel of the iterative integral (Bažant and Kim, 1979a), this non-linearity also causes an increased stiffness for load increments after a longer period under sustained compressive load. It appears that sustained compressive stress gradually stiffens concrete, perhaps promoting creation of bonds as in hydration. On the other hand, tensile stress should cause a decrease of stiffness, although experimental data to verify this are still lacking.

For the integral-type constitutive law with a kernel that includes a stress-dependent adaptation parameter, the structural creep analysis can be again carried out in a step-by-step manner, with iterations in each time step in which the value of the adaptation parameter is updated from one iteration to the next until a certain tolerance is met (Bažant and Kim, 1979a; Bažant *et al.*, 1982). The integral-type formulation would of course be inefficient for large structures, and for that purpose it would be preferable to develop a rate-type form of the constitutive equation with a stress-dependent adaptation parameter.

Remark. An efficient description of the adaptation non-linearity is possible with the model in the Addendum to Chapter 2).

3.3.5 Non-linearity at high stress and multiaxial viscoplasticity

Modelling of this non-linearity is important for an accurate determination of the safety of a structure against collapse under a long-time load, especially when long-time buckling is involved. The uniaxial formulation of this non-linearity, with applications to long-time stability of columns, has been the most researched type of non-linearity in creep of concrete. Most of the previous studies have used a non-linear generalization of the rate-of-creep method (Dischinger method), which corresponds to the Maxwell rheologic model with stress-dependent viscosity, modeling behaviour that is generally termed viscoplasticity. Although the Maxwell model is inadequate in the service stress range, it appears more realistic for high stresses near collapse. This is no doubt due to a transition from linear aging viscoelasticity to a plastic type of response as the stress is increased from the service stress range to the failure range.

Various types of the constitutive law based on the aging Maxwell model with stress-dependent viscosity were used in uniaxial analyses of column buckling by Manuel and McGregor (1967), Bažant and Tsubaki (1980), Mauch and Holley (1963), and others. Although some approximate analytical solutions were attempted in the early work, the recent more realistic analyses of non-linear column buckling utilized step-by-step integration in time, with iterations in each step. This algorithm appears to converge, although near the collapse load the convergence is, of course, slow (partly because the creep problem must be solved under load rather than displacement control).

For the purpose of finite element modeling of nuclear vessels subjected to accidental high-temperature loads, some triaxial forms of the non-linear creep law of concrete have been recently developed by Marchertas and Kennedy (1983). It seems reasonable to assume that this triaxial formulation should represent a time-dependent viscous analogy to a triaxial plasticity-type model for the non-linear deformations of concrete. Such a generalization can be obtained if the plastic proportionality parameter used in a plasticity model is replaced with a stress-dependent viscosity, while discarding the continuity condition of the theory of plasticity based on the subsequent loading surfaces which formulate the hardening–softening rule. These formulations have been coded in finite elements, using step-by-step integration in time, with iterations in each step. Reasonable agreement with some limited test data for high-temperature behaviour was demonstrated; however, a much more extensive experimental verification as well as theoretical development will probably be needed in the future.

3.4 EFFECTS IN AGING LINEARLY VISCOELASTIC STRUCTURES

Aging linearly viscoelastic material behaviour gives rise to a variety of effects in structures, which are by now quite well understood and easier to analyse than the non-linear effects. We will now briefly review the current knowledge of these effects.

3.4.1 Relaxation and shrinkage stresses

Relaxation of a structure is the stress or internal force variation induced by a sudden enforced displacement or inelastic strain such as thermal expansion. If the structure is homogeneous, i.e. its creep properties are the same at every point, all the stresses in the structure vary in proportion, a property which follows from McHenry's analogy (Section 3.2.7). For a non-homogeneous structure, other effects are superimposed and the stress variations at various points of the structure are not proportional. However, the stress variation obtained from the assumption of approximate homogeneity usually dominates.

For a homogeneous structure, it suffices to obtain the elastic stresses (or internal forces) caused by the enforced displacements, and then reduce them in proportion to the relaxation function $R(t, t')$. According to the age-adjusted effective modulus method,

$$\frac{\sigma(t)}{\sigma(t_0)} = E(t_0)R(t, t_0) = 1 - \frac{\phi(t, t_0)}{1 + \chi(t, t_0)\phi(t, t_0)} \tag{3.32}$$

in which t_0 is the age at imposed deformation. However, direct use of $R(t, t_0)$ may be preferable here, since χ is derived from $R(t, t_0)$.

When shrinkage deformations are inhibited by the constraints of a redundant structure, the deformations are imposed gradually rather than suddenly. Again, if the structure is homogeneous the solution may be easily obtained from McHenry's analogy. In general, this leads to the integration of a Volterra's integral equation, easily carried out in a step-by-step manner. It is found, however, that relatively good results are obtained if the evolution of shrinkage strain is assumed to be proportional to the creep coefficient $\phi(t, t_0)$, in which case the age-adjusted effective modulus method may be applied. This yields for the internal shrinkage forces the formula

$$X(t) = \frac{X^1_{sh}\varepsilon_{sh}(t)}{1 + \chi(t, t_0)\phi(t, t_0)} \tag{3.33}$$

in which X^1_{sh} = elastic shrinkage force corresponding to unit shrinkage, and $\varepsilon_{sh}(t)$ = shrinkage strain at age t.

It must be emphasized that this formula should be applied only for the cross-section resultants (axial force or bending moment) and not for the stresses at various points. This is because the applicability of this formula to stresses would require the shrinkage strain $\varepsilon_{sh}(t)$ to be the same not only for all the cross-sections of the structure, in the mean, but also for all the points of each cross-section, which is, of course, not true due to the shrinkage delay from moisture diffusion. As previously discussed (Sections 2.3 and 3.3.1), the actual shrinkage stress values at individual points differ greatly from the values obtained from Eq. (3.33) according to the formulae of the theory of bending.

Figure 3.4 illustrates the accuracy of Eq. (3.33), as compared to the exact aging viscoelastic solution (Bažant and Najjar, 1973). The shrinkage stress predictions according to other simplified methods are also plotted, and are seen to be inferior to the age-adjusted effective modulus method (see also Chiorino *et al.*, 1984, pp. 161–3). Nevertheless, the predictions according to the improved Dischinger

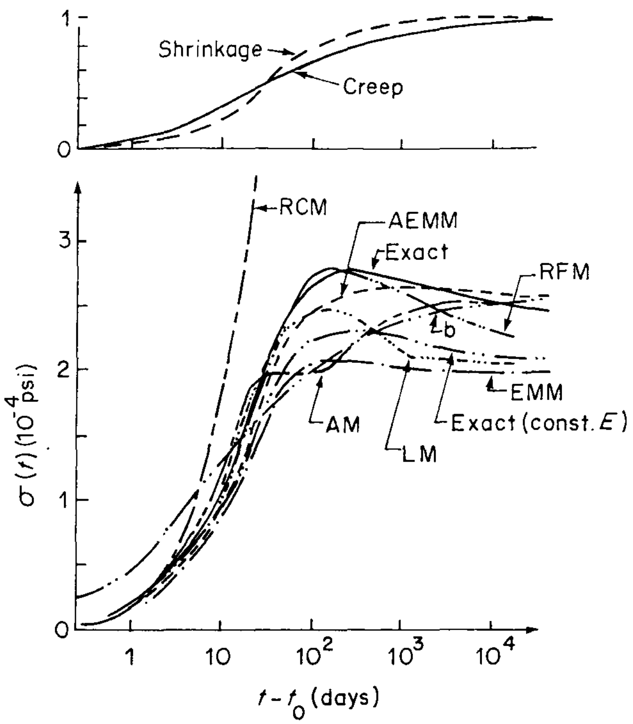


Figure 3.4 Shrinkage stresses

method (CEB–FIP Model Code, 1978), which require solution of a differential equation, are still quite good. Křístek *et al.* (1982) obtained another simple solution assuming that the delayed elastic response is linear in the load duration (however, they demonstrated superiority only to the rate-of-creep method).

3.4.2 Composite and inhomogeneous cross-sections of beams

In a cross-section in which the creep properties of the material are not the same at all the points, creep causes stress redistributions. Further time-dependent stress variations are caused when the cross-section is built in stages and some parts carry load before the remaining parts are cast or attached.

The exact analysis of a composite or inhomogeneous cross-section according to aging linear viscoelasticity leads to a system of two Volterra integral equations if the cross-section resultants (axial force and bending moment) are known. Applying the operator formulation, Huet (1970) adopted this approach and formulated (in 1973–75) the general solution in a powerful matrix operator form. A general solution was given by Bažant (1971), and in the operator form by

Lazić and Lazić (1977, 1984b), and also, in terms of the so-called reduced relaxation functions, by Mola (1982a, b).

In today's climate, however, it is easy to obtain an accurate solution with a computer by step-by-step methods; e.g., Bažant (1962, 1966b, Bažant and Najjar (1973) 1973, 1975); Neville *et al.* (1983); Chiorino *et al.* (1984). For a simple approximate solution, it is usually sufficient to apply the method of the aging coefficient, χ, although with certain types of compliance function an appreciably improved accuracy may be obtained by the method of the improved aging coefficient, χ_F (Lazić and Lazić, 1981, 1985; Lazić, 1982).

When the structure is redundant, the cross-section resultants are normally not known. The exact analysis leads to the system of Volterra integral equations coupled with a differential equation of the beam, which was solved directly by Bažant (1970). This can also be reduced to a system of Volterra integral equations which contain no unknown functions, only the redundant forces. By solving this system, one obtains the cross-section resultants, and then the above-mentioned method of cross-section analysis should be applied. This method is complicated, although the exact solutions for redundant composite structures can be obtained for the rate-of-creep method (Lazić and Lazić, 1982b), as well as for non-aging elasticity (Lazić, 1975). An accurate numerical solution can be carried out more easily by using the incremental quasi-elastic form of the stress–strain relation for the time steps, which leads to a series of linear elasticity problems for an elastic redundant structure with composite cross-sections (e.g., Bažant, 1964a, b, 1970, Huet, 1980). Again, however, exact solutions are usually not needed for practice because the age-adjusted effective modulus method was shown to be quite accurate for these problems (see the comparison in Fig. 3.5).

The method of the improved aging coefficient χ_F as well as Lazić's theorems can be utilized, too. In Lazić (1982), the general case was treated by using the improved aging coefficient with practical applications in various structural creep problems. Under the assumption that the redundant forces depend on the relaxation function linearly, Lazić's theorems lead to a system of algebraic equations for the redundant forces (Lazić, 1985a). This method was shown to be quite accurate compared to the exact viscoelastic solutions.

The solution of the composite cross-section is analogous to its elastic analysis for arbitrary initial strains. As an extension of the creep solutions according to incremental laws (see Chapter 2), as well as the solution by the Trost method (1967), and an early special approximate version of the age-adjusted effective modulus method, the general case was treated by using the age-adjusted effective modulus method by Bažant and Najjar (1973). More recently, Dilger (1982) and Neville, Dilger and Brooks (1983) appraised this approach as the most powerful one and demonstrated practical applications in various situations. We will now briefly outline this approach.

Consider a reinforced uncracked concrete cross-section with a single layer of steel. It is convenient to introduce a transformed cross-section characterized by

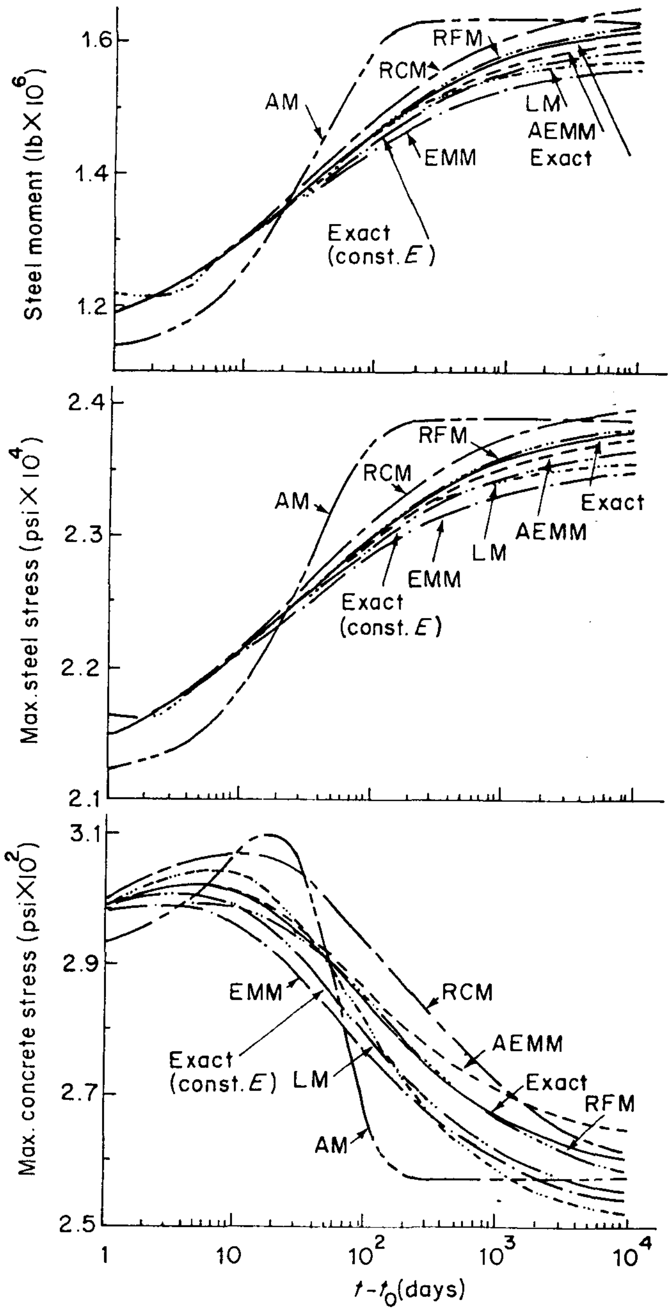


Figure 3.5 Changes of stresses and girder moment due to creep in steel–concrete composite beam (100 psi = 0.6895 N/mm^2; 10^6 in × 1 b = 112.5 kN × m) $t_0 = 7$ days (after Bažant and Najjar, 1973)

the creep moduli ratio

$$n^* = E_s/E''_c(t, t_0) = n_0[1 + \chi(t, t_0)\phi(t, t_0)] \tag{3.34}$$

in which E_s = elastic modulus of steel, and $n_0 = E_s/E_c$ = elastic moduli ratio based on the standard concrete modulus E_c at age 28 days. The calculation consists of two stages (Sattler, 1950; Bažant, 1962, 1966a, 1975; Dilger, 1982). First, unrestrained creep and free shrinkage is allowed to take place in each part of the cross-section. Elastic stresses which cancel these deformations are then calculated on the basis of transformed cross-section properties, and their resultants over the entire cross-section, consisting of axial force N^* and bending moment M^*, are evaluated. Subsequently, these resultants are applied in the opposite (negative) sense to the transformed composite cross-section and the stresses produced by this loading are then superimposed. If the structure is redundant (Bažant and Najjar, 1973), one must first calculate the structural loads which are in equilibrium with the cross-section forces $-M^*$ and $-N^*$, and then superimpose the effect of these loads calculated for the redundant elastic structure with the transformed elastic composite cross-sections. When the steel in the cross-section is prestressed, one must include the prestress in the cross-section resultants M^* and N^*.

The aforementioned procedure can also be applied to composite sections made of concretes with different properties or different ages, although the error of the age-adjusted effective modulus might not be so small in this case (Chiorino, 1984, p. 266). Instructive examples were given by Chiorino (1984, pp. 208–17) and a general solution was presented in Bažant (1962, 1966a) and in Huet's works during 1975–80 (Huet, 1978, 1980).

It should be noted that the composite action of the cross-section alters the stress relaxation in prestressing steel, compared to a solution of the so-called intrinsic relaxation, i.e. the stress decrease in steel calculated without taking the composite action into account. A correction factor α_r which multiplies the intrinsic relaxation in order to obtain the correct stress relaxation in steel in a composite cross-section was calculated in a step-by-step procedure by Tadros *et al.* (1977a, 1979) and was given in a chart form by Dilger (1982).

The aforementioned calculation procedure for composite cross-sections can also be used for an approximate linear treatment of the differences in creep and shrinkage caused by drying or temperature change in the cross-section (Bažant *et al.*, 1975). It may also be adapted to analyse the effect of cracking or strain-softening in certain parts of the cross-section, occurring simultaneously with creep (Křístek and Bažant, 1987).

To illustrate the foregoing cross-section analysis with the age-adjusted effective modulus method, the formula for the loss of compression force in a thin slab of a steel–concrete composite statically determinate beam may be given:

$$\Delta N_c(t) = -\frac{\phi(t, t_0)N_c(t_0) + \Delta\varepsilon_{sh}(t)E_c(t_0)A_c}{1 + \chi(t, t_0)\phi(t, t_0) + A_c/nA_s(1 + e^2A_s/I_s)} \tag{3.35}$$

in which A_s and I_s = area and central moment of inertia of the steel girder, e =

eccentricity of the thin slab centroid with respect to the steel girder centroid, $n = E_s/E_c(t_0)$, A_c = area of the concrete slab (whose centroidal moment of inertia I_c is assumed negligible). The moment change in the steel girder then equals $eN_c(t)$. The creep deflection of the composite statically determinate girder can then be computed from the moment in the steel girder.

An equally simple formula is possible for the prestress loss in a statically determinate girder with a single tendon. In this case, however, the gain of accuracy compared with the classical effective modulus method is relatively small.

3.4.3 Inhomogeneous redundant beam structures

When a beam structure consists of parts that have significantly different ages, stress redistributions occur due to creep. Other types of structural inhomogeneity, which can be analysed linearly as a good approximation, are differences in the cross-section size or in the degree of drying along the structure. Overall, the stresses in an inhomogeneous structure are transferred due to creep from the parts which creep more into the parts which creep less.

The age differences, which typically arise in segmental construction of box girder bridges, were initially analysed on the basis of the rate-of-creep method (see e.g. Finsterwalder and Knittel, 1955; Bažant, 1961), in which case the problem was reduced to a system of first-order differential equations for the redundants. However, as demonstrated later (Bažant and Najjar, 1973), the age-adjusted effective modulus method is both more accurate and simpler, consisting of an elastic analysis of a frame structure with a non-uniform elastic modulus.

A simple illustrative example is the shear force produced at midspan of a fixed-end beam consisting of two identical cantilevers which are of equal length and are connected at midspan by a hinge. This force, X, is obtained by solving the following compatibility equation based on the age-adjusted effective modulus method:

$$\begin{aligned}\{[1+\chi(t,t_0)\phi(t,t_0)]f_a+[1+\chi(t-\Delta,t_0)\phi(t-\Delta,t_0)]f_b\}X(t)\\+[\phi(t,t_0)-\phi(t_1,t_0)]\delta_a+[\phi(t-\Delta,t_0)-\phi(t_1-\Delta,t_0)]\delta_b=0\end{aligned} \qquad (3.36)$$

in which f_a, f_b are the unit flexibilities of the left and right cantilever, δ_a and δ_b are their deflections due to applied loads, and Δ = age difference (the left cantilever is older). Each cantilever is assumed to be of uniform age and to be loaded when its age is t_0; t_1 is the age of the older cantilever when the two cantilevers are joined by a hinge at the midspan. For a detailed exposition, see, e.g. Bažant and Najjar (1973), Bažant (1975), Chiorino *et al.* (1984).

In the foregoing type of problem, the stresses in the older cantilever are constant from time t_0 to time t_1, after which they vary gradually. This type of stress variation is not linear in the creep coefficient $\phi(t, t_0)$, as is assumed in the age-adjusted effective modulus method, and therefore the error of this method is

larger. A certain useful correction for this type of stress history was formulated by Křístek *et al.* (1982).

3.4.4 Steel–concrete systems (cable-stayed bridges)

A beam or frame in which concrete members interact with steel members is a special case of an inhomogeneous structure, in which one part of the structure exhibits creep, the other does not. The most important structures of this type are perhaps the cable-stayed bridges with a concrete girder. Due to creep in the girder, the internal forces are gradually transferred from the concrete beam into the cable stays, which causes the forces in the cables to grow in time. A special case occurs when the initial cable forces are such that the initial deflections of the girder at the cable support points are zero. In that case there the cable forces vary only due to axial creep shortening and shrinkage of the girder, but not due to bending creep of the girder (provided that homogeneous linearly viscoelastic material properties are assumed). Again, analysis according to the age-adjusted effective modulus method is relatively simple. Also pertinent is a general solution of redundant homogeneous structures with elastic members presented by Chiorino *et al.* (1986), based on the works of Mola (1979, 1981, 1982a, b) on reduced relaxation functions.

3.4.5 Change of structural system

A change of the structural system is often involved in the construction procedure of a modern prestressed concrete structure and has been analysed in many works; see e.g. Bažant (1962, 1964a, b, 1966a, 1975), Bažant and Najjar (1973), Neville *et al.* (1983), Chiorino *et al.* (1984), Chiorino and Mola (1982) (restraints introduced at various times), Chiorino *et al.* (1986) (delayed elastic restraints), etc. For instance, bridge spans are cast initially as simply-supported beams, and later, while these beams already carry its dead load, they are connected above the supports. If they were not connected, the simply-supported beams would continue deflecting due to creep and the relative rotations of the beams above the supports would gradually grow. However, the growth of the relative rotations is restrained after the simply-supported beams are connected. This is equivalent to enforcing on the simply-supported beams an opposite rotation above the midspan, causing a negative bending moment which gradually builds up at the midspan location (Fig. 3.6a).

Another typical example comes from cantilever construction of prestressed concrete bridges. The two halves of the bridge span are initially built as cantilevers and later, while they already carry the dead load, they are connected at midspan. This prevents further growth of relative rotations and relative deflections of the two cantilevers at their ends, thus causing a gradual build-up of a positive bending moment at the midspan. If the creep in the two connected cantilevers is not the same, for example due to differences in age, then a shear force

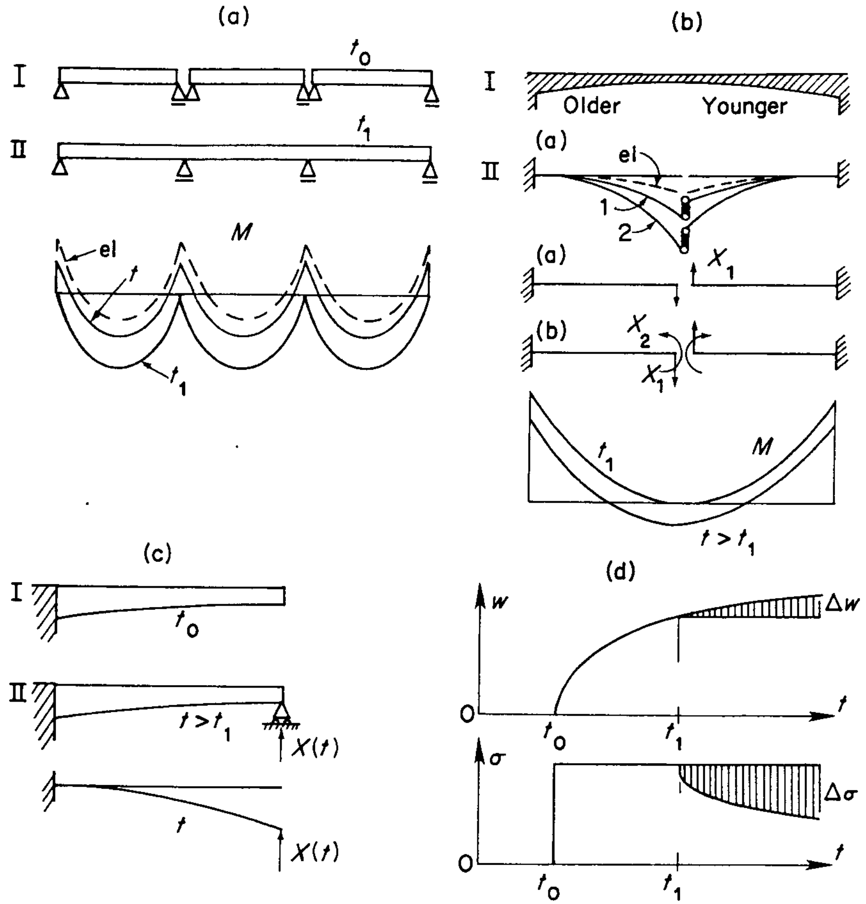


Figure 3.6 Examples of changes in the structural system

also gradually builds up at the midspan. Creep and shrinkage effects considering the load histories from a sequence of construction phases for a prestressed concrete bridge were analysed by Haas (1982) (Fig. 3.6b).

If a hinge is used at midspan in order to avoid the positive bending moment, then a shear force gradually builds up after the instant of joining if the two cantilevers are of different ages. This is normally the case in segmental construction where the same traveller truss is used for casting or assembly of the bridge segments. The use of a hinge at midspan, however, is not a good design practice because deflections are considerably larger than for a span with a monolithic connection at midspan. A change of slope at midspan, which is likely to develop due to the random nature of creep, creates serious serviceability problems.

Another example, also from segmental bridge construction, is the procedure in which the end span is built as a cantilever from the pier towards the abutment. Subsequently, a bearing is inserted between the end of the cantilever and the abutment. If it were not inserted, the cantilever end would continue deflecting, and since this is prevented by the bearing, a vertical reaction gradually builds up at the abutment (Fig. 3.6(c)).

Examples of changes in the structural system are not limited to bridge construction. We may cite for example shored construction of slab buildings, in which the shores are removed, then replaced by reshores, which are later removed again (Aguinaga-Zapata and Bažant, 1986).

Let u_{el} be the displacement or rotation of the structure in the sense of the later imposed constraint, evaluated for the initial structural system *I*. Let t_0 be the age of concrete at load application, and t_1 the age at the time of adding the constraint. The displacement which would take place in the sense of the constraint if the constraint were not added is $u_{el}[\phi(t,t_0)-\phi(t_1,t_0)]$. We can consider this displacement to be imposed on the structure in the opposite sense. The stresses produced by this gradually increasing displacement are reduced due to creep by the factor $1+\chi(t,t_1)$. From this type of consideration, one can conclude that generally an internal force or stress in a redundant structure after a change of structural system is expressed as (Bažant, 1975; Křístek and Bažant, 1987):

$$X(t) = X^{\mathrm{I}} + (X^{\mathrm{II}} - X^{\mathrm{I}})\frac{\phi(t,t_0)-\phi(t_1,t_0)}{1+\chi(t,t_1)\phi(t,t_1)} \tag{3.37}$$

in which X^{I} and X^{II} are the values of the internal force calculated for an elastic structure with structural system I or II respectively. The long-time deflection due to creep, $u(t)$, after the structural system I is changed into the structural system II at time t_1 may be calculated as

$$u(t) = u^{\mathrm{I}}\phi(t,t_0) + (u^{\mathrm{II}} - u^{\mathrm{I}})[\phi(t,t_0)-\phi(t,t_1)] \tag{3.38}$$

where u^{I}, u^{II} = elastic deflections corresponding to structural systems I and II, respectively. The first term represents the usual creep deflection without the effect of the change of structural system, and the second term is the creep deflection due to the change of structural system at time $t_1(t_1 \geqslant t_0)$.

Compared to the exact aging viscoelastic solution, the foregoing formula (Eq. 3.37) is very accurate if the time-lag $t_1 - t_0$ is small. If it is large, then the evolution of displacement

$$u(t) = u^{\mathrm{I}}\phi(t,t_0) + (u^{\mathrm{II}} - u^{\mathrm{I}})[\phi(t,t_0)-\phi(t_1,t_0)]$$

differs more substantially from the curve of $\phi(t,t_0)$, which means that the assumptions of Bažant's theorem are becoming less realistic, as is illustrated with the typical stress history shown in Fig. 3.6(d). However, a correction may be introduced in a two-step analysis in which the time at which the structural system is changed is adjusted on the basis of a preliminary analysis of stress relaxation, which was done by Křístek *et al.* (1982).

3.4.6 Cyclically built structures

Numerous successive changes in the structural system occur when a bridge girder is built using the span-by-span construction technique. For example, a launching truss may be used to cast *in situ* a span-long section of the girder, which usually extends from one inflection point of the elastic deflection curve to the next. Alternatively, each span may be assembled on a supporting truss from precast segments, and then prestressed. In either case, the degree of structural redundancy increases each time when a new span resting on a new pier is added. At the same time, the age of concrete in each span is different. The changes in the structural system as well as the differences in age cause certain appreciable, although usually not very large, redistributions of internal forces. Their calculation according to aging viscoelasticity is possible with a computer; however, it is quite complicated.

A considerable reduction in the number of unknowns may be achieved by taking advantage of the periodicity of construction. For this purpose, one needs to assume the bridge to consist of infinitely many spans, an assumption which seems to be acceptable for all the spans except the first and last. The problem may then be reduced to a system of several Volterra integral equations coupled with a periodicity condition for the unknowns. A solution of this type has been presented and significance of the results discussed (Bažant and Ong, 1983).

3.4.7 Shored construction of slab buildings

Another type of cyclical construction are the slab-column systems of tall buildings, constructed floor by floor. In this type of construction, redistributions of loads between the floors are considerably affected by the use of temporary shoring of several floors near the top, frequently followed by reshoring.

Until recently, this problem used to be analysed elastically, by the method of Grundy and Kabaila. A solution according to aging linear viscoelasticity is, however, quite feasible with a computer, and has been carried out (Aguinaga-Zapata and Bažant, 1986). The results confirm that creep and shrinkage cause in this case only relatively small deviations from the elastically calculated load distributions among the top floors; however, the consideration of creep and of the construction sequence is important for deflections.

3.4.8 Box girder bridges and shear lag

In order to maintain the proper vertical alignment, deflections of segmentally built box girder bridges need to be carefully predicted and compensated for during construction. The analysis needs to take into account the differences in age from segment to segment and the sequential applications of dead loads and

prestress. The problem has recently been analysed by the layered finite element method (Bažant and Kim, 1987).

Large-span box girder bridges have shown some propensity to develop excessive deflections or cracking well before the end of their design lifetime (typically 50 years), as revealed by the problems experienced with some older bridges of this type. Therefore, it is advisable to calculate the statistical distribution of the predicted deflections and stresses, and design not for their means but for extreme values such as 95 per cent confidence limits.

Another important aspect of creep in box girder bridges is the effect of shear lag. This effect is well known for short-time elastic behaviour; however, it is equally important for long-time behaviour (Křístek and Bažant, 1986) (Fig. 3.7). The linear aging viscoelastic analysis of this effect can be based on McHenry's analogy and can utilize an elastic program for the folded plate theory.

The shear lag is particularly important for bridges of a large width-to-span ratio. It affects deflections and also modifies the effect of a change in the structural system, such as a change from simply-supported girders to a continuous girder.

3.4.9 Stresses in shells, containments, tanks, and slabs

In circular tanks resting on a stiff foundation, as well as nuclear containment shells, the shrinkage in the cylindrical wall is generally much higher than the shrinkage in the foundation slab. The resulting shrinkage stresses in the shell wall need to be reduced for the effect of creep (Bažant *et al.*, 1975; Fig. 3.8). Furthermore, the differences in drying conditions at various points across the thickness of the cylindrical wall result in significant differences in creep, which produce residual self-equilibrated stresses. In a simplified manner, these stresses have been analysed according to linear aging viscoelasticity.

Bridge slabs are sometimes made by laying precast parallel beams of box cross-section next to each other and then connecting them with shear keys. When the creep properties of the adjacent box beams are not the same, for example due to age differences, one needs to take into account the transverse redistribution of the load among the parallel beams.

3.4.10 Shear deflections and torque

Similarly to elastic behaviour, shear deformations in thin-walled beams can significantly contribute to creep deflections. A simplified analysis of this problem, based on the truss analogy, has been made by Dilger and Abele (1974); see also Neville *et al.* (1983). A similar analysis of creep deformations is possible for torsion of thin-walled girders. Shear lag in prestressed concrete box girder was analysed by Křístek and Bažant (1985) (see also Křístek and Bažant, 1987).

As far as shear friction at the steel–concrete interface is concerned, there are recent test results on cracked concrete available which show a considerable

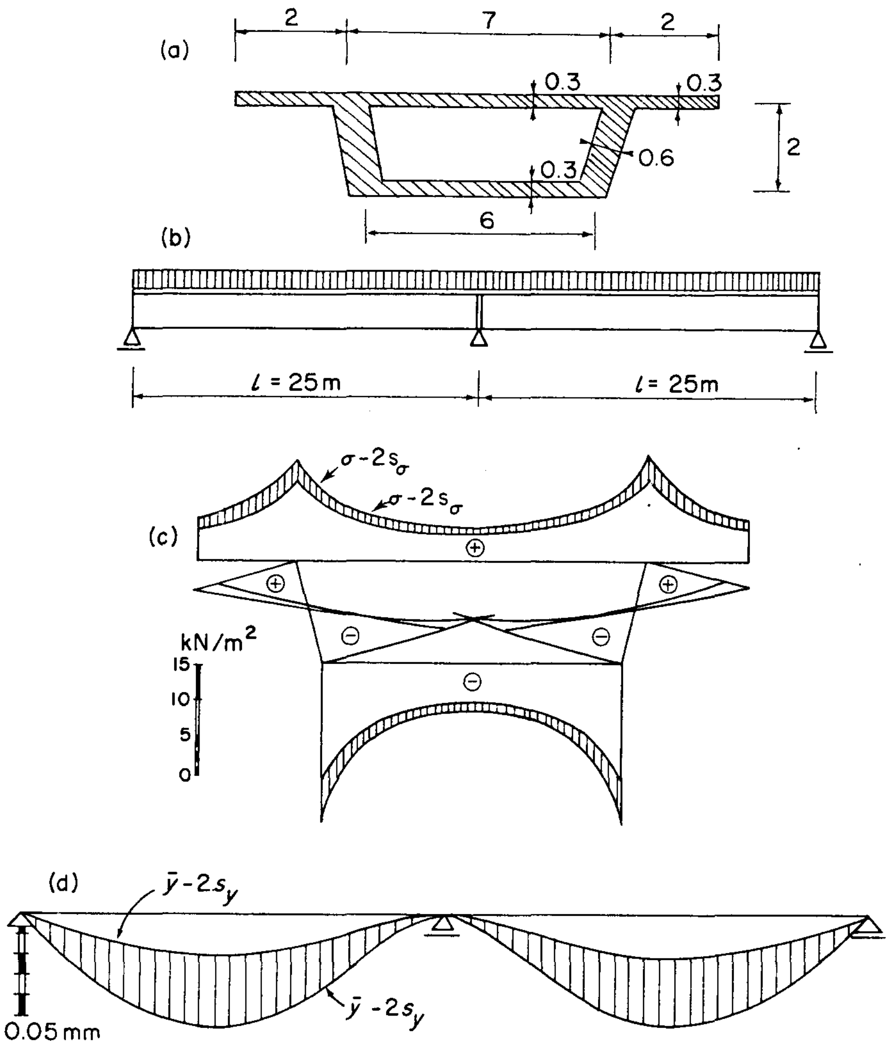


Figure 3.7 Effect of creep on a box girder bridge (Křístek and Bažant, 1985). (a) Cross-sectional dimension; (b) longitudinal configuration; (c) longitudinal normal stresses, mean ± standard deviation; (d) deflections, mean ± standard deviation

increase of the shear displacement and crack opening during constant shear stress (Frénay, 1985). The relations among the shear stress, normal stress (or normal confining stiffness), shear displacement, and crack opening depend on the shear reinforcement ratio, the mix of concrete, and the logarithm of loading duration. Although stresses and displacements are related non-linearly, those relations can be approximated by an average relation for design purposes. Without further

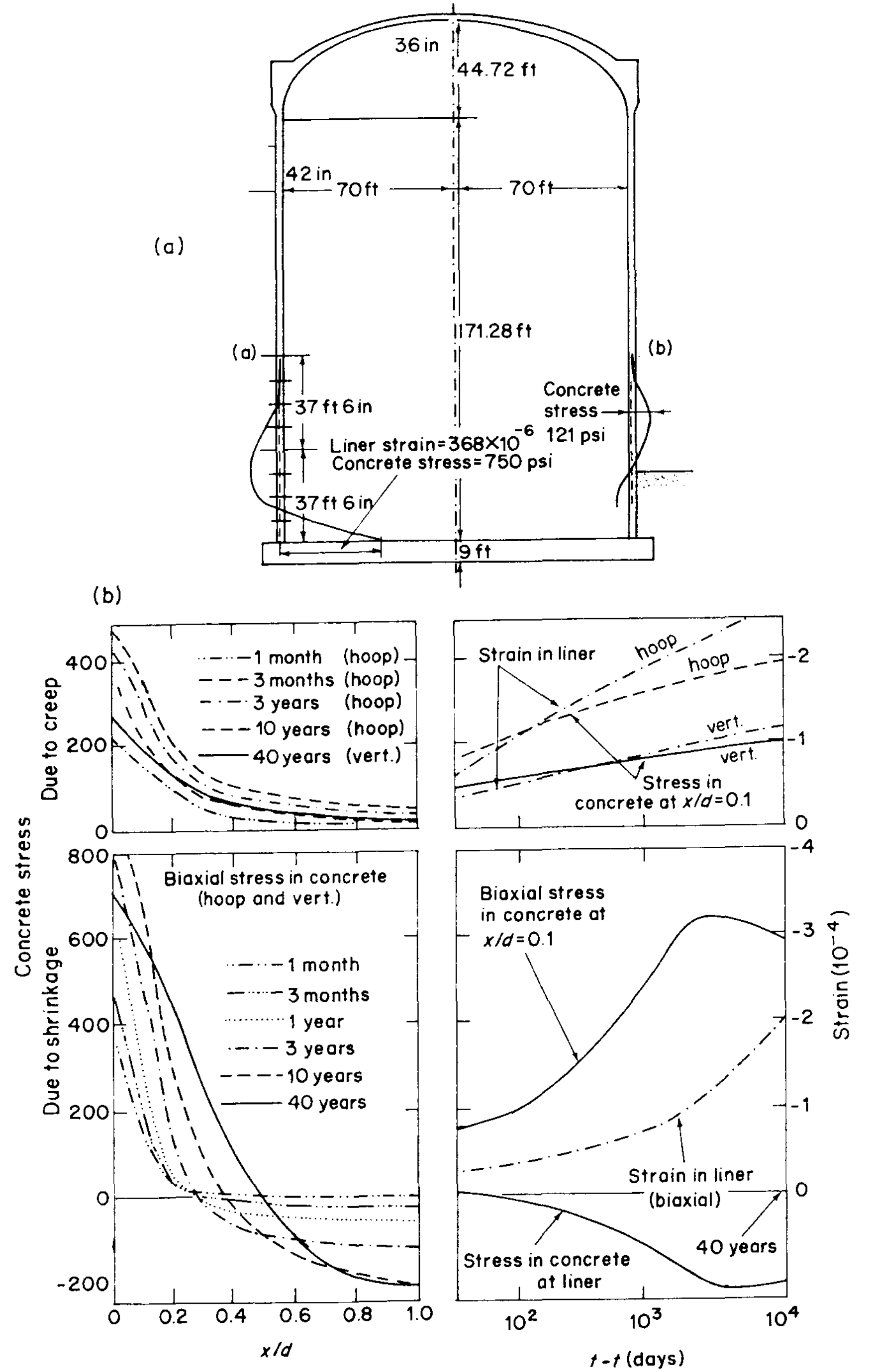


Figure 3.8 (a) Vertical section of typical containment shell and effects of shrinkage difference between shell and foundation slab (1 ft = 0.305 m; 1 in = 25.4 mm; 1 psi = 6.89 kN/m²). (b) Approximate concrete stress and linear strains caused by shrinkage and creep in general cross-section of wall (no cracking assumed); after Bažant *et al.* (1975)

elaboration it seems that the shear creep coefficient is rather close to the uniaxial creep coefficient for plain concrete.

3.4.11 Thermal stresses in nuclear containments and other structures

In nuclear reactor containments, thermal stresses are particularly important (e.g. Rashid, 1968), and they are significantly reduced by creep (Fig. 3.9). A large body of literature has been devoted to this problem (see the Transactions of SMiRT between 1971 and 1987). Due to the relatively large thickness of these structures, the effect of drying may probably be neglected in many cases. The problem can then be analysed according to aging linear viscoelasticity, in which the creep viscosities as well as evolution of the equivalent hydration period t_e depend on temperature, for which the activation energy concepts may be used. A general-

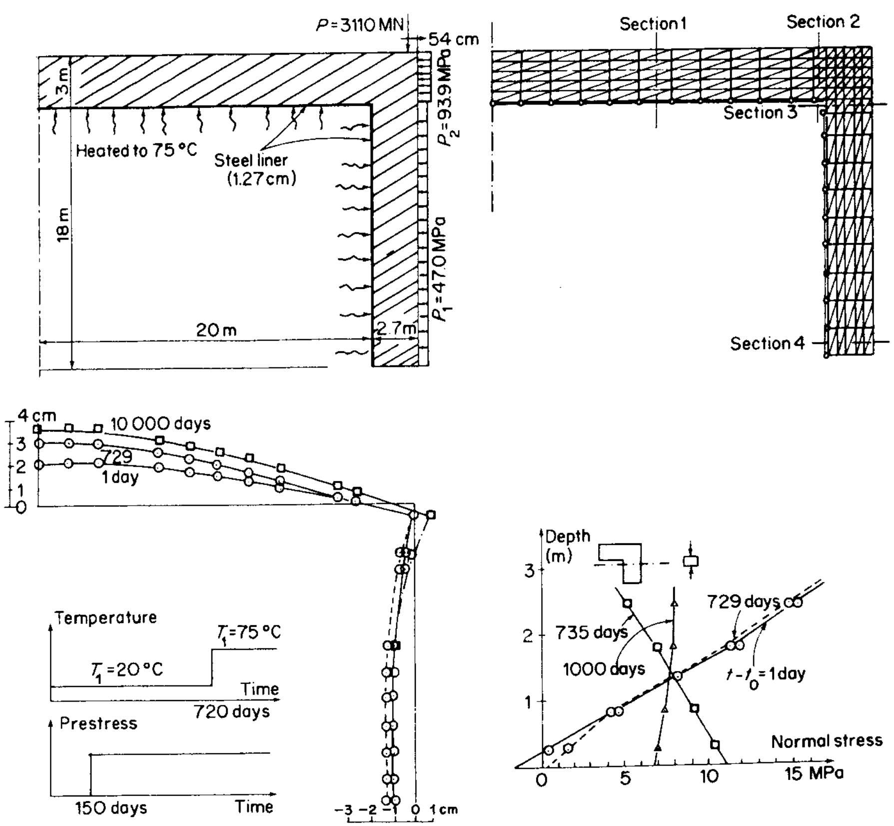


Figure 3.9 Example of a finite element creep analysis of a PCRV with age- and temperature-dependent properties (after Bažant and Rossow, 1981)

purpose finite element program (CREEP 80) which implements this analysis has been developed (Bažant *et al.*, 1981).

Detailed calculations of the effects of creep and shrinkage in large vacuum shell structures for nuclear power plants have been conducted by Huterer *et al.* (1985a, b).

3.4.12 Effect of hydration heat in dams

In concrete gravity dams, hydration heat can produce large thermal stresses. Their reduction by creep is important, and this problem provided an early impetus to the development of the creep theory (Maslov, 1940; Gvozdev, 1953; Babuška, 1958; Hanson, 1953; Zhu and Wang, 1976). The analysis must take into account the effect of temperature on creep viscosities as well as the rate of hydration, manifested in the evolution of the equivalent hydration period. Drying, however, can generally be neglected due to the large thickness of the structures. The objective of the analysis is to determine the maximum size of concrete blocks, as well as the requirements for cooling by pipes embedded in concrete. The analysis also needs to take into account the cooling of concrete before it is poured.

3.4.13 Periodic environment and spectral structural analysis

The most efficient way to treat the effect of random environmental humidity or temperature on an ageing linearly viscoelastic structure is the spectral method. The ergodic stationary random process describing the environment is represented by its power spectrum. Due to linearity of the problem, one may solve independently the response of the structure to a single periodic (sinusoidal) component of the spectrum and then superimpose the results. The diffusion equation needs also to be simplified to a linear form, with a constant diffusivity, and so must be the dependence of free shrinkage on pore humidity (Fig. 3.10). The effectiveness of this approach has recently been demonstrated (Tsubaki and Bažant, 1982; Bažant and Wang, 1984, 1985). A solution for a periodic environment was also presented by Diamantidis *et al.* (1984).

The spectral approach brings to light two important effects:

1. The depth of penetration of various spectral components of the environment depends strongly on their frequency; it increases as the frequency decreases (i.e. the period lengthens).
2. The period of the environmental component interacts only with those the relaxation times in the relaxation spectrum of concrete which are of the same order of magnitude.

Thus, neither the short-frequency components are affected by long-time

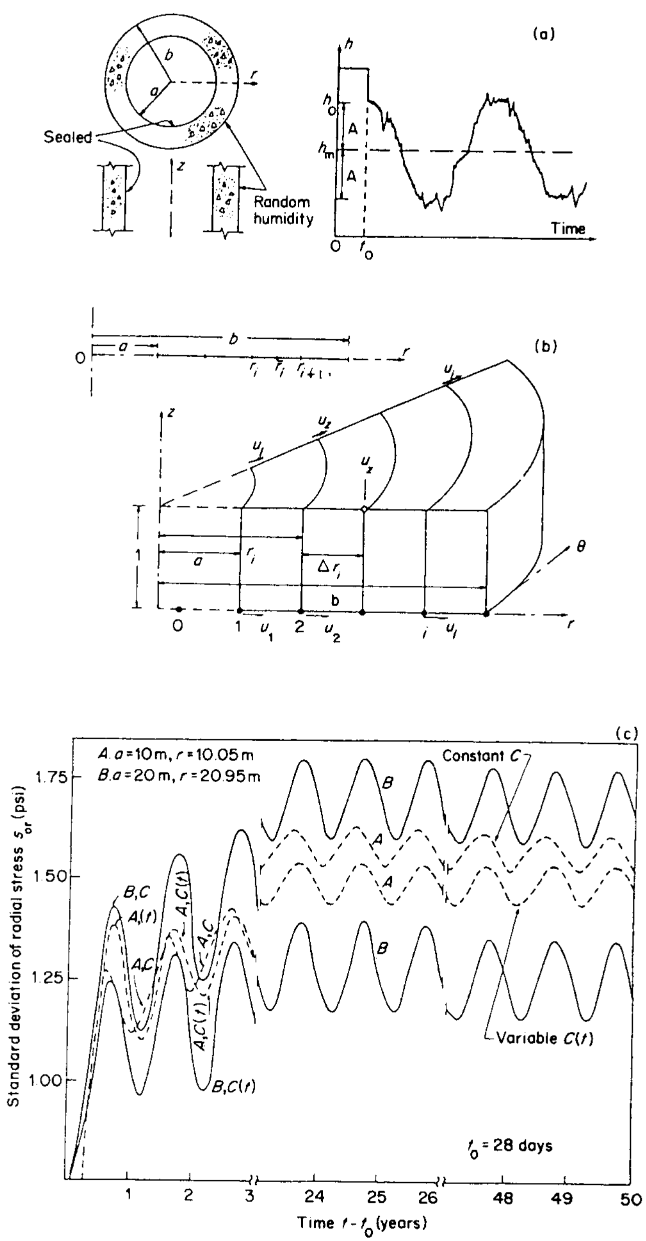


Figure 3.10 (a) Cylindrical vessel and random variation of surface humidity h; (b) one-dimensional discretization of cylindrical wall; (c) time evolution of standard deviation of radial stress (after Bažant and Wang, 1985)

creep, nor are the long-frequency components affected by short-time creep. Consequently, the effects of the random fluctuating part are very different for structures of different sizes or shapes, a fact which is ignored in the present design practice.

3.4.14 Viscoelastic stability: columns and shells

When the non-linearity of creep is taken into account, the effect of creep may be described in terms of the column or shell lifetime, expressing the time at which the buckling deflections due to initial imperfections approach infinity (based on a linear bending theory). In the linear viscoelastic approximation, however, there is no lifetime; the deflection becomes infinite either suddenly (elastically) or at infinite time. Thus, in the linear context, the design for stability must be based either on the attainment of intolerably large deflections due to initial imperfections, or on the value of the long-time critical load which causes infinite deflection at infinite time (Fig. 3.11). However, the latter approach, widely used in non-aging viscoelasticity, is less than satisfactory for concrete, since the evolution of creep properties as a function of age has no effect on the long-time critical load. This load depends only on the asymptotic creep properties at an infinite age at loading. Thus, the first approach, based on the magnitude of buckling deflections, is preferable.

In the early researches, buckling deflections were calculated from simplified creep laws by reducing the problem to a differential equation with age-dependent coefficients; this was done for the rate-of-creep method (Naerlović, 1960) as well as the Arutyunian formulation. For the latter, the problem was solved in terms of the incomplete gamma functions (Bažant, 1968b). Distéfano 1965, Distéfano and Sackman (1967), and Sackman (1963) studied the problem from the viewpoint of history integrals and Tauberian theorems. Subsequently, Bažant and Najjar (1973) presented creep buckling formulae based on the age-adjusted effective modulus method and showed that their predictions are very close to the accurate solutions obtained in step-by-step calculations using the integral-type creep law.

Using the integral operator formulation, Lazić (1985b) developed for the compliance function of any given form an accurate solution for the long-time critical load of a composite column, applicable to any boundary conditions and a constant cross-section. In two special cases, the long-time critical load linearly depends on the relaxation function (when the centres of gravity of concrete and of all the steel parts coincide and when the cross-section becomes homogeneous).

For a reinforced column of a symmetric cross-section, the creep deflection in excess of the elastic short-time deflection (Fig. 3.11; Bažant, 1968b) is given by

$$w_{\max}(t) = \frac{w^0_{\max}}{(1 - P/P^0_{\mathrm{cr}})} \frac{1 - (I_s/I_{\mathrm{tr}})}{(P''_{\mathrm{cr}}/P) - 1} \frac{\phi(t, t_0)}{1 + \chi(t, t_0)\phi(t, t_0)} \tag{3.39}$$

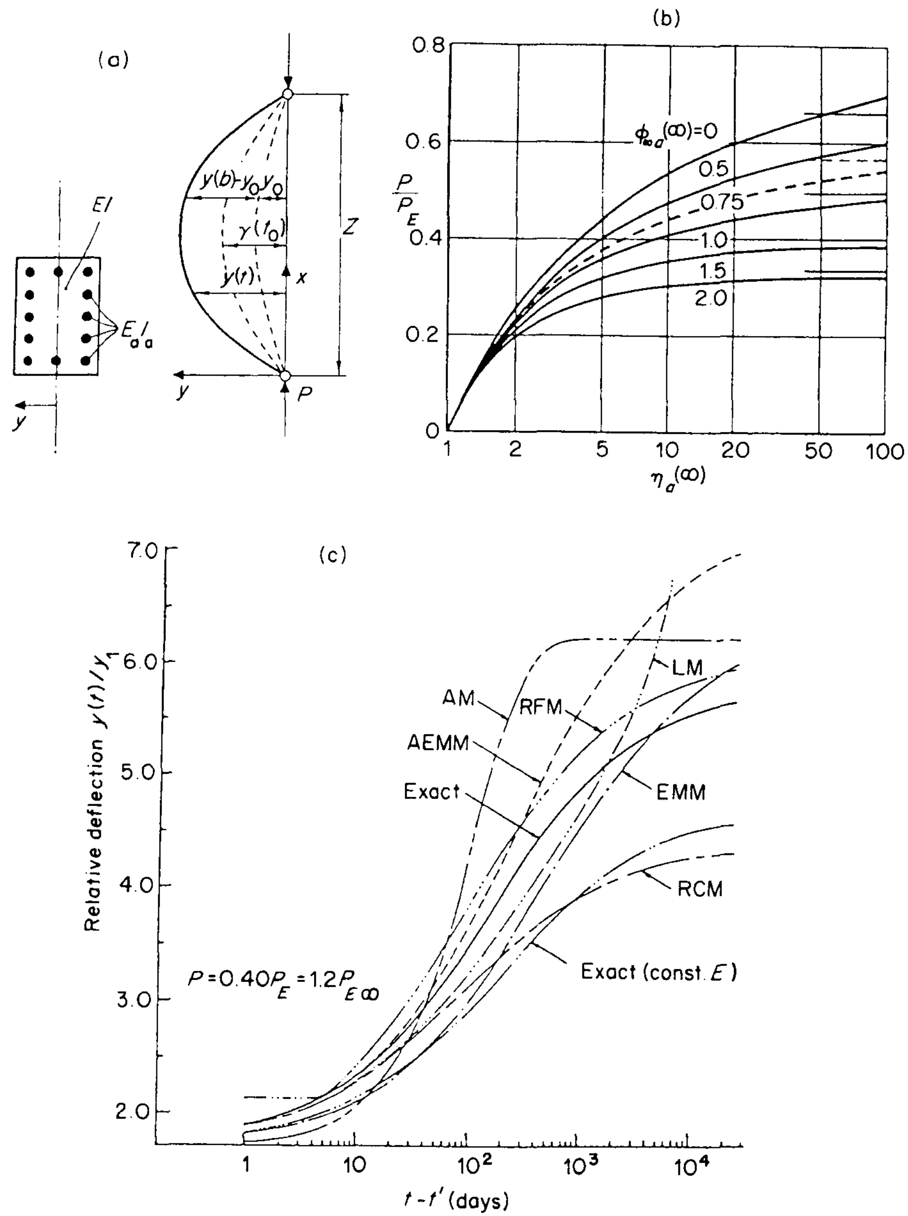


Figure 3.11 Creep buckling deflection of reinforced concrete column (a) calculated according to linear aging viscoelasticity (b, c); a—closed-form solution in terms of incomplete gamma function for aging standard solid (after Bažant, 1968); b—exact numerical step-by-step solution for general aging creep law compared with other methods (EMM = eff. modulus, AEMM = age-adjusted eff. modulus, RCM = rate of creep, RFM = ratio of flow, LM = Levi's, AM = Arutyunyuan's method (after Bažant and Najjar, 1973)

in which $w_{max}(t)$ = maximum ordinate of the deflection curve at time t, w^0_{max} = maximum ordinate of the initial imperfect shape of the column, assumed to be sinusoidal; P_{cr}, P''_{cr} = buckling loads of an elastic column with concrete modulus $E_c(t)$ or $E''(t, t_0)$; P = axial compression load of the column, t_0 = age of concrete at load application; I_s, I_{tr} = centroidal moments of inertia of steel and of the entire transformed cross-section with modulus $E''(t, t_0)$. Similar equations can be written for creep buckling deflections of continuous beams, frames, arches, plates, shells, etc., in complete analogy to elastic stability theory.

The fact that coefficient χ applies to creep buckling just as well as it does to relaxation problems confirms that this coefficient does not really introduce any correction for stress relaxation *per se* and is therefore properly called the aging coefficient (instead of the initially used term the relaxation coefficient).

Long-time stability of concrete structures is often checked by simply replacing the elastic modulus E in the corresponding elastic critical load formula with the effective modulus E_{ef}. The resulting formulae are somewhat simpler than those of the age-adjusted effective modulus method. This approach is the standard practice for checking the long-time stability of concrete shells (see, e.g. Kollár and Dulácska, 1984) and is also implied in the code formulae for columns, e.g. in the ACI Code. Compared to the use of the age-adjusted effective modulus, the use of the effective modulus is generally on the safe side in the case of buckling problems. However, it implies that the actual safety factor for a structure loaded at a high age is less than that for a structure loaded at a low age, while ideally it should be the same.

3.5 NON-LINEAR EFFECTS IN STRUCTURES

The linear analysis just reviewed often does not give an accurate picture of the structural behaviour, and if certain additional non-linear effects are not taken into account, the error may remain so large that little benefit is derived from the linear analysis. We will now review the basic types of non-linear effects.

3.5.1 Buckling and long-time stability

For buckling instability of columns under long-term loads it is important to realize that the specific creep (creep per unit stress) sharply increases as the peak stress is approached. This enables one to obtain, within a geometrically linear theory, an infinite deflection within a finite time. This lifetime serves as a more realistic indicator of long-term stability. In contrast to linear analysis, the lifetime and the critical permanent load that corresponds to this lifetime depend not only on the asymptotic creep properties for very long times, but also on the intermediate and initial creep properties, including the material aging as it interacts with the stress history.

Non-linear buckling of columns (Fig. 3.12) has been studied by Manuel and

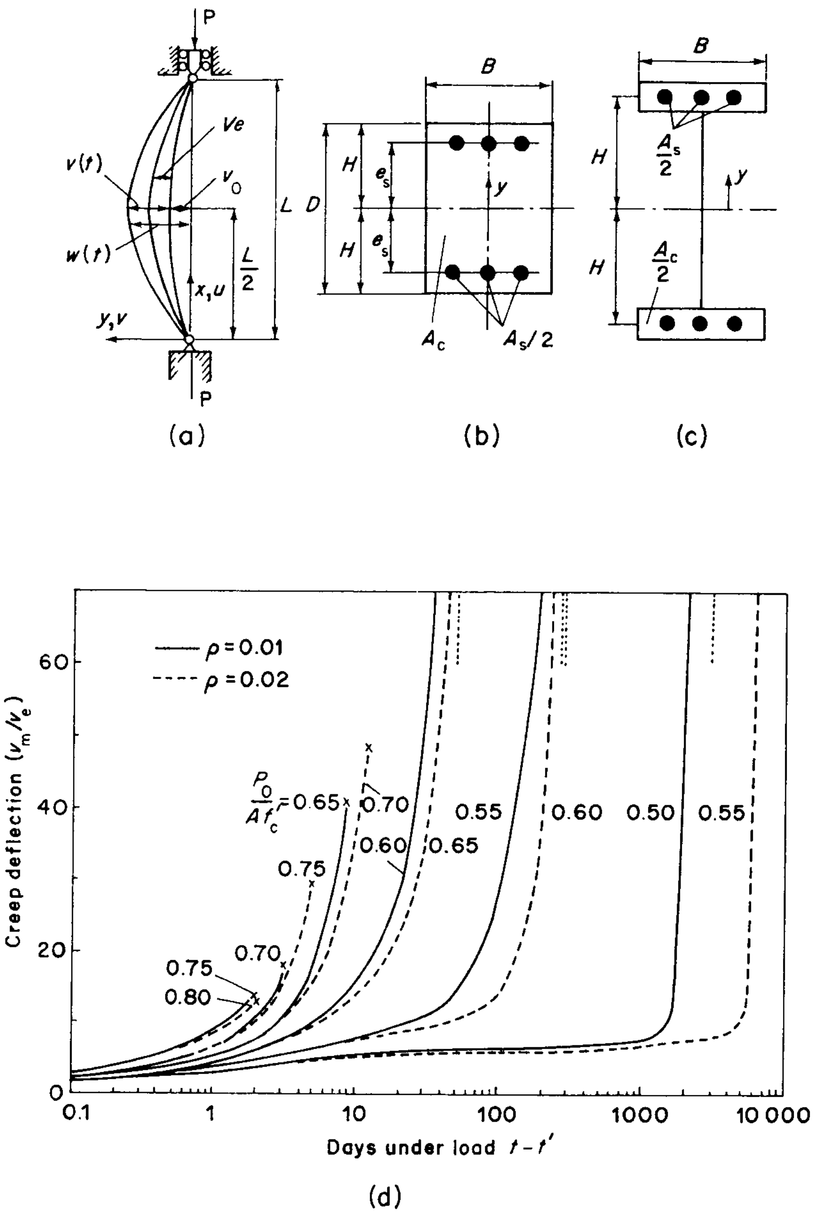


Figure 3.12 Reinforced concrete column with initial curvature (after Bažant and Tsubaki, 1980): (a) simply-supported column; (b) cross-section; (c) idealized I-section; (d) deflection at midspan, v_m [v_e = elastic deflection at midspan, $v_0/D = 0.01$, $L/D = 20$, $f_c = 6{,}700$ psi (46.2 MPa), $p = 0.01$, P_0 = constant sustained load, $A = 25$ sq in (1.61×10^4 mm^2)]

McGregor (1967), Bažant and Tsubaki (1980), Mauch and Holley (1963), and others. The main object of interest normally is a combination of long-time and short-time loads based on the specified safety factors. The distribution of loads into the short-time and the long-time components is found to have considerable influence on the results. Particularly important is the ratio of the long-time load to the short-time load; the higher it is, the greater the importance of non-linear creep. Columns with a higher dead-to-live load ratio have a higher safety factor if the total load is the same.

Non-linear creep buckling of concrete shells and slabs has apparently not yet been studied, although this is feasible with a finite element approach.

3.5.2 Effect of cracking in beams, slabs, shells, and reactor vessels

In unprestressed structures, the creep effects are relatively small due to extensive cracking, and in fully prestressed structures the cracking effects are small due to prestress. The influence of cracking on creep under service stresses is the largest for partially prestressed structures. This has been recognized by engineers, and various practical studies have been made.

The reduction of stiffness due to cracking was usually taken into account by some empirical formula such as Branson's formula used in the ACI Code. However, this formula does not distinguish between various types of cross-sections, reinforcement, distribution of bending moment, etc. A more realistic solution may be obtained by direct analysis of cracking at various locations of the cross-section. It was found (Bažant and Oh, 1984) that good agreement with test data is obtained by treating cracking as strain-softening, separately at various layers of the cross-section, and by superimposing the strain-softening deflections on those due to creep and shrinkage. Such calculations show excellent agreement with Branson's formula within its range of validity. Curiously, it was found that the tension-stiffening effect, caused by transmission of tensile stresses from steel into concrete by means of bond stresses, is apparently insignificant except perhaps at very large deformations with large distinct cracks.

For beams, the analysis of cracking by means of strain-softening can be carried out according to the bending theory; however, for more complicated structures, including slabs and shells, the finite element approach, possibly in its layered form, is normally inevitable.

Cracking and strain-softening is important for creep effects in structures that undergo drying, and explains to a large extent drying creep effects (Fig. 3.13).

3.5.3 Stress-induced shrinkage (or swelling) and thermal expansion (contraction)

Aside from the need to consider cracking produced by shrinkage and thermal stresses, one needs to include the stress-induced shrinkage and thermal expansion

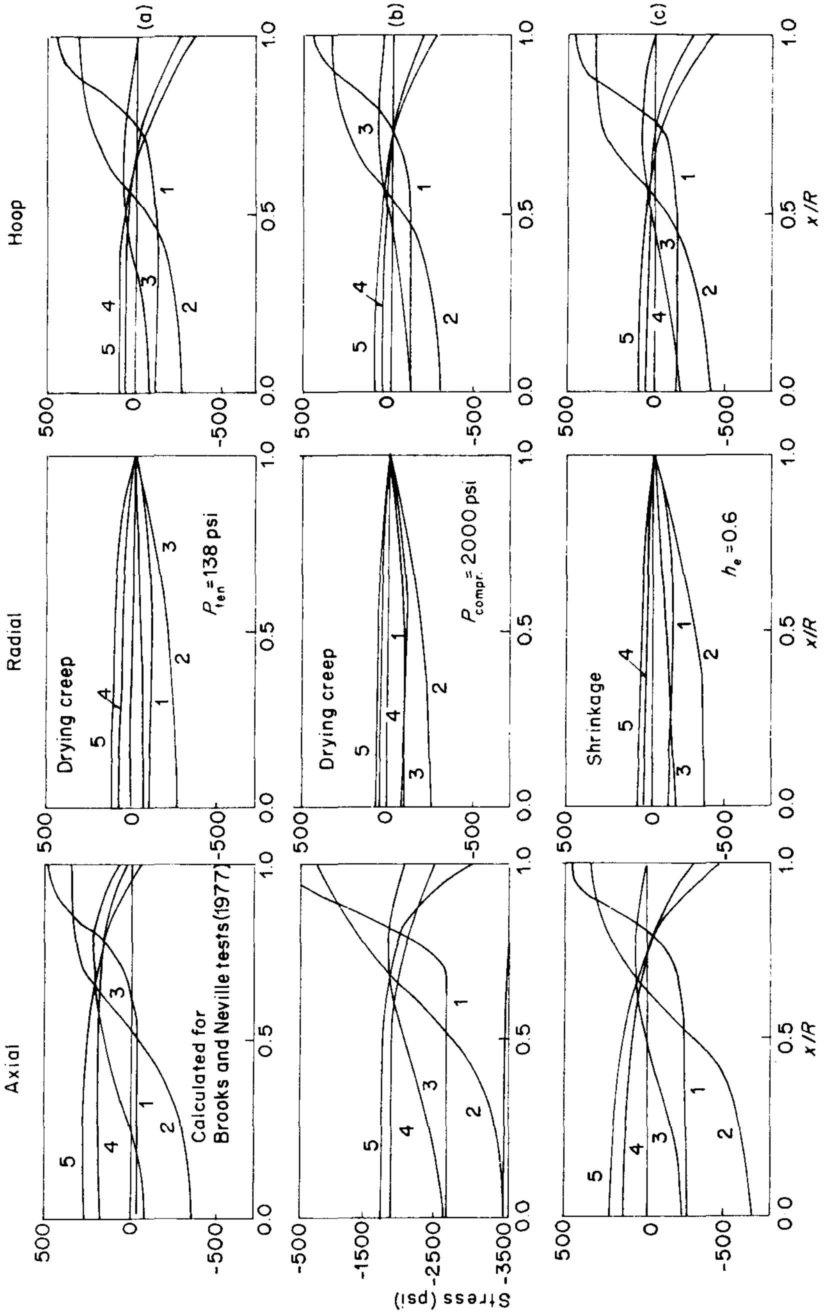


Figure 3.13 Stress distributions calculated by Bažant and Chern (1985) for Brooks and Neville's test. (a) tensile test; (b) compressive test; (c) shrinkage

to obtain a realistic picture of the environmental effects. While these effects can be easily taken into account in finite element analysis, they inevitably have to be neglected in the simpler linearized solution. It should be kept in mind, though, that the linearized analysis of random environment by the spectral approach (Bažant and Wang, 1984, 1985), as well as the simplified models based on a linear humidity distribution solution for a single component periodic environment (Diamantidis *et al.*, 1984), describe only a part of the real behaviour, and the non-linear effects will have to be considered eventually.

3.5.4 Effect of cyclic creep on bridge deflections

A periodically fluctuating stress component superimposed on a constant stress causes an increase of creep beyond that predicted according to the principle of superposition. This is of some importance for prestressed concrete bridges, in which the repeated load component is significant and is combined with the stress fluctuations due to environmental effects.

The increase of creep due to cyclic loads occurs primarily near the upper and lower faces of the beam, and is zero at the neutral axis. Thus, the beam faces undergo additional shortening which does not occur at the neutral axis. This leads to additional self-equilibrated stresses in the cross-section, tensile at the top and bottom faces and compressive at the neutral axis (Fig. 3.14).

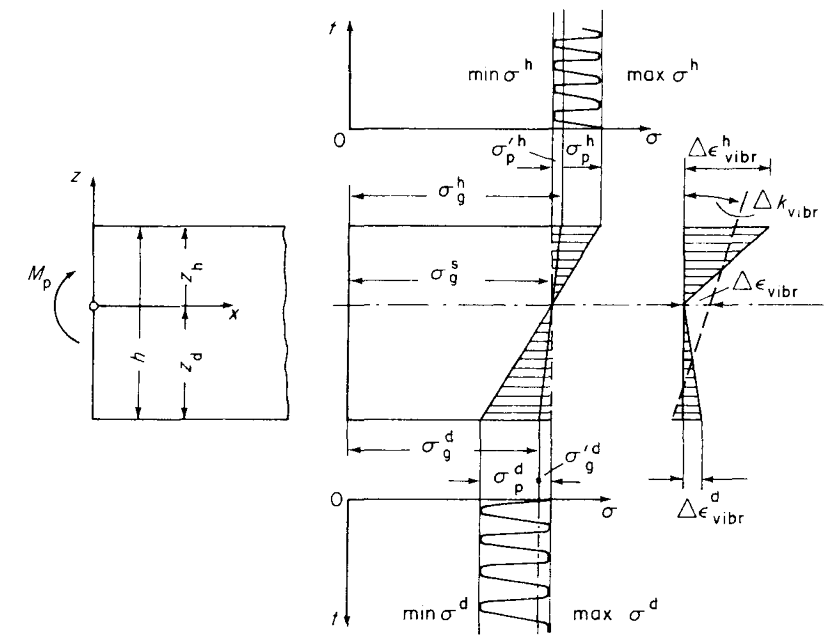


Figure 3.14 Effect of cyclic creep on beam deflection (Bažant, 1968a)

If the cross-section is non-symmetric, as is usually the case, the additional cyclic creep is larger at the face which is further from the neutral axis than it is at the opposite face. Thus, the cyclic creep causes an additional warping of the beam. In a redundant structure, this further induces additional redundant internal forces. These effects have been analysed for box girder bridges (Fig. 3.14) and were found to have an appreciable although not dominant influence.

At high stress levels beyond the working stress range, the creep response due to repeated loads becomes similar to plastic shakedown, and may exhibit a time-dependent version of incremental collapse, as well as fatigue.

3.5.5 Effect of clay consolidation in foundation

On certain sites, structures have to be founded on clay which undergoes gradual consolidation. This causes redistribution of internal forces in a redundant structure, and at the same time the gradual changes in support reactions affect the amount of clay consolidation in the foundation. Thus, the creep in the structure and the consolidation of clay (including the creep of clay) interact.

Since the dependence of clay consolidation settlements on the stress in clay is highly non-linear, the entire structure–soil interaction problem is rendered non-linear as well. The problem has been analysed using a secant linearized model for clay (Bažant, 1964c; Fig. 3.15). An iterative succession of such linear analyses can be used to obtain an accurate non-linear solution.

3.5.6 Viscoplastic analysis and high-temperature behavior

Three-dimensional viscoplastic constitutive equations which represent a generalization of plasticity models for concrete have been recently applied to some problems of nuclear reactor vessels in hypothetical high-temperature core-

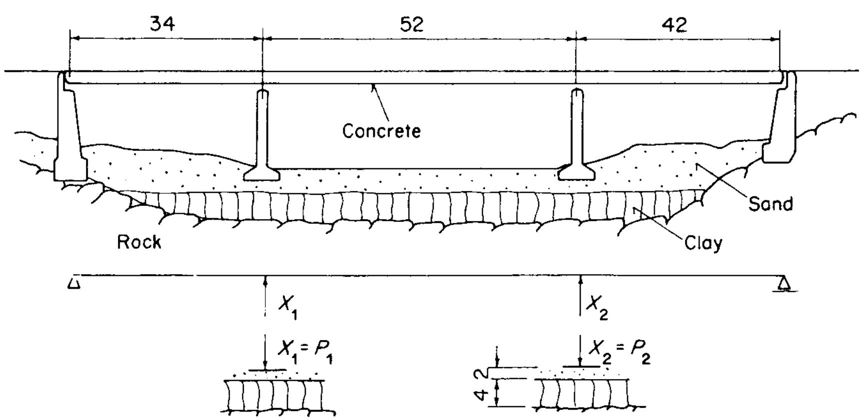


Figure 3.15 Effect of clay consolidation in foundation

disruptive accident conditions (Marchertas and Kennedy, 1983; Pfeiffer *et al.*, 1985; Thelandersson, 1983; Bažant, 1983). At high temperature, the rapid viscous flow of concrete near the strength limit causes a rapid relaxation of the peak stresses in the structure and a rapid redistribution of internal forces. This is important for the problem of hot spots in heated walls, which is of interest not only for nuclear reactor structures but also for fire resistance.

The high local stresses produced by rapid heating limited to a small area may become rapidly relaxed. However, once the stress relief capability of the stress redistribution due to structural redundancy is exhausted, the viscoplastic behaviour at high temperature is dangerous since it may lead to large deformations and rapid structural collapse.

3.6 DESIGN FOR STOCHASTIC BEHAVIOR AND OBSERVATIONS ON STRUCTURES

Eventually, further progress will require the design engineers to recognize that all the effects in structures discussed in the preceding review are in reality stochastic. As will be seen in Chapter 5, the statistical variability of creep and shrinkage is enormous, although it can be greatly reduced by certain measures such as Bayesian statistics based on short-time measurements. It will be desirable to develop design methods which are based not on the mean predictions but on certain extreme values that are exceeded only with a small specified probability such as 5 per cent.

The Latin hypercube sampling approach (see Bažant and Liu in Chapter 5, and Křístek and Bažant 1987) can be combined with all of the methods described in this chapter to obtain statistical information. However, the stochastic process aspects can hardly be analysed in this manner.

Another practical method based on response surface fitting has been developed by Wium and Buyukozturk (1985). They applied the method to calculate the variability of the long-term deformation in a multiple-span bridge. The results indicate a large variability in the response due to randomness of the factors that influence creep and shrinkage. The major source of uncertainty is found to be the variability of the material properties of the concrete.

Various observations of creep and shrinkage effects in structures have been made in the past. However, comparisons of the results of analysis with measurements on structures have so far been of limited usefulness since typically only some of the intervening effects were analysed while others have been either ignored or taken into account in an oversimplified manner. Successful verification has often been claimed; however, a fundamental question arises as to the meaning of such verification when one obtains an excellent agreement with some very limited observations or test results while at the same time the assumptions of analysis strongly disagree with known measurements of material behaviour.

3.7 CONCLUDING REMARKS

In this chapter, linear and non-linear analysis of structures for the effects of creep and shrinkage have been discussed. The basic assumptions of analysis, and the current code recommendations have been reviewed and comparisons of the methods of analysis made. Creep and shrinkage effects have been considered for different types of structures under a variety of loading conditions.

The effects of creep and shrinkage on structures can be very large, generally larger than the effects of short-time loadings. Although stress redistributions that occur due to long-term effects may frequently result in favourable structural conditions, for certain cases differential creep and shrinkage in various parts of a structure can cause deleterious cracking, followed by deterioration of the structure. For certain structural situations, creep and shrinkage may unfavourably alter the safety margin against the collapse of the structure under short-time overloads. Creep in slender structures may cause the critical loads for long-time instability to become much less than the elastic critical loads.

Considerable development has taken place in recent years in the area of non-linear creep analysis of concrete structures. However, nearly all the practical applications governed by the current codes or recommendations rely on the linearity assumption. This assumption may lead to acceptable results for service load conditions. Aging linearly viscoelastic material behaviour gives rise to a variety of effects in structures. Presently, these effects are well understood and can be analysed without difficulty. Generally, however, linear analysis does not lead to a realistic prediction of the structural behaviour for the full range of load levels. For example, effects such as material aging as it interacts with the stress history may significantly influence the lifetime and the critical permanent load that corresponds to that lifetime. A variety of non-linear effects in structures are discussed.

Finally, it is emphasized that all the creep effects in structures are in reality stochastic. The properties of concrete exhibit a large variability due to randomness of the factors that influence creep and shrinkage. These variations introduce large scatter in the response of structures, which makes it difficult to predict the long-term response accurately.

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