Microplane Model for Triaxial Deformation of Saturated Cohesive Soils

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Abstract: A microplane model, in which the constitutive properties are characterized independently on planes of various orientations (microplanes), is presented. It is found that the basic scheme previously developed for concrete is also valid for clays, but with certain modifications. The micro-macro constraint is kinematic, and the stresses on each microplane are defined as explicit functions of the volumetric and deviatoric normal and shear components of the macroscopic strain tensor on the microplane. To account for undrained behavior, a pore water pressure term that affects only the volumetric equations is introduced. This makes possible the uncoupling of the stress-strain and the pore water pressure formulations. The model is calibrated and verified by comparisons with numerous data for both drained and undrained tests, and good agreement is attained, including volume changes, pore water pressure evolution, and various stress-strain diagrams. Although the model involves nine material parameters, four (or five in some cases) can be fixed constant for all soils, and only five (or four) need to be determined by data fitting. The fact that the stress is given as an explicit function of strain makes the model suitable for finite element codes.

Introduction

During the last two decades, significant progress has been achieved in the development of realistic constitutive models for nonmetallic materials such as concrete, soils, rocks, ceramics, high-strength composites, etc. Most of the early models were based on a phenomenologic approach that used the theory of plasticity with various modifications and hardening and softening rules to describe and predict the behavior of these materials. For soils, the plasticity-based models (Mróz et al. 1979; Prévost 1978; Dafalias and Herrman 1982) were shown capable of describing some basic features of inelastic response, and one particular version of the plastic models, called the critical state theory developed by Roscoe and Burland (1968), Schofield and Wroth (1968), Atkinson and Bransby (1977), and others, has been one of the major achievements in this field. Another useful group of models, developed in the 1970s, was based on the endochronic theory, first proposed by Valanis (1971) for metals and later extended and modified for soils (Bažant and Krizek 1976; Cuellar et al. 1977; Ansal et al. 1979; Valanis and Read 1980; Bažant et al. 1983; Bažant and Kim 1986) as well as concrete. The endochronic theory models provided a good, realistic, and quite general representation of material behavior, especially hysteresis and volume changes under cyclic loading, but at the cost of considerable complexity and a large number of parameters. Combinations of plasticity with endochronic descrip-

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tion of volume changes at cyclic loading were proposed by Zienkiewicz et al. (1978).

Recently, two new types of models, based on miroscopic material characteristics, have been proposed and extended for geomaterials: (1) The classical slip theory of plasticity, developed initially by Batdorf and Budiansky (1949) for plasticity of metals, and its recent variants such as the multilaminate and microplane models (Zienkiewicz and Pande 1977; Pande and Sharma 1980, 1983; Pande and Xiong 1982; Bažant and Oh 1983, 1985; Bažant 1984; Bažant and Gambarova 1984; Bažant and Kim 1986; Bažant and Prat 1987, 1988; Prat and Bažant 1989; Carol et al. 1990); and (2) particle simulation, started by Cundall (1971) and Cundall and Strack (1979), and recently applied to heterogeneous materials, e.g., concrete, fiber composites and sand (Burt and Dougill 1977; Trent et al. 1987; Roelfstra 1987; Gili 1988; Bažant and Ožbolt 1989).

This paper attempts constitutive modeling by a simplified micromechanics approach of the type 1 mentioned previously. The micromechanical interactions between sources of inelastic deformation in statistically heterogeneous materials are basically twofold: (1) At distance; and (2) between different orientations. These interactions are of course intertwined but are more easily treated separately. The interactions at distance give rise to nonlocal properties and can be approximately characterized by nonlocal models. The nonlocal approach can be applied to the type of models we will be studying (Bažant and Ožbolt 1989) but is beyond the scope of the present study, in which attention is focused strictly on the stress-strain relation at a point of macroscopic continuum.

In this paper, the type-1 model, presently called the microplane model, will be applied to the triaxial inelastic behavior of clays. The original idea of the method is due to Taylor (1938), who proposed that the stress-strain relation be specifed independently on planes of various orientations in the material, assuming that either the stresses on that plane (now called the microplane) are the resolved components of the macroscopic stress tensor (static constraint), or the strains on the plane are the resolved components of the macroscopic strain tensor (kinematic constraint). The responses on the planes of various orientations are then related to the macroscopic response simply by superposition or, as has been done in recent works (Bažant 1984; Carol et al. 1990), by means of the principle of virtual work. In the initial application to metals, beginning with Batdorf and Budiansky (1949), only the static constraint was considered, and so it was the early applications to soils (Zienkiewicz and Pande 1977; Pande and Sharma 1980, 1983; Pande and Xiong 1982) which successfully described some basic aspects of soil behavior, other than strain softening. It appeared, however, that the microplane system under a static constraint becomes unstable when strain softening takes place (Bažant and Oh 1983, 1985; Bažant and Gambarova 1984). For this and other reasons, it is necessary to use the kinematic constraint, which will be adopted here.

In applications to concrete, the microplanes may be imagined to characterize the planes of microcracking within the microstructure of the material [Fig. 1(*a*)]. It was for this reason that the original term *slip planes*, used in the applications to metal plasticity, became unsuitable and the more general term *microplane* (permitting plastic as well as brittle response) was coined (Bažant and Oh 1983). For concrete, the microplanes are imagined chiefly

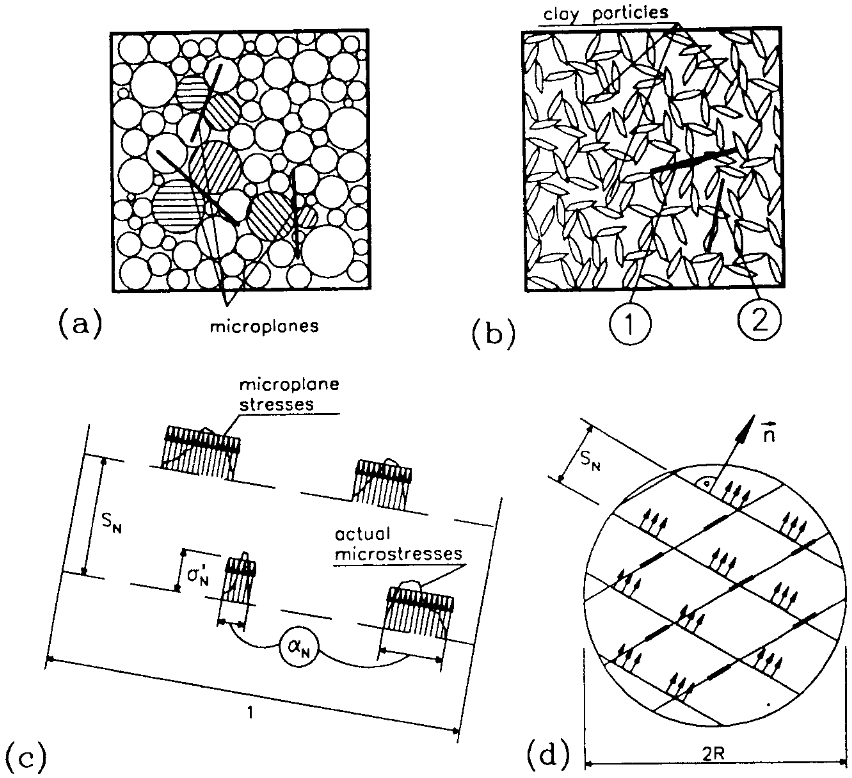

FIG. 1. (*a*) Microplanes in Granular Material; (*b*) Microplanes in Cohesive Soil: 1 = Microplane as Slip Plane between Clay Platelets; 2 = Microplane as Normal Plane to Clay Platelets; (*c*) and (*d*) Physical Interpretations of Microplane Stresses

to represent the planes of concentrated microcracking, in particular the thin contact layers between hard aggregate particles in which most of the inelastic deformation (possibly including also plastic slip) takes place. For clays [Fig. 1(*b*)], the microplanes may be imagined to represent the slip on the contact planes between clay platelets (Bažant and Kim 1986) or the planes normal to the platelets on which slip is manifested by normal strain (Bažant and Prat 1987). Although the correlation to the microstructural mechanism of inelastic deformations is largely intuitive, the microplane model has the advantage that it can distinguish among the intensities of inelastic strains at various orientations and describe how they are mutually constrained.

In our development of the model for clays, we will follow the scheme originally developed for concrete (Bažant and Prat 1988) and applied recently to the drained behavior of soils in a brief conference paper (Prat and Bažant 1989). In this paper, however, we will extend the validity of the model to undrained behavior as well. While the previous microplane models for clay dealt with creep (time-dependent deformation) of clay, the present paper will describe only time-independent deformations, but it will not be limited to deviatoric response. Generalization of the present model for creep is planned for subsequent work.

Basic Hypotheses

Hypothesis I

The strains on a microplane are the resolved components of the macroscopic strain tensor ϵ_{ij}.

This hypothesis, which represents a kinematic (tensorial) constraint, makes it possible to obtain stable response during strain softening, while the classical hypothesis of static constraint would not. Strain softening does occur in soils, e.g., for shear of overconsolidated clays. According to this hypothesis, the strain vector on a microplane whose direction cosines are n_i is $\epsilon_i^n = \epsilon_{jk} n_k$. The normal component and its vector then are (Fig. 2):

$$\epsilon_N = n_j \epsilon_j^n = n_j n_k \epsilon_{jk} \quad (1a)$$

$$\epsilon_{N_i} = n_i n_j n_k \epsilon_{jk} \quad (1b)$$

The latin lowercase subscripts refer to Cartesian coordinates x_i ($i = 1, 2, 3$), and repeated subscripts imply summation. The magnitude of the strain vector $\boldsymbol{\epsilon}^n$ on the microplane is $\|\boldsymbol{\epsilon}^n\| = \sqrt{\epsilon_j^n \epsilon_j^n} = \sqrt{n_i \epsilon_{ji} n_k \epsilon_{jk}}$. The vector of the shear strain component is $\boldsymbol{\epsilon}_T = \boldsymbol{\epsilon}^n - \boldsymbol{\epsilon}_N$, and the shear strain components and magnitude are

$$\epsilon_{T_i} = (\delta_{ij} - n_i n_j) n_k \epsilon_{jk} \qquad \epsilon_T = \|\boldsymbol{\epsilon}_T\| = \sqrt{\epsilon_{T_i} \epsilon_{T_i}} = \sqrt{n_k \epsilon_{jm} n_m (\epsilon_{jk} - n_j n_i \epsilon_{ik})} \quad (2)$$

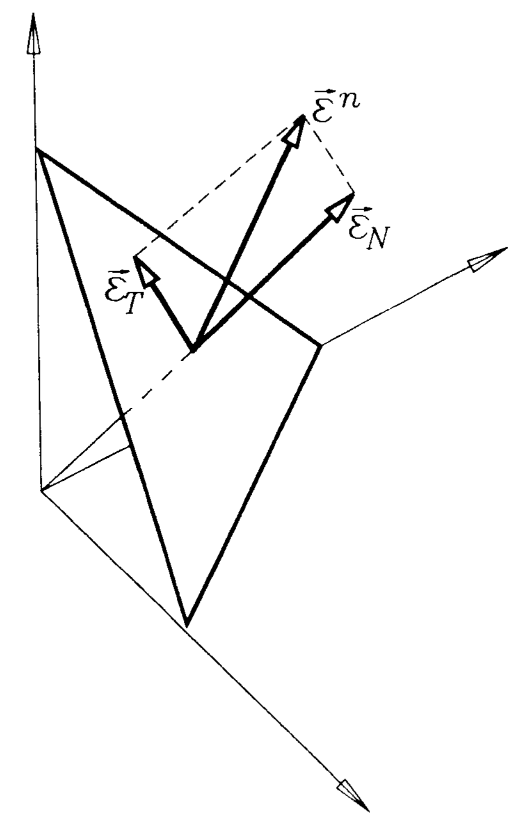

FIG. 2. Strain Components on Microplane

in which δ_{ij} = the Kronecker's delta tensor.

Physically, the microplane strains ϵ_N and ϵ_{T_i} may be imagined to represent the sum of the inelastic relative displacements on all the weak planes contained within a unit volume of the material, plus the associated elastic deformations of all the particles.

Hypothesis II

The response on each microplane depends explicitly on the volumetric strain ϵ_V, in the sense that the microplane equations must include a separate treatment of the volumetric (spherical) and deviatoric normal components of strain.

According to this hypothesis, we decompose the strain tensor ϵ_{ij} into its deviatoric and hydrostatic components

$$\epsilon_{ij} = e_{ij} + \frac{\epsilon_{kk}}{3} \delta_{ij} \quad (3)$$

where e_{ij} = the deviatoric strain tensor. Introducing now (3) into (1a), we obtain the following decomposition of the normal component of the strain vector on the microplane:

$$\epsilon_N = n_i n_j \left(e_{ij} + \frac{\epsilon_{kk}}{3} \delta_{ij} \right) = n_i n_j e_{ij} + \frac{\epsilon_{kk}}{3} \delta_{ij} n_i n_j = n_i n_j e_{ij} + \frac{\epsilon_{kk}}{3} = \epsilon_D + \epsilon_V \quad (4)$$

where $\epsilon_V = \epsilon_{kk}/3$ = the hydrostatic component of the strain tensor, called the volumetric strain, which has the same value for all microplanes; and $\epsilon_D = n_i n_j e_{ij}$ = the normal component on each microplane of the deviatoric strain tensor, called deviatoric strain.

This hypothesis means that on a microplane we must consider three components of strain: (1) Volumetric ϵ_V; deviatoric normal ϵ_D; and shear $\boldsymbol{\epsilon}_T$ (the last being a vector). The volumetric component ϵ_V is the same for all the microplanes. The deviatoric normal component ϵ_D for a given microplane is a vector whose direction is fixed once the microplane has been chosen. The shear strain component $\boldsymbol{\epsilon}_T$, however, is a vector lying in the microplane, whose direction is variable. In the present derivation, we will use the three spatial components ϵ_{T_i} of the shear vector $\boldsymbol{\epsilon}_T$, although one could also use two in-plane components referred to a set of microplane reference axes.

Hypothesis III

The volumetric response, deviatoric normal response, and the shear response on each microplane are mutually independent, i.e., decoupled. Note that this hypothesis implies independence of the volumetric and deviatoric response only on one microplane considered separately; macroscopically, they are, of course, coupled, due to interaction of the microplanes.

Hypothesis IV

The vector of shear stress $\boldsymbol{\sigma}_T$ and the vector of shear strain $\boldsymbol{\epsilon}_T$ acting on a microplane are parallel, i.e., $\sigma_{T_i} \sim \epsilon_{T_i}$. This hypothesis is introduced in the interest of reducing the number of unknowns. It would be possible, of course, to consider nonparallel σ_{T_i} and ϵ_{T_i} within the microplanes, but then it would be necessary to calculate and store two shear components of stress and strain on each microplane instead of one.

Hypothesis V

The microplane stress-strain relations for monotonic loading histories (histories with no unloading) are path-independent, i.e., they can be written as total rather than incremental stress-strain relations. This hypothesis does not imply path-independence for macroscopic monotonic loading; it means that all the macroscopic path-dependence is the consequence of various combinations of loading and unloading on different microplanes.

Effective Stress

According to the effective stress principle of soil mechanics, soil deformations depend on the so-called "effective stresses." The effective stress tensor σ'_{ij} characterizes the stresses transmitted by the solid skeleton

$$\sigma'_{ij} = \sigma_{ij} - p_w \delta_{ij} \quad (5)$$

where σ_{ij} = the total stress tensor; and p_w = the pore water pressure. The deviatoric effective stress tensor s'_{ij} is identical to the deviatoric total stress tensor s_{ij}

$$s_{ij} = \sigma_{ij} - \frac{\sigma_{kk}}{3}\delta_{ij} = \sigma'_{ij} + p_w\delta_{ij} - \frac{1}{3}(\sigma'_{kk} + 3p_w)\delta_{ij} = \sigma'_{ij} - \frac{\sigma'_{kk}}{3}\delta_{ij} = s'_{ij} \quad (6)$$

Therefore, we need to introduce the effect of the pore water pressure only in the equations that affect the volumetric behavior of the material. Since the volumetric response does not depend, by definition, on the orientation of the microplanes, the stress-strain formulation can be uncoupled from the pore-pressure formulation. The effective stress-strain equations will be formulated using the microplane theory, while the volumetric stress-strain–pore-pressure equations can be formulated macroscopically without the use of microplanes.

Microplane Constitutive Equations and Material Functions

We assume the existence of a functional relation between the strain and effective stress components on each microplane. Because of hypothesis III, the three components (volumetric, deviatoric normal, and shear) are mutually independent. Thus we can write

$$\sigma'_V = F_V(\epsilon_V) \qquad \sigma'_D = F_D(\epsilon_D) \qquad \sigma'_T = F_T(\epsilon_T) \quad (7)$$

where $\epsilon_T = \|\boldsymbol{\epsilon}_T\| = \sqrt{\epsilon_{T_r}\epsilon_{T_r}}$. Differentiation of (7) leads to

$$d\sigma'_V = F'_V(\epsilon_V)d\epsilon_V \qquad d\sigma'_D = F'_D(\epsilon_D)d\epsilon_D \qquad d\sigma'_T = F'_T(\epsilon_T)d\epsilon_T \quad (8)$$

$F_V(\epsilon_V)$, $F_D(\epsilon_D)$, and $F_T(\epsilon_T)$ = material functions that define the constitutive relation and are determined empirically.

Volumetric Stress-Strain Relation

We need to distinguish between hydrostatic compression and tension. For volumetric compression, we assume a relationship similar to the known experimental curves obtained from oedometric tests, e.g., a bilinear relation between ϵ_V and $\log \sigma'_V$.

Consider that the soil sample has an in situ vertical stress σ^0_{vert} and a horizontal stress $\sigma^0_{\text{hor}} = k_0\sigma^0_{\text{vert}}$, where k_0 = the lateral stress coefficient at rest. The maximum stress the soil has undergone is called the preconsolidation pressure p_c. A soil is called overconsolidated if $p_c > \sigma^0_{\text{vert}}$, and normally consolidated otherwise. If the soil sample is overconsolidated, we will use for virgin (initial) loading (Fig. 3)

$$\sigma'_V = \sigma^o_V e^{\epsilon_V/C^*_s} \quad (9)$$

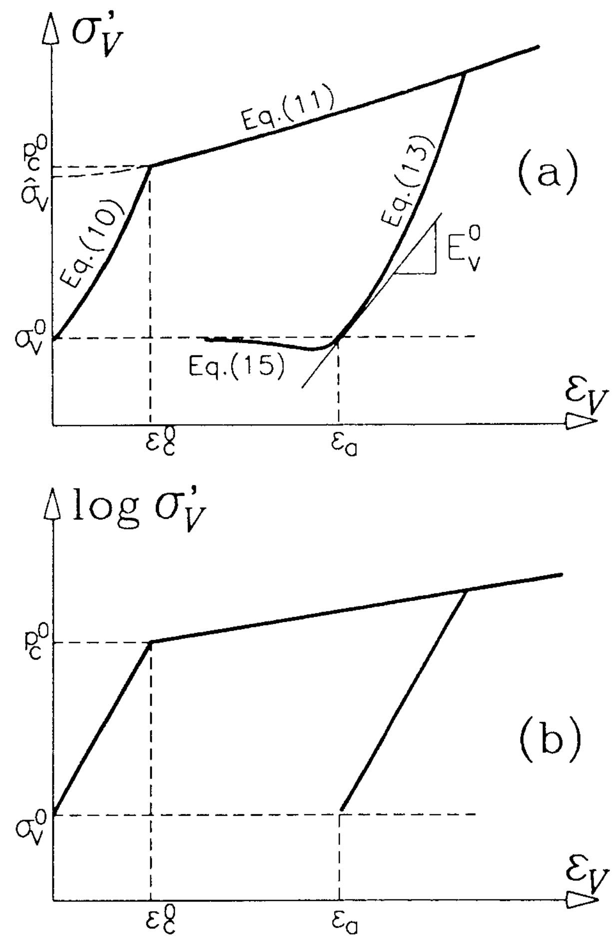

FIG. 3. Microplane Volumetric Stress-Strain Relation

where $\sigma^o_V = (1/3)(\sigma^0_{\text{vert}} + 2\sigma^0_{\text{hor}}) - p^o_w = (1/3)(1 + 2k_0)\sigma^0_{\text{vert}} - p^o_w$ = the initial effective volumetric stress in situ; p^o_w = the initial pore water pressure; and C^*_s = an empirical material parameter. If the two horizontal stresses σ^o_x and σ^o_y are unequal, we assume $\sigma^0_{\text{hor}} = (\sigma^o_x + \sigma^o_y)/2$. If the initial vertical stress is not less than the preconsolidation pressure, then the virgin loading branch can be described as

$$\sigma'_V = \hat{\sigma}_V e^{\epsilon_V/C^*_c} \qquad \text{with } \hat{\sigma}_V = \sigma_m e^{-\epsilon_m/C^*_c} \quad (10)$$

where σ_m and ϵ_m = the maximum effective volumetric stress and strain ever reached. These two state variables have the following initial values:

$$\sigma^o_m = \sigma^o_V \qquad \epsilon^o_m = C^*_s \log\frac{\sigma^o_m}{\sigma^o_V} \quad (11)$$

Note that if the soil is normally consolidated, i.e., if $p_c = \sigma^0_{\text{vert}}$, then $\epsilon^o_m = 0$.

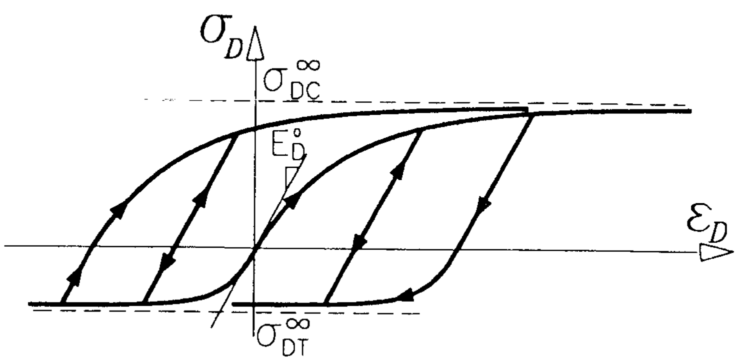

FIG. 4. Microplane Deviatoric Normal Stress-Strain Relation

The unloading branches in compression are defined so that in the (ϵ_V, log σ_V') space they are straight lines of slope $1/C_s^*$ (see Fig. 3)

$$\sigma_V' = \sigma_V^o e^{(\epsilon_V - \epsilon_a)/C_s^*} \quad (12)$$

where ϵ_a = the value of ϵ_V corresponding to the point on the unloading branch at which $\sigma_V = \sigma_V^o$

$$\epsilon_a = \left(1 - \frac{C_s^*}{C_c^*}\right)\epsilon_m + C_s^* \log\left(\frac{\sigma_V^o}{\hat{\sigma}_V}\right) \quad (13)$$

and C_c^* = an empirical material parameter.

For tension (i.e., when the current $\sigma_V' < \sigma_V^o$), we assume a stress-strain curve with a peak and a descending branch asymptotically approaching zero. The curve is shifted by a distance equal to the latest value of ϵ_a, so that continuity is maintained in the transition from compression to tension

$$\sigma_V' = \sigma_V^o + E_V^o(\epsilon_V - \epsilon_a)e^{-(1/p)|(\epsilon_V - \epsilon_a)/\epsilon_p|^p} \quad (14)$$

where $E_V^o = \sigma_V^o/C_s^*$; and p and ϵ_p = material parameters. The value of E_V^o is chosen so that continuity of slopes is maintained in the transition from compression to tension.

Finally, for tension unloading, we assume a linear branch with slope E_V^o such that

$$\Delta\sigma_V' = E_V^o \Delta\epsilon_V \quad (15)$$

The volumetric curve thus defined ensures that the ϵ_V − log σ_V' relation for compression is bilinear [Fig. 3(*b*)].

In summary, C_s^*, C_c^*, ϵ_p, and p are empirical material parameters; p_c and σ_V^o are given for each test and sample, and ϵ_a is computed from (13).

Deviatoric Stress-Strain Relation

Again, in this case, we need to distinguish between tension and compression. We assume the following relations with a horizontal (plastic) plateau (Fig. 4)

$$\sigma_D = F_D(\epsilon_D) = \sigma_{DC}^{\infty}(1 - e^{-k_{DC}|\epsilon_D|}) \qquad \text{if } \sigma_D \geq 0 \quad (16a)$$

$$\sigma_D = F_D(\epsilon_D) = \sigma_{DT}^{\infty}(1 - e^{-k_{DT}|\epsilon_D|}) \qquad \text{if } \sigma_D < 0 \quad (16b)$$

where the sign of σ_D (positive in compression, negative in tension) is chosen

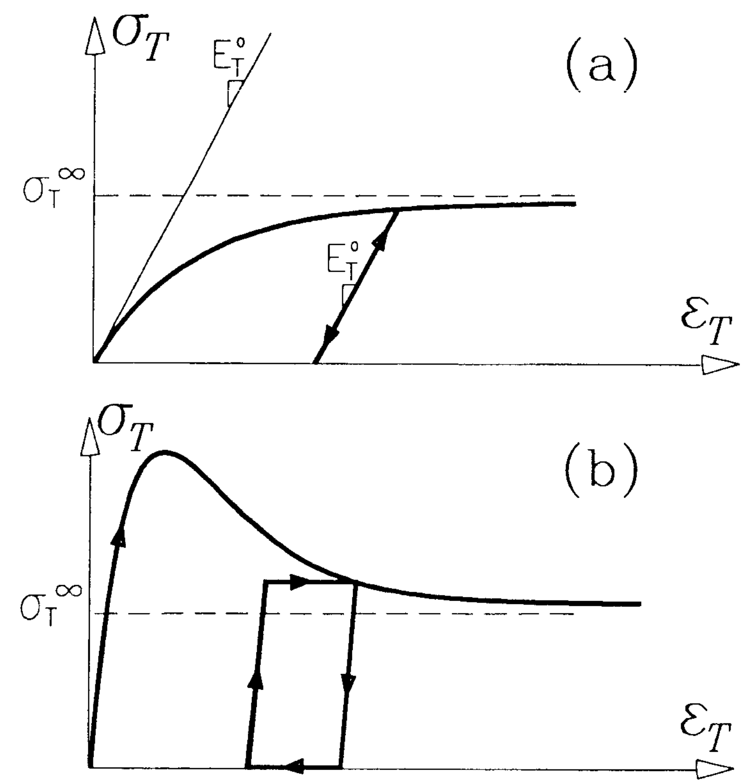

FIG. 5. Microplane Shear Stress-Strain Relation: (*a*) Normally Consolidated Clay; (*b*) Overconsolidated Clay

according to the usual convention in soil mechanics. In (16*a*) and (16*b*) σ_{DC}^{∞}, σ_{DT}^{∞}, k_{DC}, and k_{DT} are empirical material constants, not entirely independent if we enforce continuity of slopes at the origin. In that case, the following relation must hold: $|\sigma_{DC}^{\infty} k_{DC}| = |\sigma_{DT}^{\infty} k_{DT}| = E_D^o$, where E_D^o = the initial elastic modulus. Eqs. (16*a*) and (16*b*) apply only for loading on the microplane. For unloading, we assume on each microplane linear elastic behavior with elastic modulus E_D^o. It must be noted that the relationships defined by (16*a*) and (16*b*) act as the envelopes for future loading-unloading-reloading cycles (Fig. 4).

Shear Stress-Strain Relation

The shear stress-strain relation must show a dependency on the overconsolidation ratio $r_{OCR} = p_c/\sigma_{ven}^0$, to take into account its effect on the macroscopic behavior (Fig. 5). Thus, we adopt the following equation:

$$\sigma_T = F_T(\epsilon_T) = \sigma_T^{\infty}[1 + (a\epsilon_T - 1)e^{-k_T\epsilon_T}] \quad (17)$$

where

$$a = a_0(r_{OCR} - 1) \quad (18)$$

and σ_T^{∞}, k_T, and a_0 are empirical parameters. Parameter a depends on r_{OCR}: for normally consolidated clays, $a = 0$, and the curve always has a positive slope with an asymptotic horizontal plateau [Fig. 5(*a*)]. For overconsolidated clays, $a > 0$, and the stress-strain curve exhibits a peak followed by a descending branch asymptotically approaching a residual value σ_T^∞ [Fig. 5(*b*)]. The unloading rules are shown also on Figs. 5(*a* and *b*). Note that if the initial shear modulus E_T^o is known, then the exponent k_T in (17) can be computed as $k_T = E_T^o/\sigma_T^\infty - a$.

Eq. (17) represents only a relation between the norms of the shear stress and shear strain vectors. However, since we have assumed (hypothesis IV) that these vectors are parallel, we can easily obtain the components of the stress vector as

$$\sigma_{T_r} = \sigma_T \frac{\epsilon_{T_r}}{\epsilon_T} \qquad (19)$$

where $\sigma_T = \|\boldsymbol{\sigma}_T\| = F_T(\epsilon_T)$; and $\epsilon_T = \|\boldsymbol{\epsilon}_T\|$.

Pore-Water Pressure

We have shown in (6) that the pore pressure must be introduced only into the volumetric equations, and therefore, the effective stress-strain model (for which we use the microplane model) can be uncoupled from the pore water pressure model, which is scalar and does not necessitate the microplane approach. For drained tests, in which no pore pressure is allowed to develop, the effective and total stress tensors coincide, and therefore $\sigma_V' = \sigma_V$. Thus, the total volumetric stress can be simply written as

$$\sigma_V = \sigma_V' = F_V(\epsilon_V) \qquad (20)$$

However, in undrained tests the total volumetric stress must be computed as

$$\sigma_V = \sigma_V' + p_w = F_V(\epsilon_V) + p_w \qquad (21)$$

To analyze the undrained behavior of a saturated cohesive soil, we will use the model developed by Ansal et al. (1979) as an extension of Bažant and Krizek's (1975) formulation for a two-phase medium. The basic assumptions are: (1) Both free and bound water behave elastically; (2) their compressibilities are the same; and (3) both can carry only volumetric stress.

If no drainage is allowed, the volume change of the solid skeleton is equal to the volume change of the pore water. Then

$$\epsilon_V = \frac{\sigma_V'}{3K} + \epsilon'' \qquad (22)$$

where ϵ'' = the accumulated inelastic volumetric strain, and

$$p_w = \frac{3C_w}{n}\epsilon_V \qquad (23)$$

where n = the porosity; and C_w = the water compressibility. Eq. (23) can be combined with (21) and (22) to obtain the final equation for the pore water pressure

$$p_w = \frac{C_w K}{nK + C_w}\left(\frac{\sigma_V}{K} + 3\epsilon''\right) \qquad (24)$$

where $K = 2G(1 + \nu)/3(1 - 2\nu)$ = bulk modulus. Similar to Ansal et al. (1979), we assume the following variation of the shear modulus G along the stress path:

$$G = G_0\left(1 + \frac{\sigma_V' - \sigma_V^o}{10\sigma_V^o} + \frac{3\epsilon''}{10n}\right) \qquad (25)$$

The accumulated inelastic strain ϵ'' is obtained by integrating the inelastic volumetric strain increment $d\epsilon''$, which is a measure of the time-independent densification dilatancy and can be written as

$$d\epsilon'' = \frac{C(1 + 2{,}500\epsilon_V)}{(1 + 1{,}000J_2^\xi)\left(1 + \dfrac{\sigma_V}{4p_a}\right)(1 + 9{,}000\epsilon'')} d\xi \qquad (26)$$

in which C = a material parameter; $J_2^\xi = (1/2)\epsilon_{ij}\epsilon_{ij}$; σ_V = the total volumetric stress; p_a = the atmospheric pressure; ϵ'' = the accumulated inelastic volumetric strain (densification dilatancy); and $d\xi = \sqrt{(1/2)d\epsilon_{ij}d\epsilon_{ij}}$ = path length increment in the strain space. The use of $d\xi$, borrowed from the endochronic models (Bažant and Krizek 1976; Cuellar et al. 1977; Bažant et al. 1983), has been shown to be particularly effective for cyclic loading.

Macroscopic Constitutive Law

The basic structure we will use for the effective stress-strain law follows that of the latest version of the microplane model for concrete (Carol et al. 1990). The constitutive law is written in terms of the current stresses and strains (and not in terms of their increments), which allows the model to be explicit with all the numerical advantages shown by Carol et al. (1990).

Since the microplanes are constrained kinematically (hypothesis I), it is impossible for the effective microplane stresses σ_N' and σ_{T_r}' to equilibrate the effective macrostresses σ_{ij}' exactly. We need, though, to enforce the micro-macro equilibrium in an overall, approximate sense. As shown in Bažant (1984), this can be achieved by applying the principle of virtual work to a small unit sphere of the material. We imagine that in the microstructure the microplanes represent the force-transmitting weak planes of concentrated deformation, i.e., the interparticle contact zones.

Let $\nu_N d\Omega$ be the number of all the microplanes for which the normals lie within a small solid angle $d\Omega$ centered around the orientation vector **n** [the average spacing of these nearly parallel planes is $s_N = 1/\nu_N$, Fig. 1(*c* and *d*)]. Assume that the stresses are constant over area fraction $\alpha_N\nu_N d\Omega$ of all these planes (representing the extent of the contact zones) and zero over the remaining area fraction $(1 - \alpha_N)\nu_N d\Omega$ [Fig. 1(*c*)]. By virtue of introducing coefficients ν_N and α_N, the microplane stresses can be interpreted as the averages of the actual stresses transmitted through the contact zones. The virtual work of the effective microstresses on all the microplanes with normals lying within the solid angle $d\Omega$, done in a small volume element ΔV_0, is $(\sigma_N'\delta\epsilon_N + \sigma_{T_r}'\delta\epsilon_{T_r})\alpha_N\nu_N\Delta V_0$ in which $\delta\epsilon_N$ and $\delta\epsilon_{T_r}$ are small variations of the microplane strains. The element ΔV_0 is small enough so that the changes of σ_{ij}' and ϵ_{ij} across this element are negligible, but large enough so the number of grains on contact zones within the element is large. According to the principle of virtual work, this must be equal to the work of macroscopic effective stresses σ_{ij}' within this element, i.e.,

$$\Delta V_0\sigma_{ij}'\delta\epsilon_{ij} = \int_\Omega \Delta V_0(\sigma_N'\delta\epsilon_N + \sigma_{T_r}'\delta\epsilon_{T_r})\Psi_N d\Omega \qquad (27)$$

where $\Psi_N = \alpha_N\nu_N$; and $d\Omega = \sin\theta d\theta d\phi$, with θ and ϕ = angular spherical coordinates. The integration is carried out over a unit hemisphere of all orientations, $\theta \in (0,\pi)$, $\phi \in (0,\pi)$, rather than the full sphere because the diametrically opposite points on the sphere correspond to the same microplane. In previous works, a function differing from function Ψ_N by a constant multiplicative factor was used, since the microplane stresses were not physically interpreted as the average of the actual stresses transmitted through the contact zones, but as some stresses proportional to them. Function Ψ_N is a weight function of the orientation **n**, which in general can introduce anisotropy of the material in its initial state. If, however, such a function is unknown, we can take approximately Ψ_N = constant, which implies isotropy. Substituting now $\delta\epsilon_N = n_i n_j \delta\epsilon_{ij}$, $\delta\epsilon_{T_r} = (\delta_{ij} - n_r n_i) n_k \epsilon_{jk}$ and setting $\sigma'_N = \sigma'_V + \sigma'_D$, we get

$$\sigma'_{ij}\delta\epsilon_{ij} = \int_{\Omega} [n_i n_j(\sigma'_V + \sigma'_D) + (\delta_{ri} - n_r n_i) n_j \sigma'_{T_r}]\Psi_N d\Omega \delta\epsilon_{ij} \quad \ldots\ldots\ldots\ldots\ldots (28)$$

This variational equation must hold for any variations $\delta\epsilon_{ij}$. However, since the strain tensor is symmetric, the variations of its nine components are not independent because $\delta\epsilon_{ij} = \delta\epsilon_{ji}$. Taking into account the fact that the volumetric strain and stress is the same for all microplanes, the volumetric term in (28) can be taken out of the integral. Then, according to Carol et al. (1990), the macroscopic effective stress tensor can be expressed as

$$\sigma'_{ij} = \sigma'_V\delta_{ij} + \int_{\Omega} n_i n_j \sigma'_D \Psi_N d\Omega + \int_{\Omega} \frac{1}{2}(n_i\delta_{rj} + n_j\delta_{ri} - 2n_i n_j n_r)\sigma'_{T_r}\Psi_N d\Omega \quad \ldots (29)$$

where σ'_V, σ'_D, and σ'_{T_r} = the microplane effective stresses computed according to (7). The macroscopic total stress tensor σ_{ij} then results from the principle of effective stresses of soil mechanics

$$\sigma_{ij} = \sigma'_{ij} + p_w\delta_{ij} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30)$$

where σ'_{ij} is obtained from (29) and the pore water pressure p_w from (24). The latter equation can be rewritten as

$$p_w = \alpha\sigma_{kk} + \beta\epsilon'' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (31)$$

with $\alpha = C_w/3(nK + C_w)$ and $\beta = 9K\alpha$. Thus

$$\sigma_{ij} = \sigma'_{ij} + \alpha\sigma_{kk}\delta_{ij} + \beta\epsilon''\delta_{ij} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32)$$

Denoting $B_{ij} = \sigma'_{ij} + \beta\epsilon''\delta_{ij}$ (which is a known tensor), we obtain the following system of equations for the unknowns σ_{ij}:

$$\sigma_{ij} - \alpha\sigma_{kk}\delta_{ij} = B_{ij} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (33)$$

The solution of this system of equations gives the values of the macroscopic total stress tensor and, therefore, the value of the pore water pressure as well.

In practical calculations, the integration over a unit hemisphere is replaced by a summation over a finite number of microplanes of N_m orientations (N_m = 21 is normally necessary); see Bažant and Oh (1986). A finite element program can be written in the manner described by Bažant and Ožbolt (1989) and Carol et al. (1990).

Verification with Experimental Data

The purpose of constitutive modeling is to find the simplest possible way to mathematically represent the often complex behavior of the materials. One

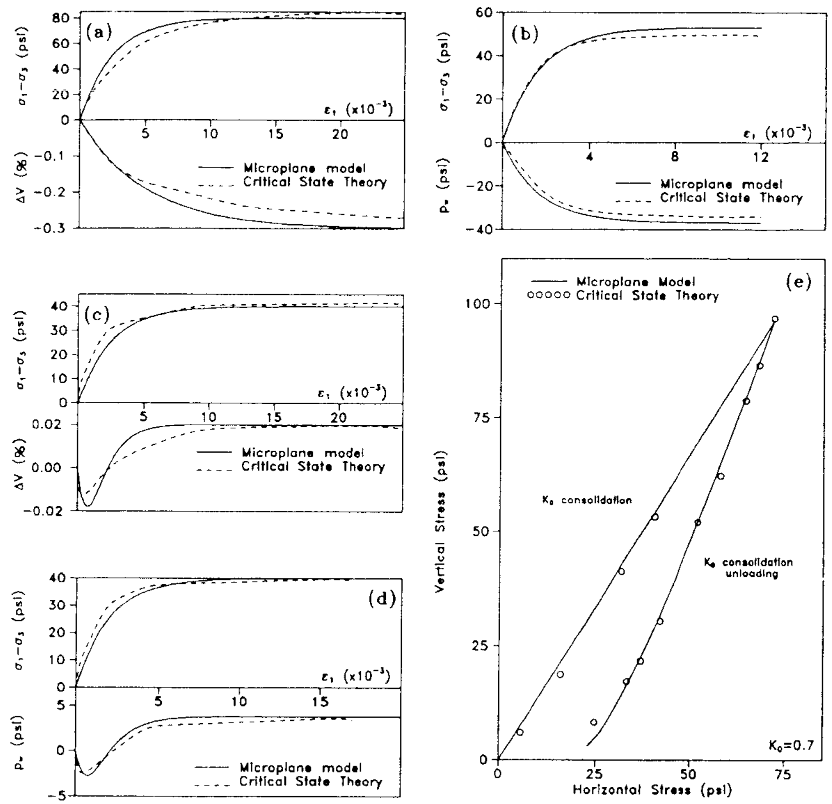

FIG. 6. Comparison of Microplane Model and Critical State Theory for Standard Triaxial Tests on Normally Consolidated Clay: (*a*) Drained Compression; (*b*) Undrained Compression; (*c*) Drained Extension; (*d*) Undrained Extension; (*e*) k_0 Consolidation

must look for a model with the fewest possible parameters without sacrificing accuracy, so that the numerical and experimental results are in good agreement. The microplane model has been proven to be a good material model in that sense; one can obtain accurate predictions of the material behavior with a reduced number of parameters. In this section, we will show that this applies also to the model for soils, and we will justify the validity of the model by comparisons with several experimental data available in the literature.

As has been shown in previous works (Bažant and Prat 1988; Carol et al. 1990), there exists a relation between the macroscopic modulus of elasticity E, Poisson's ratio ν, and the three initial elastic moduli on the microplanes (E_V^o, E_D^o, and E_T^o). Assuming that the relation between the shear and volumetric moduli is $\eta_0 = E_T^o/E_V^o$, we can obtain the microplane moduli according to the following formulas:

$$E_V^0 = \frac{E}{1-2\nu} \qquad E_D^0 = \frac{E_V^0}{3}\left[\frac{5(1-2\nu)}{1+\nu-2\eta_0}\right] \qquad E_T^0 = \eta_0 E_V^0 \quad \ldots\ldots\ldots\ldots (34)$$

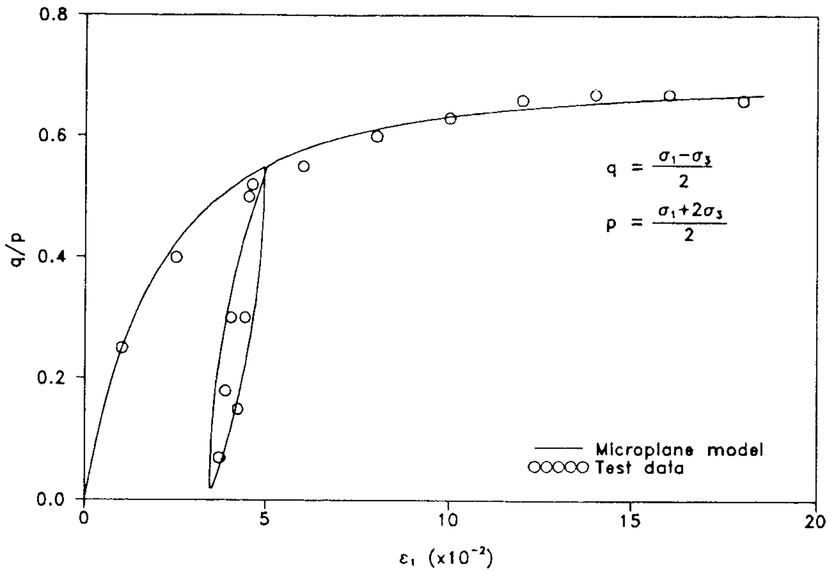

FIG. 7. True Triaxial Tests of Kaolin Clay (Wood 1975)

Thus, a convenient way to organize the input data is to assume known E and ν, suitably choose the first material parameter η_0, and calculate the microplane moduli from (34).

The initial volumetric modulus E_V^o is closely related to the parameters C_s^* and C_c^*. From (9) and (10), it can be shown that for normally consolidated soils ($r_{\mathrm{OCR}} = 1$)

$$C_c^* = \frac{\sigma_V^o}{E_V^o} \qquad C_s^* = \xi_0 C_c^* \quad \text{(35a)}$$

and for overconsolidated soils

$$C_s^* = \frac{\sigma_V^o}{E_V^o} \qquad C_c^* = \frac{C_s^*}{\xi_0} \quad \text{(35b)}$$

where $\xi_0 = C_s^*/C_c^*$ = the second material parameter. The rest of the parameters are: (1) Parameters of the volumetric stress-strain relation in tension (14): p (exponent) and ϵ_p (peak strain); (2) parameters of the deviatoric stress-strain relation (16*a*) and (16*b*): σ_{DC}^∞ and σ_{DT}^∞; (3) parameters of the shear stress-strain relation (17) and (18): σ_T^∞ and a_0; and (4) parameter C of the inelastic volumetric strain (26).

Note that the exponents in (16*a*), (16*b*), and (17), k_{DC}, k_{DT}, and k_T, can be obtained as a combination of the other parameters and the initial elastic moduli, and therefore are not independent material parameters. Thus, the total number of parameters in the model is nine. However, after extensive verification, it has been found that some of the parameters show almost no variation from test to test, and so they may be fixed. The general values of these parameters are:

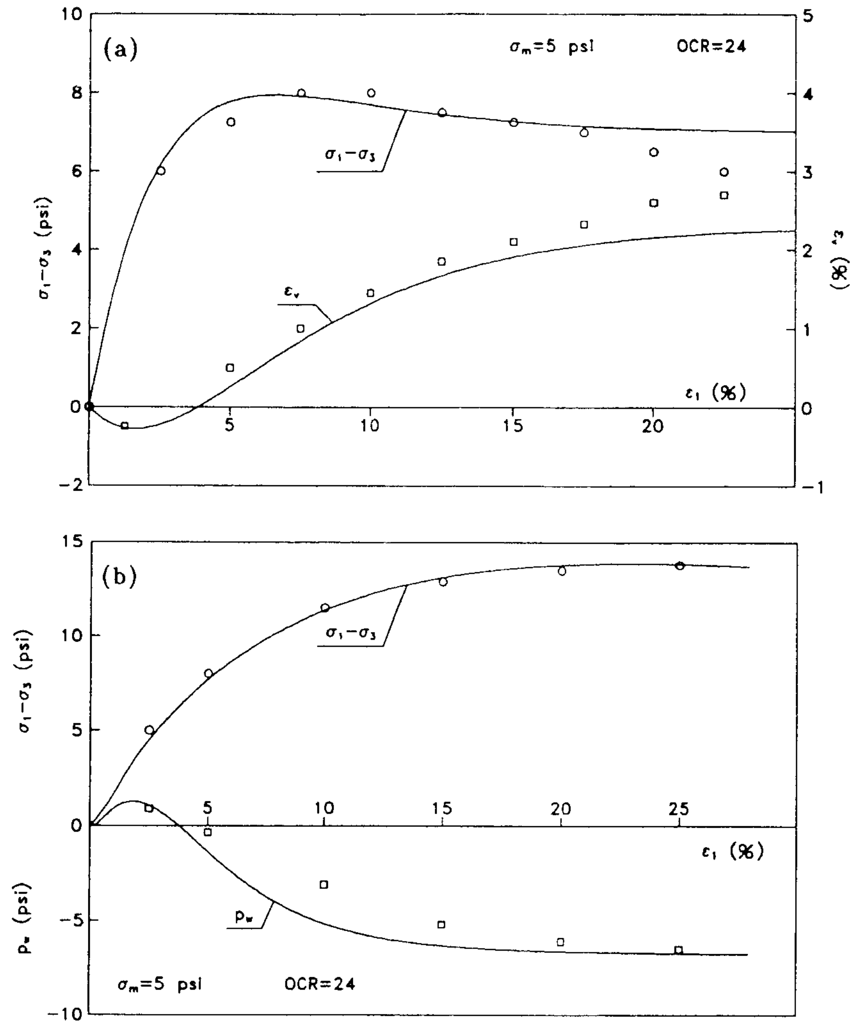

FIG. 8. Triaxial Tests on Overconsolidated (r_{OCR} = 24) Weald Clay (Henkel 1956): (*a*) Drained; (*b*) Undrained

$$\eta_0 = 0.5, \qquad \xi_0 = 0.1, \qquad \epsilon_p = 0.0001, \qquad p = 1 \quad \text{(36)}$$

The reduced number of parameters is then five (four if the soil is normally consolidated, because in that case parameter a_0 is meaningless). Five is a small enough number for practical purposes.

Fits of several typical test data from the literature are shown in Figs. 6–13. The measurements are shown as the data points, while the results of the model are plotted as solid curves, unless otherwise indicated in the figures. The values of the Poisson's ratio ν, Young's modulus E, initial void ratio e_0 (or porosity n), water compressibility C_w, preconsolidation pressure p_c, in situ vertical stress σ_{vert}^0, and coefficient of lateral pressure at rest k_0 are

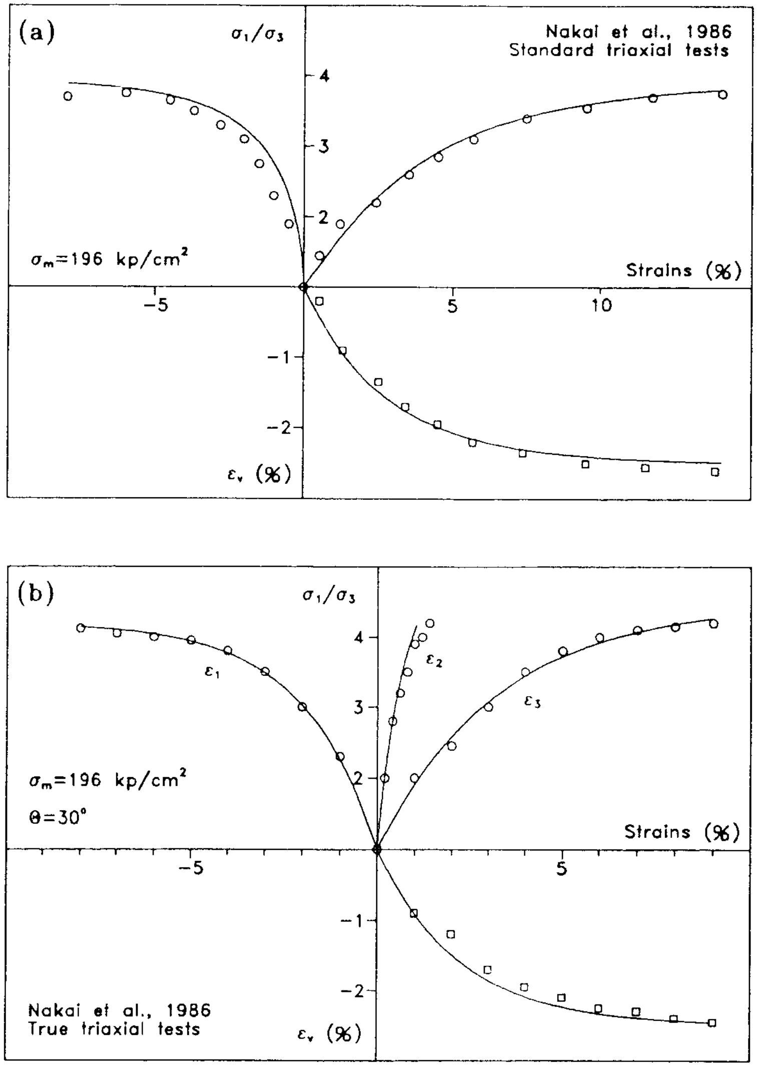

FIG. 9. Drained Triaxial Tests on Normally Consolidated Clay (Nakai et al. 1986): (*a*) Standard Triaxial Tests; (*b*) True Triaxial Tests

fixed prior to data fitting, according to the information provided with the experimental results.

Fig. 6 shows the comparison between the present microplane model and the results obtained with the critical state model, as reported by Pande and Sharma (1983). The parameters used are σ_{DC}^{*} = 25.7 psi (0.1772 MPa), σ_{DT}^{*} = 8.3 psi (0.0572 MPa), σ_{T}^{*} = 20.1 psi (0.1386 MPa), and C = 3.5. These results correspond to a normally consolidated soil under triaxial compression and extension, in drained and undrained conditions. It can be observed from the figures that the results obtained with the microplane model

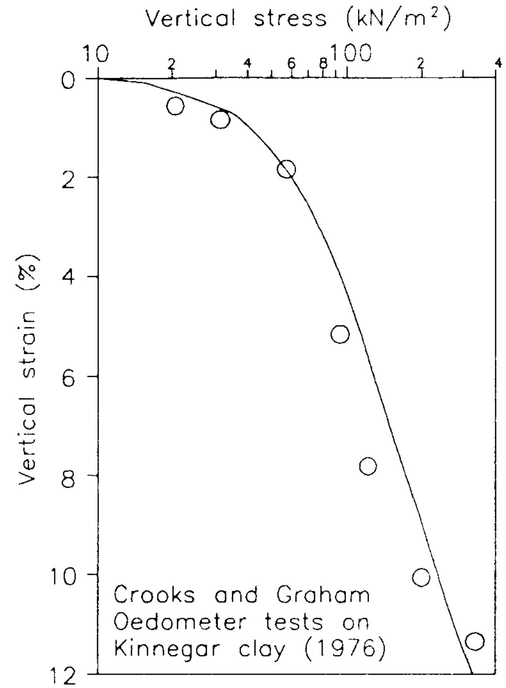

FIG. 10. Plane Strain Tests of Kaolin Clay (Hambly 1972)

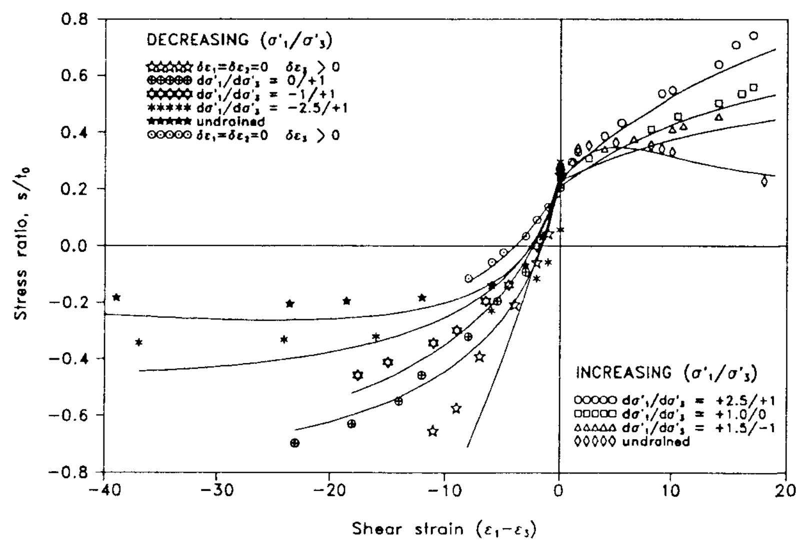

FIG. 11. Oedometric Tests (Crooks and Graham 1976)

agree rather closely with those obtained with the well-known critical state theory of plasticity (Schofield and Wroth 1968).

Fig. 7 shows the results obtained by Wood (1975) on cubic tests of kaolin clay, with unloading-reloading cycle. The microplane model, it is shown, can reproduce the experimental data rather well, including the unloading-reloading phenomena. The parameters used in this case are $\sigma_{DC}^{\infty} = 36.7$ psi (0.253 MPa), $\sigma_{DT}^{\infty} = 10.1$ psi (0.0696 MPa), $\sigma_{T}^{\infty} = 20.5$ psi (0.1413 MPa), and $C = 2.5$.

Fig. 8 shows the test data on overconsolidated clays ($r_{OCR} = 24$) obtained by Henkel (1956). The parameters used are $\sigma_{DC}^{\infty} = 41.2$ psi, $\sigma_{DT}^{\infty} = 9.3$ psi, $\sigma_{T}^{\infty} = 37.1$ psi, $C = 2.9$, and $a_0 = 20$. The results from the microplane

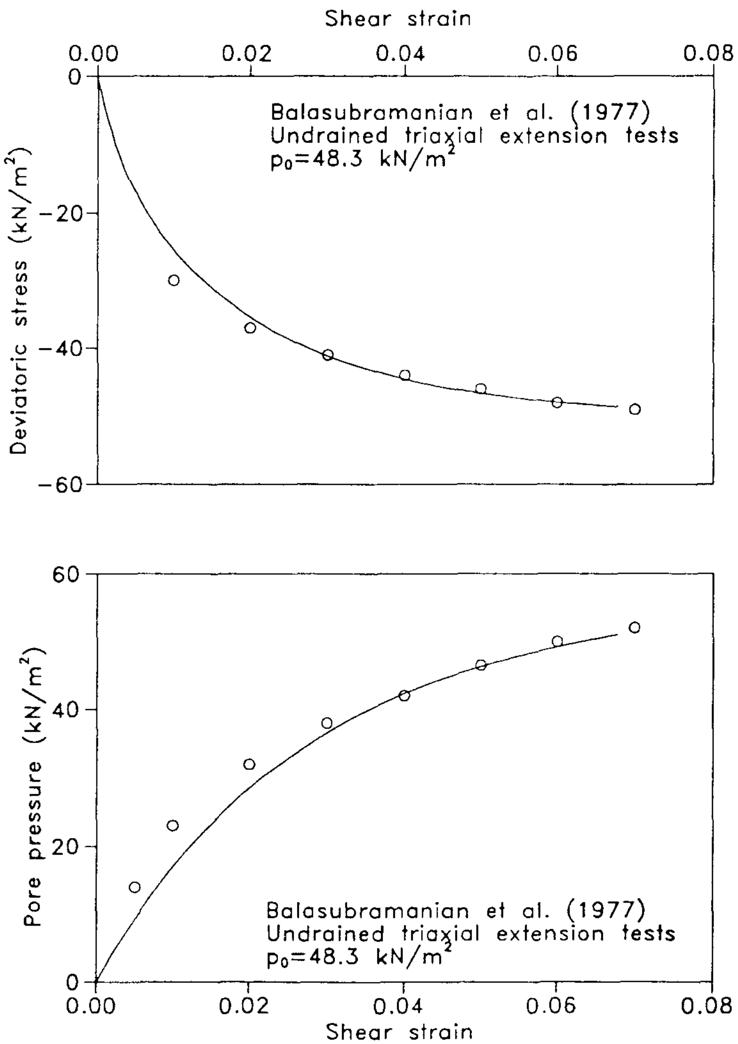

FIG. 12. Undrained Triaxial Extension Tests (Balasubramanian and Uddin 1977)

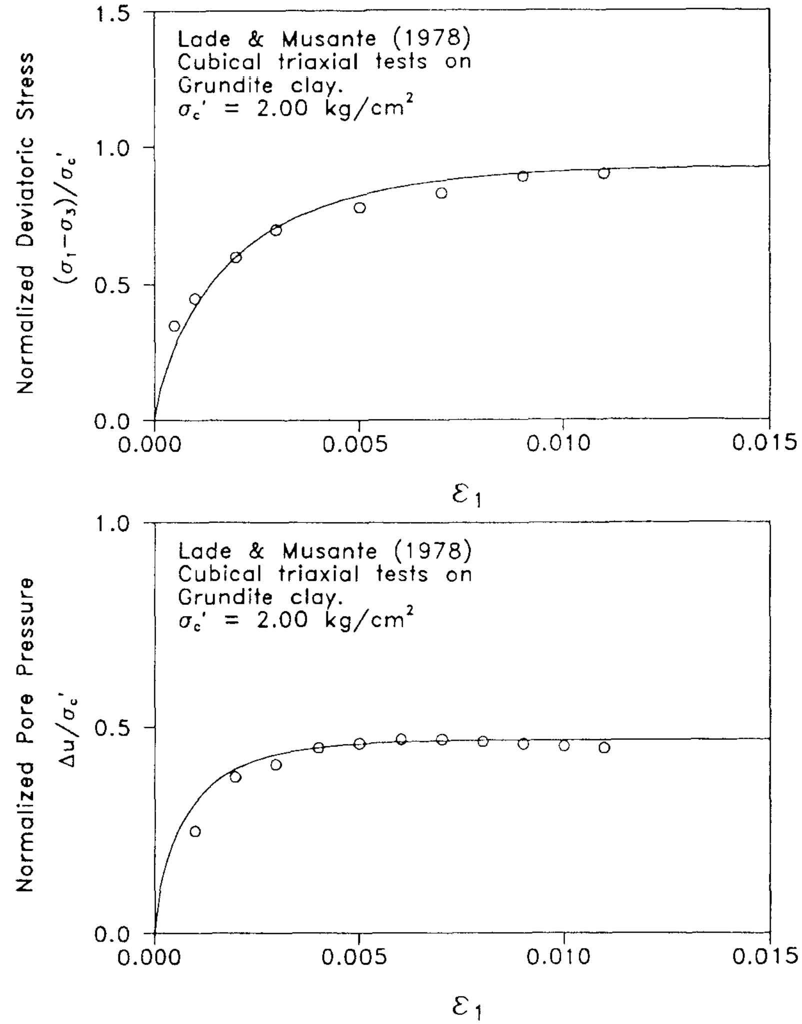

FIG. 13. Cubical Triaxial Tests on Undrained Clay (Lade and Musante 1978)

model can reproduce the main features of the behavior of the material, such as slight strain softening and dilatancy in the case of the drained samples, and the variation of the pore water pressure in the case of undrained samples.

Fig. 9 shows data from standard [Fig. 9(*a*)] and true [Fig. 9(*b*)] triaxial tests of clays obtained by Nakai et al. (1986). For both types of tests, the microplane model can correctly reproduce the main trends of the different strain-stress relations. The parameter values for this case are $\sigma_{DC}^{\infty} = 40.3$ psi, $\sigma_{DT}^{\infty} = 8.9$ psi, $\sigma_{T}^{\infty} = 31.7$ psi, and $C = 3.0$.

Fig. 10 shows several data from plane-strain tests carried out by Hambly (1972). The parameters used are $\sigma_{DC}^{\infty} = 35.2$ psi (0.245 MPa), $\sigma_{DT}^{\infty} = 7.6$ psi (0.0524 MPa), $\sigma_{T}^{\infty} = 20.5$ psi (0.141 MPa), $C = 2.5$, and $a_0 = 15$. The data correspond to different states of stress and strain, with different proportion between stresses or strains, and with undrained as well as drained conditions. The theoretical curves show an acceptable agreement with the experimental data in all cases.

Fig. 11 shows comparisons with oedometric tests carried out by Crooks and Graham (1976). It can be seen that the microplane model can be adjusted closely to the experimental results, including the value of the preconsolidation pressure. The parameters used are σ_{DC}^{∞} = 40.3 psi (0.278 MPa), σ_{DT}^{∞} = 9.7 psi (0.0669 MPa), σ_{T}^{∞} = 37.1 psi (0.256 MPa), and C = 2.9.

Finally, it has been found that the present model can provide good agreement with undrained triaxial extension tests performed by Balasubramanian and Uddin (1977), shown in Fig. 12 [parameters: σ_{DC}^{∞} = 15.6 psi (0.1076 MPa), σ_{DT}^{∞} = 6.5 psi (0.0448 MPa), σ_{T}^{∞} = 16.5 psi (0.1138 MPa), and C = 2.9], and with undrained cubical triaxial tests on Grundite clay carried out by Lade and Musante (1978), shown in Fig. 13 [parameters: σ_{DC}^{∞} = 49.3 psi (10.334 MPa), σ_{DT}^{∞} = 13.7 psi (10.0945 MPa), σ_{T}^{∞} = 51.3 psi (10.3537 MPa), and C = 3.5].

Summary and Conclusions

1. The microplane formulation, previously developed for brittle materials (concrete and rocks) has been extended to include the triaxial inelastic behavior of soils. It has been found that a formulation similar to the one presented for concrete can be used for soils and can describe all the behavior normally modeled by the critical state theory, as well as strain softening of overconsolidated soils, plain strain and stress and other multiaxial complex loading histories, and especially anisotropy.

2. The work presented is also an extension of a simplified model developed earlier for deviatoric creep of drained soils. A pore water pressure term is included in a manner that gives undrained as well as drained behavior as special cases. The model seems capable of reproducing well the main trends of both types of behavior, including the pore-pressure variation (for undrained tests) and the volume change (for drained tests).

3. The number of parameters that must be adjusted to fit complex test data (five at the most, four in some cases) is small enough for practical purposes.

4. The present formulation can be used as a general constitutive equation in nonlinear finite element programs.

5. The present model is explicit in the sense that no iterations are required to obtain a stress state from a given initial strain state and strain increment. Therefore, its use in a finite element code is easy and efficient, as is a similar model recently developed for concrete (despite the need to integrate over a hemisphere, typical of the microplane formulations).

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