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TECHNICAL NOTES
NOTES TECHNIQUES

Comment on recent analysis of concrete creep linearity and applicability of principle of superposition

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This technical note is a specific contribution related to the subject of the ConCreep-5 Conference reported in this issue, page 370.

A recent RILEM symposium paper [1] based on J.-H. Shen's dissertation at Darmstadt University [2] concluded that 'some well-known concrete creep functions are mathematically unsuitable for the application of the superposition principle' and 'the creep formulations in many building codes should therefore be reconsidered'. These conclusions served as the justification for a subsequent proposal for a nonlinear creep formulation in [3]. If these conclusions were correct, it would necessitate a sweeping change in the building codes and recommendations for creep analysis of concrete structures. However, the mathematical argument that led to these conclusions is not valid. It is important to clarify that, because some engineers might be tempted to accept these conclusions without checking the correctness of the mathematical argument.

In [2] and [1], various simple functions used for the description of creep curves, such as the power function, hyperbolic function or logarithmic function are considered. The function is differentiated at constant stress. The original function is then used to eliminate some variables. This yields, for the strain history, a differential equation which is satisfied by the function for the creep curve at constant stress. It is noted that this differential equation is nonlinear (in strain). On that basis it is argued that the superposition principle is inapplicable and that the nonlinear differential equation obtained should be used to predict the response at general time-variable stress histories.

This argument is erroneous. Since the nonlinear differential equation was obtained under the assumption of constant stress, it does not follow that this equation should be valid generally at variable stress. The differential equation obtained is only one of an infinite number of possible differential equations or integral equations whose solution for constant stress gives the original function for the creep curve at constant stress.

To illustrate this, Shen's argument in the case of a power function goes as follows. At constant uniaxial stress σ, the corresponding creep strain is written as

$$\varepsilon = \sigma C(t - t_0)^m \tag{1}$$

in which t is the time (age of concrete), t_0 is the time at the moment of applying the constant stress, and m and C are constants (Equation 5.59 in [2] and Equation 9 in [1]). It is then noted that Equation 1 satisfies the following differential equation

$$\frac{\dot{\varepsilon}}{\sigma C} = m\left(\frac{\varepsilon}{\sigma C}\right)^{(m-1)/m} \tag{2}$$

see Equation 5.61 of [2] and Equation 10 of [1]. Then it is claimed that, because of the nonlinearity of this differential equation, the superposition principle cannot be applied to the power function in Equation 1. This claim is not justified. The reason is that the superposition principle is used to obtain the creep response at variable stress, whereas Equation 2 has been obtained under the assumption of constant stress and is therefore proven valid for constant stress only.

Shen [1, 2] then makes arguments of the same kind for the hyperbolic function (Equations 5.65–5.67 of [2] and Equations 15–21 of [1]), the logarithmic function (Equations 5.68–5.70 of [2] and Equations 22–24 of [1]), the exponential function (Equations 25–30 of [1]), and so forth. All these arguments are invalid.

The error in Shen's argument can be demonstrated simply by bringing it *ad absurdum* in various ways. For example, one can check that Equation 1 satisfies not only the differential equation in Equation 2 but also the

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following differential equations:

$$\dot{\varepsilon} = Cm\sigma(t - t_0)^{m-1} \tag{3}$$

$$\dot{\varepsilon} = \frac{m}{t - t_0}\varepsilon \tag{4}$$

$$\dot{\varepsilon} = C^2 m \frac{\sigma^2}{\varepsilon}(t - t_0)^{2m-1} \tag{5}$$

$$\dot{\varepsilon} = Cm\sigma(t - t_0)\left(\frac{\varepsilon}{C\sigma}\right)^{(m-2)/m} \tag{6}$$

$$\dot{\varepsilon} = \frac{Cm\sigma}{(t - t_0)^2}\left(\frac{\varepsilon}{C\sigma}\right)^{(m+1)/m} \tag{7}$$

$$\dot{\varepsilon} = Cm\sigma(t - t_0)^{r-1}\left(\frac{\varepsilon}{C\sigma}\right)^{(m-r)/m} \tag{8}$$

where r is any number. Using Shen's kind of argument, one could claim that any one of these equations should apply at variable stress. This would obviously be inadmissible because these differential equations are different and give different responses to a given history of variable stress. Thus, the material behaviour would be non-unique. In fact, one can deduce from Equation 1 by this kind of erroneous argument an infinite number of different differential equations. Moreover, since creep can equivalently be described by integral equations, one can show similarly that different integral equations could be deduced by this kind of erroneous argument.

Among the foregoing differential equations, Equations 3 and 4 are linear, while the others are nonlinear. Equation 3 may be generalized to variable stress as $\dot{\varepsilon}(t) = \sigma(t)(t - t_0)^{m-1}$. But this is not the only possible generalization that yields Equation 1 when σ is constant. Another such generalization to variable stress has the form of a linear Volterra integral equation obtained by applying to Equation 1 the principle of superposition:

$$\varepsilon(t) = \int_{t_0}^{t} [E^{-1} + C(t - \tau)^m]\, d\sigma(\tau) \tag{9}$$

(Stieltjes integral), where E is the elastic modulus. Indeed, substituting the time variable stress $\sigma(t)$ as a Heaviside step function $\sigma(t) = \sigma_0 H(t - t_0)$ one obtains Equation 1 from Equation 9.

Alternatively, the power function in Equation 1 can be obtained as the integral of a system of first order differential equations that correspond to the Kelvin chain model with a certain retardation spectrum. To obtain Equation 1 from the retardation spectrum exactly, one must consider the number of these differential equations (and the number of the retardation times) to be infinite, which is the well known case of a continuous retardation spectrum. This retardation spectrum is obtained, according to Tschoegl [4] (or [5]), by means of Widder's explicit formula for the inversion of a Laplace transform [6]. In engineering practice, the linear formulation for variable stress which yields the power function in Equation 1 need not be known exactly. It suffices to represent the power function with an error not exceeding approximately 1%. In that case, considering a time range of four orders of magnitude of the elapsed time $t - t_0$ (sufficient for most practical purposes), one may use a Kelvin chain model with only four Kelvin units whose elastic moduli are given in Bažant and Prasannan [7] (which is referenced as No. 11 in [2]).

The nonlinear differential equations 5–8 can be generalized to variable stress by other ways than by replacement of constant σ with variable $\sigma(t)$. For example, Equation 1 may be obtained by substitution of $\sigma(t) = \sigma_0 H(t - t_0)$ or $[\sigma(t)]^2 = \sigma_0^2 H(t - t_0)$ into each of the following nonlinear integral equations:

$$\varepsilon^2(t) = C^2 \int_{t_0}^{t} (t - \tau)^{2m}\, d[\sigma^2(\tau)] \tag{10}$$

$$\varepsilon^2(t) = C^2 \int_{\tau=t_0}^{t} \int_{\xi=t_0}^{t} (t - \tau)^m (t - \xi)^m\, d\sigma(\tau)\, d\sigma(\xi) \tag{11}$$

$$\varepsilon^2(t) = C^2 \int_{\tau=t_0}^{t} \int_{\xi=t_0}^{t} \left(t - \frac{\tau + \xi}{2}\right)^{2m} d\sigma(\tau)\, d\sigma(\xi) \tag{12}$$

$$[\dot{\varepsilon}(t)]^2 = C^2 m^2 \int_{\tau=t_0}^{t} \int_{\xi=t_0}^{t} \left(t - \frac{\tau + \xi}{2}\right)^{2m-1} d\sigma(\tau)\, d\sigma(\xi) \tag{13}$$

Equations 11 and 12 have the form of the quadratic term in the Fréchet expansion of a nonlinear functional. An infinite number of different integral equations with single integrals or multiple integrals, all of which can be reduced to Equation 1, are possible.

A compliance function in the form of a power function (Equation 1) is of course widely used in non-ageing linear viscoelasticity of polymers. The principle of superposition has been verified by extensive numerical results as valid for many polymers. Now it should be noted that if Shen's argument ([1–3]) were applicable to concrete, it would also be applicable to polymers. Linear viscoelasticity would have to be abandoned as the theory for polymers. This consequence is absurd, and the validity of the principle of superposition in the viscoelasticity of polymers is further confirmation of the incorrectness of Shen's argument.

Another questionable aspect in applying Equations 1 and 2 to concrete is that they do not take into account the aging. To take it into account, one must replace constant C by a function $C(t_0)$. Then, however, Equation 2 cannot be considered as a general constitutive law applicable to variable stress $\sigma(t)$. There is simply no way of generalizing the function $C(t_0)$ to arbitrary time histories if these are first order differential equations. For example, if t_0 is considered as the time of first loading for a variable time history, one inevitably violates the requirement of continuity. Consider, as one history, stress $\sigma = 1$ to be applied at age $t_0 = 100$ days. Then consider, as one history, stress $\sigma = 0.001$ to be applied at age 10 days, and then increased to $\sigma = 1$ at age 100 days. The second history must yield nearly identical response, but using $t_0 = 10$ days one would get a very different response.

From the fundamental viewpoint, Shen's argument is based on a misconception. It does not distinguish between a possible mathematical representation of a particular curve, describing the response to a particular loading history, and a constitutive equation which must apply to every loading history. The adjective 'nonlinear' is used incorrectly: not in the sense of nonlinear dependence of the response history upon the loading history (which includes dependence on the stress magnitude) but in the sense of the temporal description of a particular response. In fact, speaking of the 'linearity of the creep function', as stated in [2] and [1], makes no sense in the theory of constitutive modelling of materials. The notion of a creep (compliance) function (defined as ε/σ_0 where σ_0 is the constant applied stress) implies linearity to begin with. One may speak of the linearity or nonlinearity of material behaviour or the constitutive model, but not of the creep function. Thus, it makes no sense to claim (p. 207 of [1]) that 'linearity of the creep function is a prerequisite for the application of the superposition principle.'

The concept of linearity is the basic result of functional analysis which emerged around 1900 from the works of Volterra, Fréchet, Riesz and Hadamard [8–15], not quoted by Shen. The most general and indeed the only way to express the linearity of a functional (the constitutive model for material behaviour) is (for uniaxial stress) to express the response (strain history) by means of a simple integral of the input (stress history) and a function of the current time and the time as integration variable. No linearity of the dependence of this function in any sense (such as Shen's) is required. The nonlinearity of a functional (i.e., of time-dependent constitutive model) is expressed either by the dependence of the kernel (i.e. the creep function) on the input variable (stress), in addition to time (as in the foregoing Equation 10), or by multiple history integrals [8], as in the foregoing Equation 11 or 13. Many studies of such nonlinear time-dependent constitutive models exist [16–22].

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