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Cohesive crack model for geomaterials: Stability analysis and rate effect

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The cohesive (or ficticious) crack model, characterized by softening stress-displacement relations, provides a good description of fracture of quasibrittle materials such as conrete, rock, or tough ceramics. The cohesive crack model is formulated in terms of compliance influence functions and the failure is analyzed as a stability problem. The size effect is determined by means of an eigenvalue problem. In this problem, the structure size for which a given relative crack length yields the maximum load is the eigenvalue. The model is further generalized to time dependence. The opening displacement is considered as a function of the cohesive stress and the opening rate of the crack. Finally, applications to rock and concrete are discussed.

1 INTRODUCTION

Geomaterials, such as rock or concrete, are not perfectly brittle, but quasi-brittle. Unlike perfectly brittle materials, there are significant zones of softening damage in front of the crack tip. This negates the applicability of linear elastic fracture mechanics (LEFM). To understand the material behavior, the softening damage zone must be taken into account.

The cohesive crack model is one of the fracture mechanics approaches to such behavior. The crack is described by a softening stress-displacement relation. Since the cohesive stress is not proportional to the crack opening displacement (separation of crack surfaces), the cohesive crack model is a nonlinear fracture model. Its basic concept was proposed by Barenblatt (1962) and Dugdale (1960), but their initial implementations were simplified and linearized. Barenblatt assumed that the cohesive stress is independent of the crack opening displacement. Dugdale made an even simpler assumption that the cohesive stress is constant (as in plastic yielding). The cohesive crack model of Hillerborg, Modéer and Petersson (1976) (also Petersson, 1981), as well as Bažant's (1976, 1982) crack band model (also Bažant and Cedolin, 1979; Bažant and Oh, 1983) are examples of truly nonlinear cohesive crack models. Only such models are considered in this paper.

The cohesive crack model is fundamentally different from the classical fracture mechanics. This theory is based on a critical quantity that controls the onset of crack growth, such as Griffith's (1924) surface energy to Irwin's (1948, 1957) critical stress intensity factor, critical crack tip opening displacement (Bilby, Cottrell and Swinden, 1963), and critical J-integral (Rice, 1968). In the cohesive crack model, the failure of a quasi-brittle structure is defined as the stability limit. It is possible to formulate a certain energy potential, and then all the critical quantities mentioned above are defined, directly or indirectly, by the first variation of the energy. Meanwhile the condition of stability limit is defined by the second variation of the energy.

This paper reviews and elucidates some recent advances in the cohesive crack model. First, an energy

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potential of the cohesive crack model is formulated in terms of cohesive stresses or opening displacements, and for different loading conditions. Based on this potential, the conditions of stability limit are derived as an eigenvalue problem. Furthermore, it is demonstrated how the solution of such an eigenvalue problem can be used to obtain the size effect law of the cohesive crack model. In the particular case of a linear softening law, this eigenvalue problem is linear. As a result, the maximum values of the load or the loading parameter (for similar structures of various sizes) can be expressed explicitly in terms of the eigensolution. The cohesive crack model is further generalized to take into account the rate effect in crack propagation. A rate-dependent softening law is introduced and numerical examples of the rate effect on the peak load are briefly described. Finally, the conference presentation also reviews some recent results on measurements of the rate effect in fracture of limestone and concrete and the effect of loading rate on the size effect.

2 COHESIVE CRACK MODEL BY COMPLIANCE FUNCTIONS AND POTENTIAL

For an elastic structure, such as a three-point-bend beam shown in Fig. 1, the process zone equation and load-line displacement can be written as

$$g[\sigma(x)] = -\int_{a_0}^{a} C^{\sigma\sigma}(x,x')\sigma(x')dx' + C^{\sigma P}(x)P \quad (1)$$

$$u = -\int_{a_0}^{a} C^{P\sigma}(x)dx + C^{PP}P \quad (2)$$

where a = total crack length which includes the process zone and the stress-free crack; the lower integration limit a_0 = the initial crack (notch) length; P = the loading parameter, u = the corresponding load-line deflection, and $C^{\sigma\sigma}(x,x'), C^{\sigma P}(x), C^{P\sigma}(x)$ and C^{PP} are compliance influence functions (Green's functions), with proper symmetry condition according to linear elastic reciprocity. In the time-independent cohesive crack model, the crack-opening displacement w is expressed as a function of σ,

$$w = g(\sigma) \quad 0 \leq w \leq w_c \quad (3)$$

where w_c is the critical separation (or crack opening)

The stress in the cohesive crack model is everywhere finite. Thus the total stress intensity factor K at the tip of the process zone must be zero, which is what we call the crack-tip equation:

$$K = K_P - K_\sigma = k_P P - \int_{a_0}^{a} k_\sigma(x)\sigma(x)dx = 0 \quad (4)$$

where $k_\sigma(x)$ is the stress intensity factor at the process zone tip due to a pair of unit forces acting in the process zone at the position x; k_P is the stress intensity factor due to a unit load. Equations (1) and (4) represent the fundamental equations of the cohesive crack model. If a is given, their solution yields the loading parameter and the cohesive stress.

Equations (1) and (4) can be equivalently expressed in the variational form:

$$\delta_0 G = G_0\delta\sigma = \int_{a_0}^{a} \delta\sigma(x)\Big[g(\sigma(x)) + \int_{a_0}^{a} C^{\sigma\sigma}(x,x')\sigma(x')dx' - C^{\sigma P}(x)P\Big]dx = 0 \quad (5)$$

$$\delta_a G = G_a\delta a = \frac{(K_P - K_\sigma)^2}{E'}\delta a = 0 \quad \forall \quad \delta a \quad (6)$$

From thermodynamics it is clear that a potential for a structure with a cohesive crack must exist provided that the stress-displacement relationship for the carck is monotonic. The crack does not suffer any unloading, the rest of the body is elastic, the loads are conservative, and the loading process is either isothermal or isentropic (or adiabatic). Then, according to the first law of thermodynamics, the energy potential of the system is the sum of the energy potentials of its individual parts. The elastic part of the structure obviously has a potential, and so do the conservative loads. As for the crack, the cohesive (crack-bridging) stress can be obtained as the derivative of the area under the curve (Helmholtz's free energy) and the opening displacement can be obtained as the derivative of the area to the left of the curve (Gibb's free energy, or complementary energy), provided there is no unloading and the curve descends monotonically. These areas represent the energy potentials for the crack.

A strictly mathematical derivation of the potential by a variational method is useful, since it allows deducing the field equations and possibly constructing approximate solutions.

It is easy to prove that the symmetry condition $\delta_a\delta_\sigma G = \delta_\sigma\delta_a G$ holds true. Invoking Green's theorem.

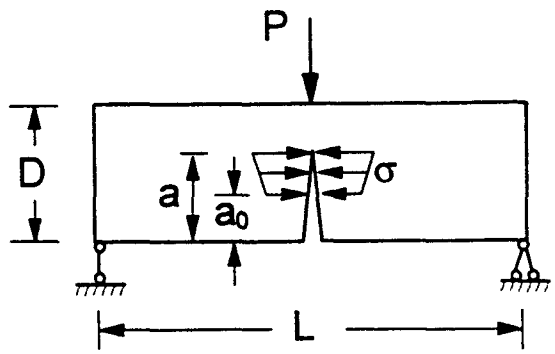

Figure 1: Geometry definitions of three-point beams

we can chose the integration path to obtain the potential G as follows

$$G = \int_0^{\sigma} G_{\sigma}(\sigma, a)\delta\sigma + \int_{a_0}^{a} G_a(0, a)\delta a \qquad (7)$$

Thus

$$G = \int_{a_0}^{a} \Gamma^*(\sigma)dx + \frac{1}{2}\int_{a_0}^{a}\int_{a_0}^{a} C^{\sigma\sigma}(x, x')\sigma(x) \sigma(x')dx'dx - P\int_{a_0}^{a} C^{\sigma P}(x)\sigma(x)dx + \frac{1}{2}\left[C^{PP}(a) - C^{PP}(a_0)\right]P^2 \qquad (8)$$

Here

$$\Gamma^*(\sigma) = \int_{f_t}^{\sigma} g(\sigma')d\sigma' \qquad (9)$$

which is called the complementary surface energy. In thermodynamic sense, the meaning of G for a static isothermal process of loading and fracture growth is the Gibbs free energy of the system.

Similar expressions can be obtained when the compliance influence functions are replaced by the stiffness influence functions and the displacements are used as the basic unknowns (Bažant and Li, 1994). The foregoing energy principle for a structure with a cohesive crack model was initially proposed by Li and Liang (1992) based on physical argument. The mathematical derivation of the potential outlined here and various forms of the potential function for a cohesive crack model have been given by Bažant and Li (1994).

3 STABILITY LIMIT AND SIZE-EFFECT LAW AS EIGENVALUE PROBLEM

To determine the stability limit of the cohesive crack model, one needs to calculate the so-called path derivative of equation (1) along an equilibrium path. The total first variation of the process zone equation must vanish. Under load control, the limit of crack stability corresponds to $\delta P = 0$; therefore

$$\int_{a_0}^{a}\left\{\frac{dg[\sigma(x)]}{d\sigma}\delta(x - x') + C^{\sigma\sigma}(x, x')\right\} \frac{\delta\sigma(x')}{\delta a}dx' = 0 \qquad (10)$$

This means that the bilinear form $G_{\sigma\sigma}$ is singular at the peak load. The contribution of the variation with respect to a is zero because the total stress intensity factor is zero.

The size effect law generally describes the scaling of certain physical quantity among geometrically similar structures of different sizes. For the size effect on the maximum load or loading parameter, the foregoing criterion of structural stability can be transformed into an eigenvalue problem. We introduce the dimensionless variables (keeping in mind the structure is of a unit thickness):

$$\bar{C} = CE', \quad \alpha = \frac{a}{D}, \quad \bar{\sigma} = \frac{\sigma}{f_t}, \quad \bar{w} = \frac{w}{w_c}, \qquad (11)$$
$$\bar{P} = \frac{P}{Df_t}, \quad \bar{D} = \frac{D}{L_0}, \quad \xi = \frac{x}{D}$$

Here $L_0 = EG_f/f_t^2$ = characteristic size of the process zone, and f_t = direct tensile strength of the material. For the sake of brevity of notations, we will drop the overbars with the understanding that all the variables are dimensionless. We can now write the singularity condition of $G_{\sigma\sigma}$ the condition that nonzero eigenfunction $v(\xi)$ exist and correspond to the smallest possible D satisfying

$$D\int_{\alpha_0}^{\alpha} C^{\sigma\sigma}(\xi'\xi)v(\xi')d\xi' = -2v(\xi)\frac{dg[\sigma(\xi)]}{d\sigma} \qquad (12)$$

where size D appears as an eigenvalue. Since only geometrically similar structures or specimens are considered, α is a constant. For a general softening law, $dg/d\sigma$ is a function of σ, therefore (12), as an eigenvalue problem, must be solved simultaneously with (1) and (4). The numerical method for solving this problem has been presented by Li and Bažant (1994).

For linear softening laws, however, $dg/d\sigma = -1$ = constant, and so the eigenvalue problem becomes independent of (1) and (4). We write the process zone equation in the dimensionless form as follows,

$$\frac{2}{D} - \frac{2}{D}\sigma(\xi) = -\int_{\alpha_0}^{\alpha} C^{\sigma\sigma}(\xi, \xi')\sigma(\xi')d\xi' + C^{\sigma P}(\xi)P \qquad (13)$$

Multiplying this with the eigenfunction $v(\xi)$ and then integrating with respect to ξ, we can obtain the maximum load as

$$P = \frac{2}{D}\frac{\int_{\alpha_0}^{\alpha} v(\xi)d\xi}{\int_{\alpha_0}^{\alpha} C^{\sigma P}(\xi)v(\xi)d\xi} \qquad (14)$$

The eigenvalue problem (12) and the peak load solution (14) provide a powerful formulation for solving the size-effect curve of the cohesive crack model. Similar forms of peak load solutions have been given in a discrete form by Li and Hong (1993).

If the device (e.g., the testing machine) that controls loading has a finite compliance C^M, the structure is stable only when the slope dP/du of the load deflection diagram is steeper than $-C^M$. So the limit of stability is characterized by $dP/du = -C^M$ (Bažant and Cedolin, 1991). Denoting u_T as the deflection that is controlled, we can write the eigenvalue problem as:

$$2v(\xi) = D\int_{\alpha_0}^{\alpha} \tilde{C}^{\sigma\sigma}(\xi', \xi)v(\xi')d\xi' \qquad (15)$$

where the modified compliance function is defined as

$$\bar{C}^{\sigma\sigma} = C^{\sigma\sigma} - C^{\sigma P}(C^{PP} + C^{M})^{-1}C^{P\sigma} \qquad (16)$$

The maximum value of the total deflection is

$$u_T = (C^{PP} + C^{M})\frac{\int_{\alpha_0}^{\alpha} v(\xi)d\xi}{\int_{\alpha_0}^{\alpha} C^{\sigma P}(\xi)v(\xi)d\xi} \qquad (17)$$

This expression includes as a special case ($C^M = 0$) the displacement controlled test in a rigid machine. Typical size effect curves for the maximum load and for the maximum load-line deflection are shown in Fig. 2 for three-point-bend beams. *The nominal stress is* $\sigma_N = 3PL/2bD^2$. It should be noted that the maximum deflection solution is valid only when $w(\alpha_0) \le w_c$. When the

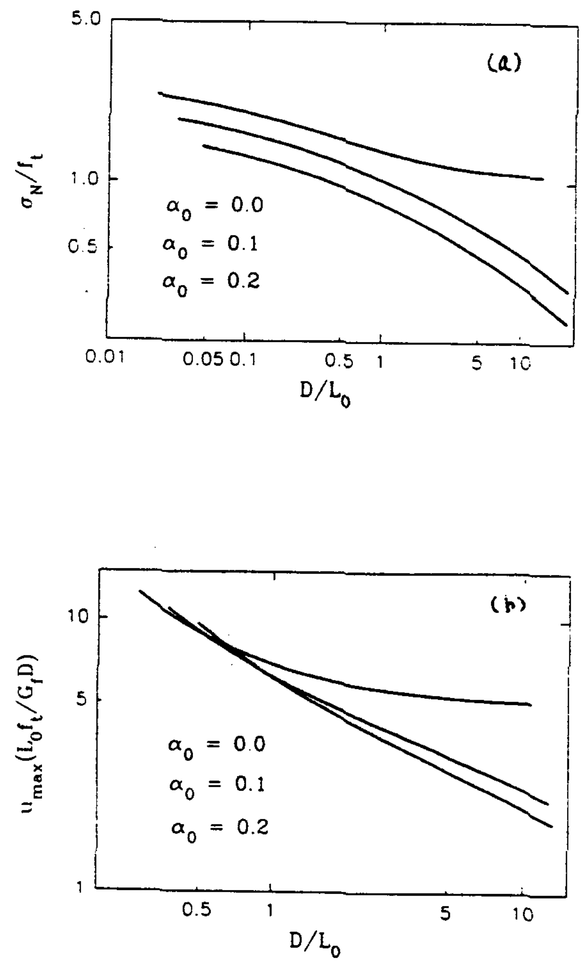

Figure 2: Typical size effect curves for peak loads (a) and maximum load line deflections (b)

equality holds, the corresponding size is called the critical size of the structure. Any structure of a size smaller than the critical size does not exhibit snap-back instability. The eigenvalue problems for the maximum deflection for loading by a soft machine as well as by a rigid machine, and the concept of critical size, are analyzed in detail by Li and Bažant (1994).

4 TIME-DEPENDENT COHESIVE CRACK MODELS

When the crack surfaces separate with different speeds, the softening law changes. If the bond ruptures are considered as a thermally activated process controlled by activation energy, and if certain simplifying assumptions are introduced, the following rate-dependent softening law can be derived (Bažant, 1993):

$$w = g\left[\sigma - \kappa \sinh^{-1}\left(\frac{\dot{w}}{\dot{w}_0}\right)\right] \qquad (18)$$

where $\dot{w}_0$ is the reference crack opening rate and κ is a dimensionless empirical parameter, considered in the range of 0.01 to 0.05. Larger values of κ makes the cohesive crack model more sensitive to the load rate change. The dimensionless process zone equation can be written as

$$\frac{2}{D}g\left[\sigma(\xi,t) - \kappa \sinh^{-1}\left(\frac{\dot{w}(\xi,t)}{\dot{w}_0}\right)\right] = - \int_{\alpha_0}^{\alpha} C^{\sigma\sigma}(\xi,\xi')\sigma(\xi',t)d\xi' + C^{\sigma P}(\xi)P(t) \qquad (19)$$

where the crack opening rate must be obtained from the rate form of the elastic relation:

$$\frac{2\dot{w}(\xi,t)}{D} = - \int_{\alpha_0}^{\alpha} C^{\sigma\sigma}(\xi,\xi')\dot{\sigma}(\xi',t)d\xi' + C^{\sigma P}(\xi)\dot{P}(t) \qquad (20)$$

The relation between the dimensionless stress and the stress rate is expressed as

$$\sigma(\xi,t) = 1 + \kappa \sinh^{-1}\left(\frac{\dot{w}(\xi,t_0(\xi))}{\dot{w}_0}\right) + \int_{t_0(\xi)}^{t} \dot{\sigma}(\tau)d\tau \qquad (21)$$

where $t_0(\xi)$ is the time instant at which the process zone tip first reaches the point ξ. The crack tip equation (4) remains the same. In the constant loading-rate tests (Bažant, Bai, and Gettu, 1993), the rate of crack mouth opening displacement, δ_{CMOD}, is used as the control variable, and so it must also be related to σ and P:

$$\delta_{CMOD} = - \int_{\alpha_0}^{\alpha} C^{c\sigma}(\xi)\sigma(\xi,t)d\xi + C^{cP}P(t) \qquad (22)$$

where $C^{c\sigma}(\xi)$ and C^{cP} are δ_{CMOD} caused by by a unit force at ξ in the process zone and by a unit load, respectively. A typical rate effect plot is shown in Fig.

3. For very slow rates δ_{CMOD}, the solutions of the rate-dependent model approach the rate-independent ones. A detailed fracture analysis based on (18)–(22) is planned for publication. Some solutions based on a similar crack-band model have been presented by Wu and Bažant (1993).

5 APPLICATIONS TO ROCK AND CONCRETE AND SIZE EFFECTS

The conference presentation also discusses the evidence of rate effects from fracture tests on limestone, granite and concrete and the influcence of the loading rate on the observed size effect. Measurements of the size effect on geometrically similar fracture specimens at different loading rates (Bažant, Gettu, and Bai, 1993; Bažant and Gettu, 1992) make it possible to calibrate the present type of rate-dependent fracture model. The data used for calibration are further supplemented by measurements of load relaxation after the displacement increase is arrested in the softening regime. This relaxation may be attributed to the creep on the bulk of the specimen, because it is observed in concrete, which creeps, but not in limestone, which does not creep. A further interesting effect is that, in concrete, the slower the loading, the higher the brittleness, as manifested by a shift of the size effect plot toward the linear elastic fracture mechanics asymptote. This effect is observed for concrete, but not for limestone, which does not creep. Another valuable experiment is to suddenly increase or decrease the loading rate in the post-peak softening regime of a fracture specimen. Such tests (Bažant, Gu and Faber, 1993) revealed that the softening can be reversed to hardening by a sufficiently high increase of the loading rate. These rate effects have been modeled by a simplified rate-dependent fracture model with creep, based on a rate-dependent generalization of the R-curve concept with creep (Bažant and Jirásek, 1993), and also by a simplified finite element model based on the crack band approach (Wu and Bažant, 1993). These results are also reviewed in the conference presentation.

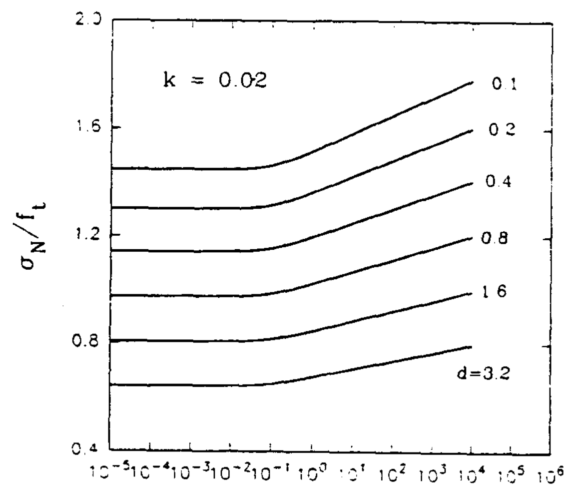

Figure 3: Typical rate effect of the cohesive crack model

CONCLUSION

The present analysis shows on the basis of variational analysis that a potential for a structure with a cohesive crack which undergoes no unloading must exist. The maximum load of a structure with a cohesive crack is formulated on the basis of a stability criterion and it is shown that the structure size for which a given relative crack length yields the maximum load can be solved as an eigenvalue of an eigenvalue problem for simultaneous integral equations. The formulation is then generalized to a rate-depedent cohesive crack model, in which the cohesive (crack-bridging) stress depends not only on the crack opening displacement but also on its rate.

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