Comment on Hyperbolicity of Wave Problem for Valanis' Global Damage Theory

G. Pijaudier-Cabot[1] **and Z. P. Bažant**[2]

Introduction

In a very interesting and stimulating recent paper, Valanis (1991) proposed a new type of nonlocal damage theory, called the global damage theory. Some conclusions of this paper, however, require further analysis, which is the purpose of this technical note.

Offering as motivation some questionable and unfounded critical comments on the nonlocal damage theory originally proposed by Pijaudier-Cabot and Bažant (1987) and Bažant and Pijaudier-Cabot (1988), Valanis formulated a different theory and claimed it to be superior because, as he stated, it exhibits unconditional hyperbolicity of the wave problem. On closer scrutiny, though, Valanis' claim is found to be invalid, and for further development of nonlocal damage models it is important to understand why.

One-Dimensional Wave Propagation

The nature of the problem can be most simply demonstrated by considering one-dimensional wave propagation along a bar of uniform cross section. In one dimension, the constitutive relation proposed by Valanis (1991) reads:

$$\sigma = E_s\epsilon, \quad E_s = \phi^2 E_0 = (1 - D)E_0 \tag{1}$$

where σ = stress, ϵ = strain, E_0 = elastic (initial) modulus, E_s = secant modulus, D = continuum damage (see also Lemaitre, 1992), and $\phi = \sqrt{1 - D}$ = nondimensional damage parameter called by Valanis the "integrity variable"; $0 \le D \le 1$ and $0 \le \phi \le 1$. The evolution of damage is defined in a nonlocal form by

$$\dot{\phi} = -E_0\phi\epsilon^2\dot{\xi} \tag{2}$$

$$\dot{\xi}(x, t) = A \int_{V_k} k(x, x')\dot{\epsilon}(x', t)dx' \tag{3}$$

if $\dot{\epsilon} > 0$, $\epsilon > 0$, and if the integral in Eq. (3) is positive; otherwise $\dot{\xi}(x, t) = 0$. A is the cross-section area of the bar, and $k(x, x')$ is a given empirical damage influence function. Since this function must remain invariant with a translation of the coordinate system, it must have the form $k(x - x')$. This influence function is positive, continuous, and differentiable over a subset denoted as V_k of the material domain which is the integration domain for $\dot{\epsilon}(x', t)$.

Consider small perturbations of an initial equilibrium state characterized by the strain ϵ_1 and the integrity variable ϕ_1, both assumed to be initially homogeneous (uniform) over the entire bar. The equation of motion reads

$$\frac{\partial \dot{\sigma}}{\partial x} = \rho \frac{\partial^2 \dot{u}}{\partial t^2} \tag{4}$$

where ρ is the mass density of the material, $\dot{\sigma}$ is the rate of stress, and $\dot{u}$ is the velocity of the small displacement perturbation. The rate of stress can be expressed as a function of the rate of strain using the constitutive relations. According to the method of linear comparison solid (Hill, 1959), we assume that the perturbation is such that $\dot{\epsilon} \ge 0$ everywhere. From Eqs. (2) and (3)

$$\dot{\phi}(x) = -E_0\phi_1\epsilon_1^2\dot{\xi}(x) = -AE_0\phi_1\epsilon_1^2 \int_{V_k} k(x - x') \frac{\partial \dot{u}(x')}{\partial x'} dx'. \tag{5}$$

From Eq. (1), the rate constitutive relation is

$$\dot{\sigma} = E_1\dot{\epsilon} + 2\dot{\phi}\phi_1 E_0\epsilon_1 \tag{6}$$

where E_1 is the secant modulus at the initial state ($E_1 = \sigma_1/\epsilon_1$). Substitution of (6) into (4) requires the computation of the spatial derivative of $\dot{\phi}$. It is now important to realize that the domain V_k depends on x (note that x' is the coordinate of the material points within the integration domain). Domain V_k

[1] Professor of Civil Engineering, Laboratoire de Mécanique et Technologie, ENS Cachan/CNRS/Universite Paris VI, 61 Avenue de Président Wilson, 94235 Cachan cedex, France.

[2] Walter P. Murphy Professor of Civil Engineering, Northwestern University, Evanston, IL 60208. Mem. ASME.

Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for publication in the ASME JOURNAL OF APPLIED MECHANICS. Manuscript received by the ASME APPLIED MECHANICS DIVISION, Mar. 19, 1993; final revision, Aug. 1, 1994. Associate Technical Editor: J. G. Dvorak.

shifts with x, therefore the spatial derivative of the integral in Eq. (5) is

$$\frac{d\left[\int_{V_k} k(x-x')\frac{\partial \dot{u}(x')}{\partial x'}dx'\right]}{dx} = \int_{V_k}\left[\frac{\partial k(x-x')}{\partial x}\frac{\partial \dot{u}(x')}{\partial x'} + \frac{\partial\left[k(x-x')\frac{\partial \dot{u}(x')}{\partial x'}\right]}{\partial x'}\right]dx'. \quad (7)$$

Alternatively, the integral term can be recast by using for the material points in V_k the coordinate $z = x' - x$ which is independent of x. Noticing that the influence function is symmetric, we thus obtain

$$\int_{V_k} k(x-x')\frac{\partial \dot{u}(x')}{\partial x}dx' = \int_{-l}^{l} k(z)\frac{\partial \dot{u}(x+z)}{\partial x}dz. \quad (8)$$

Variable z spans from $-l$ to l which is the length of domain V_k representing a basic property of the influence function. This variable is independent of the coordinate system provided the entire domain V_k is contained within the bar, and we will assume this condition in all our calculations. The calculation of the derivative of the integral in Eq. (8) with respect to x is straightforward and the result is, of course, identical to Eq. (7). Substitution of Eq. (6) with (5) and (7) into Eq. (4) yields

$$\frac{\partial^2 \dot{u}(x)}{\partial x^2} - 2AE_0\epsilon_1^2(x)\int_{-l}^{l} k(z)\frac{\partial^2 \dot{u}(x+z)}{\partial x^2}dz = \frac{1}{c_s^2}\frac{\partial^2 \dot{u}(x)}{\partial t^2} \quad (9)$$

where $c_s = \phi_1\sqrt{E_0/\rho}$ = wave velocity corresponding to the secant modulus (which represents the velocity of the unloading waves). In Valanis (1991) paper, the second term of the integrand in Eq. (7) is missing, and consequently the wave equation in Eq. (43) of that paper is not correct. Since the claim about unconditional hyperbolicity was based on this equation, that claim is also not correct. We will now examine the correct solution.

Possible Solutions

Let us now consider the harmonic wave:

$$\frac{\partial \dot{u}(x,t)}{\partial t} = Be^{-in(x-ct)} \quad (10)$$

where n is the wave number, c is the wave velocity, and $i^2 = -1$. Substituting Eq. (10) into Eq. (9), we find that the factor $Bne^{-in(x-ct)}$ cancels out from the equation and the equation of motion yields the wave velocity

$$c^2 = c_s^2[1 - 2A\epsilon_1^2\phi_1\bar{k}(n)] \quad (11)$$

with

$$\bar{k}(n) = \int_{-l}^{+l} k(z)e^{-inz}dz. \quad (12)$$

Setting $k(z) = 0$ for all $z \notin [-l, +l]$, we can regard $\bar{k}(n)$ as the Fourier transform of the damage influence function.

Equation (11) is a wave dispersion equation. This means that the velocity c is not constant but depends on the wave number n (or on the wavelength $2\pi/n$). Wave propagation in the strain-softening nonlocal material is possible as long as the expression for c^2 in Eq. (11) is real and positive. When c^2 becomes negative, wave propagation is impossible. The critical wave number n_c for which the phase velocity is zero is the solution of the equation

$$\bar{k}(n_c) = \frac{1}{2A\epsilon_1^2\phi_1}. \quad (13)$$

If the damage influence function is chosen as a bell-shaped function in the form

$$k(z) = \exp\left(\frac{|z|^2}{2l_i^2}\right) \quad (14)$$

where l_i is a characteristic length of the nonlocal continuum, the wave velocity of a small perturbation harmonic wave is

$$c^2 = c_s^2(1 - 2A\epsilon_1^2\phi_1 e^{-n^2l_i^2/2}),$$

$$\text{with} \quad n_c = \frac{\sqrt{2}}{l_i}\sqrt{\log\left(\frac{1}{2A\epsilon_1^2\phi_1}\right)}. \quad (15)$$

Therefore, the critical wavelength $2\pi/n_c$ is proportional to the characteristic length as obtained by Pijaudier-Cabot and Bodé (1992). Furthermore, it can be checked that a critical wave number is real and greater than zero only if the material undergoes strain softening. Dispersion of waves in a nonlocal continuum is a well-known and expected property. Sluys pointed out recently (1992) the importance of dispersion in nonlocal strain-softening continua: propagation of waves with a short enough wavelength is always possible even when the material undergoes softening. This can be easily checked from Eq. (15); when $n \to \infty$ the phase velocity tends to that of the elastic unloading waves c_S, which is real.

Hyperbolicity of the Wave Equation

Ellipticity or hyperbolicity of the equation of motion cannot be decided directly. This equation is an integro-differential rather than a partial differential equation. The equation of motion can be transformed into a differential equation if the possible solution belongs to the ensemble of functions $v(x)$ for which $\partial^2 v/\partial x^2$ exists and

$$\int_{-l}^{+l} k(z)\frac{\partial^2 v(x+z)}{\partial x^2}dz = \left[\int_{-l}^{+l} k(z)F(z)dz\right]\frac{\partial^2 v(x)}{\partial x^2} \quad (16)$$

where $F(z)$ is a known function. The ensemble of such functions may not contain all the possible solutions, but the harmonic solutions considered in the previous section verify this property. For this type of functions, the equation of motion becomes

$$\frac{\partial^2 v}{\partial x^2}(x)(1 - 2AE_0\phi_1\epsilon_1^2 I^*) = \frac{1}{c_s^2}\frac{\partial^2 v}{\partial t^2},$$

$$\text{with} \quad I^* = \int_{-l}^{+l} k(z)F(z)dz. \quad (17)$$

If $v(x)$ is harmonic, I^* is the Fourier transform of the damage influence function. For this partial differential Eq. (17), the calculation of the characteristics is straightforward: the variation of the first derivatives of v is combined with the equation of motion

$$\frac{\partial^2 v}{\partial x^2}(1 - 2AE_0\phi_1\epsilon_1^2 I^*) = \frac{1}{c_s^2}\frac{\partial^2 v}{\partial t^2} \quad (18)$$

$$d\left(\frac{\partial v}{\partial t}\right) = \frac{\partial^2 v}{\partial t^2}dt + \frac{\partial^2 v}{\partial t\partial x}dx,$$

$$d\left(\frac{\partial v}{\partial x}\right) = \frac{\partial^2 v}{\partial x^2}dx + \frac{\partial^2 v}{\partial t \partial x}dt \tag{19}$$

and the characteristic determinant K of this system is

$$K = \frac{dx^2}{c_s^2} - (1 - 2AE_0\phi_1\epsilon_1^2 I^*)dt^2. \tag{20}$$

The characteristic lines in the (x, t) space are given by the equation

$$\left(\frac{dx}{dt}\right) = c_s\sqrt{(1 - 2AE_0\phi_1\epsilon_1^2 I^*)} \tag{21}$$

which yields, for harmonic waves, exactly the wave velocity.

Hyperbolicity of the partial differential Eq. (17) is lost when the characteristic lines lie in the imaginary region, i.e., when the wave number is bellow the critical wave number for harmonic waves defined in Eq. (13). Therefore, we conclude that the wave equation is not unconditionally hyperbolic for Valanis' nonlocal model. When $v(x)$ is harmonic (a possible admissible solution), the integro-differential equation of motion (9) reduces to an elliptic partial differential equation if wave number is bellow the critical wave-number. However, nonlocality of the type considered in this note enables waves with a short wave length to propagate in the softening regime (Pijaudier-Cabot and Benallal, 1992).

Conclusions

(1) Wave propagation in a medium described by the global damage theory is dispersive. This feature is restricted neither to Valanis' theory nor to the integral-type nonlocal constitutive relations in general. As pointed out by Sluys (1992), all the constitutive models proposed in the literature that incorporate an internal length are dispersive: the micro-polar continua, the gradient-dependent models and also rate-dependent models. For these models, Sluys demonstrated that hyperbolicity of the wave equation is always preserved.

(2) The wave equation of Valanis' global damage theory is not unconditionally hyperbolic. This result is at variance with the assertion in Valanis' paper (1991) which was based on an incorrect wave equation. The derivative of the damage controlling variable $\xi(x, t)$ with respect to coordinate x was not calculated taking into account the variation of the limits of the nonlocal averaging integral. The question of well-posedness is, however, still an open issue since finite bodies and initial conditions ought to be included in such considerations.

Acknowledgment

Partial financial support of the Commission of the European Communities through the Brite-Euram Programme (Project BE-3275) to the first author is gratefully acknowledged. The work of the second author was partially supported under AFOSR grant 91-0140 to Northwestern University.

References

Bažant, Z. P., and Pijaudier-Cabot, G., 1988, "Nonlocal Continuum Damage: Localization, Instability, and Convergence," ASME JOURNAL OF APPLIED MECHANICS, Vol. 55, pp. 287–293.

Hill, R., 1959, "Some Basic Principles in the Mechanics of Solids without a Natural Time," *Journal of Mechanics and Physics of Solids*, Vol. 7, pp. 209–225.

Lemaitre, J., 1992, *A course on Damage Mechanics*, Springer-Verlag, New York.

Pijaudier-Cabot, G., and Bažant, Z. P., 1987, "Nonlocal Damage Theory," *ASCE Journal of Engineering Mechanics*, Vol. 113, pp. 1512–1533.

Pijaudier-Cabot, G., and Benallal, A., 1993, "Strain Localization and Bifurcation in a Nonlocal Continuum," *International Journal of Solids and Structures*, Vol. 30, pp. 1761–1775.

Pijaudier-Cabot, G., and Bodé, L., 1992, "Localization of Damage in a Nonlocal Continuum," *Mechanics Research Communications*, Vol. 19, pp. 145–153.

Sluys, L. J., 1992, "Wave Propagation, Localization and Dispersion in Softening Solids," Doctoral dissertation, Technical University of Delft, The Netherlands.

Valanis, K. C., 1991, "A Global Damage Theory and the Hyperbolicity of the Wave Problem," ASME JOURNAL OF APPLIED MECHANICS, Vol. 58, pp. 311–316.