

# SIZE EFFECT IN PENETRATION OF SEA ICE PLATE WITH PART-THROUGH CRACKS. I: THEORY

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**ABSTRACT:** The paper analyzes the vertical penetration of a small object through a floating sea ice plate. The analysis takes into account the fact that the bending cracks reach only through part of the ice plate thickness and have a variable depth profile. The cracks are modeled according to the Rice-Levy nonlinear softening line spring model. The plate-crack interaction is characterized in terms of the compliance functions for the bending moments and normal forces in the crack plane, which are computed by an energy-based variational finite-difference method. The radial crack is divided into vertical strips, and a numerical algorithm with step-by-step loading is developed to calculate the vertical growth of the crack in each strip for a prescribed radial crack length increment. The initiation of crack strips from the surface of the plate is decided on the basis of a yield strength criterion with a fracture based flow rule. Systems of up to 300 nonlinear equations are solved by the Levenberg-Marquardt optimization algorithm. The maximum load is reached when the circumferential cracks begin to form. Numerical calculations, comparison of the results with test data, and a study of scaling laws are relegated to the companion paper, which follows in this issue. Numerical calculations show a typical quasi brittle size effect such that the plot of  $\log \sigma_N$  versus  $\log h$  (where  $\sigma_N$  = nominal stress at maximum load and  $h$  = plate thickness) is a descending curve whose slope is negligible only for  $h < 0.2$  m and then gets gradually steeper, asymptotically approaching  $-1/2$ . The calculated size effect agrees with the existing test data, and contradicts previous plasticity solutions.

## INTRODUCTION

Sea ice plates subjected to a vertical load applied on a small area from above or below typically fail by propagation of radial cracks in a star pattern [Fig. 1(a)]. The maximum load, which represents the failure load under load control conditions, is reached when circumferential cracks start to form (Frankenstein 1963).

The penetration problem is important for many operations in the Arctic Ocean such as airplanes landing on the ice, vehicles traveling on the ice, or submarines penetrating through the ice from below. Of particular interest is the size effect on the nominal strength, which governs extrapolation of small-scale laboratory tests to such field situations.

The problem has been studied extensively for a long time (Bernstein 1929; Frankenstein 1963, 1966; Kerr 1975, 1996). Because small-scale laboratory tests show sea ice to be notch insensitive, the applicability of fracture mechanics to sea ice has been doubted for a long time. So it is not surprising that the penetration problem had until recently been analyzed on the basis of the strength criterion and plastic limit analysis (Kerr 1996). However, the doubts started to dissipate after Dempsey (1989, 1990; DeFranco and Dempsey 1990) suggested that this conclusion was due merely to insufficient size of the specimens. The applicability of fracture mechanics to sea ice on a large scale has recently been demonstrated by the in-situ experiments of Dempsey and coworkers (Adamson et al. 1995; Dempsey et al. 1995a,b; Dempsey 1996; Mulmule et al. 1995; Mulmule and Dempsey 1997).

In the early studies, the load capacity of a floating sea ice plate was determined by elastic analysis coupled with the tensile strength criterion (Bernstein 1929). Nevel (1958) used the strength criterion, assuming that there is a large number of

radial bending cracks splitting the ice plate into small-angle wedges that can be treated as beams of variable cross section.

Sea ice, however, is a quasibrittle material which, when failing in tension, exhibits no plastic yield plateau but postpeak softening. Such behavior alone, as a matter of principle, implies the plasticity solutions to be unrealistic except for very small sizes. [For general information, see Bažant (1983, 1984, 1993); Bažant and Planas (1998). For specific information regarding sea ice, see Bažant and Kim (1985); Bažant and Gettu (1991).] From the practical viewpoint, the main limitation of plastic limit analysis is that it cannot capture the size effect on the nominal strength of the structure. Only fracture mechanics can do that. The size effect is not statistical (Bažant et al. 1991), nor fractal (Bažant 1997a,b), but is caused by the fact that, with increasing size (plate thickness, in this case), the stored and subsequently released energy increases faster than the energy consumed and dissipated by fracture. This is called the quasibrittle size effect.

An additional reason why correct solutions must be based on fracture mechanics is that the maximum load is reached only after stable growth of large cracks. In that case, a deterministic size effect due to energy released by large cracks necessarily takes place and prevails over the statistical size effect of Weibull type (Bažant et al. 1991; Bažant and Planas 1998). If the plate failed right at crack initiation, the use of fracture mechanics would not be necessary and the strength criterion might be appropriate, provided that the stress redistribution due to a microcracking zone formed before attaining the maximum load could also be neglected. In that case, which is not the real situation, the size effect of Weibull type would have to be expected (Bažant et al. 1991; Bažant 1997a,b; Bažant and Chen 1997; Bažant and Planas 1998).

Fracture mechanics has been applied to the penetration problem by Slepian (1990), Bažant (1992a,b), Bažant and Li (1994a,b), Li and Bažant (1994), Bažant et al. (1995), and Dempsey et al. (1995a,b). To take the effect of the radial bending cracks approximately into account, linear elastic fracture mechanics (LEFM) was introduced by Bažant and Li (1994a,b). They used Nevel's (1958) one-dimensional approximation to calculate the energy release caused by radial crack propagation. Li and Bažant (1994) carried out a two-dimensional analysis and formulated a method to determine the number of initiating radial cracks.

In the aforementioned studies, the radial cracks were as-

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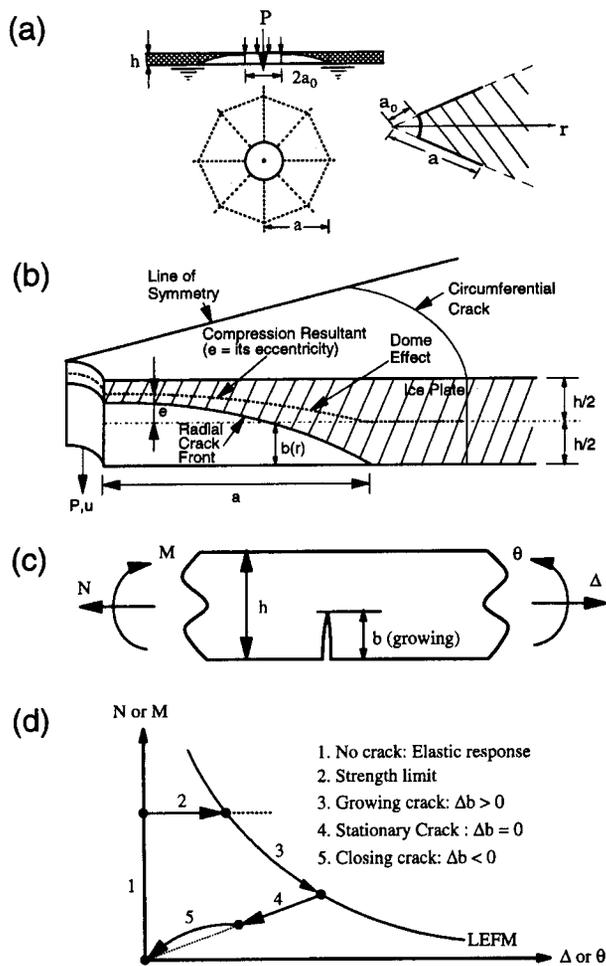


FIG. 1. Diagram of: (a) Vertical Cross Section of Floating Sea Ice Plate, Crack Pattern Viewed from Top and Plate Wedge between Two Adjacent Radial Cracks; (b) Half of Plate Wedge with Part-Through Crack Profile and Schematic Representation of Compression Resultant Elevation Showing Dome Effect; (c) Bending Moment and Normal Force Acting on Cracked Cross Section, and Additional Rotation and Displacement due to Crack; (d) Stages in Evolution of Vertical Crack Strips

sumed to be fully opened through-cracks. However, the horizontal expansion that is associated with bending cracks causes the cracks to open through only a part of the thickness of the plate, as experimentally observed by Frankenstein (1963). This expansion induces compressive forces in the plate and thus engenders a dome effect, which plays an important role in helping to carry the vertical load [see the idealized dome shape in Fig. 1(b)]. The importance of the dome effect in plates, and a similar effect in beams called the arching action, was recognized by many authors. [Regarding concrete, see Ockleston (1958); Park (1964, 1965); Desayi; and Kulkarni (1977); Park and Gamble (1980); Black (1975); Braestrup and Morley (1980). Regarding floating ice, see Sodhi (1995a,b); Kerr (1996)]. But these studies were made under the assumption of plastic behavior.

A plate with part-through cracks is actually a three-dimensional fracture problem. However, based on the simplifying idea of an embedded softening line spring, proposed by Rice and Levy (1972) in a study of the fracture of metal plates, the problem can be reduced to a two-dimensional one. Since the opening depth of the radial bending cracks is unknown in advance, the functions defining the compliances of the line springs must be solved together with the plate problem.

In the present paper, the problem will be solved by means of a system of integral equations based on the compliance

functions of the floating plate wedge formed by two adjacent cracks. This formulation was given by Bažant et al. (1995), who also incorporated the nonlinear line spring model. The formulation will be extended by introducing a method to calculate the initiation of new crack segments. In a subsequent paper (Bažant and Kim 1998), a numerical solution of the problem will be obtained using a nonlinear optimization algorithm. The size effect will be analyzed and the dome effect will be demonstrated.

## FLOATING PLATE WITH PART-THROUGH RADIAL CRACKS

Consider an infinitely extending elastic plate of thickness  $h$ , floating on water of specific weight  $\rho$  [Fig. 1(a)]. The water acts exactly as an elastic foundation of Winkler type. The differential equation of equilibrium of the plate in terms of the vertical downward deflection  $w$  as a function of rectangular coordinates  $x$  and  $y$  may be written as  $D\nabla^4 w + \rho w = 0$ , where  $D = Eh^3/12(1 - \nu^2) =$  cylindrical stiffness of the plate;  $\nu =$  Poisson's ratio; and  $E =$  Young's modulus. The load is assumed to be applied only on the boundary of the plate.

It is convenient to introduce a length constant for the plate,  $L = (D/\rho)^{1/4}$ , called the flexural wavelength. It represents the length over which an end disturbance in a semiinfinite plate decays to  $e^{-1}$  of the end value. Introducing the dimensionless coordinates  $X = x/L$  and  $Y = y/L$ , we may rewrite the differential equation of the plate as

$$\left(\frac{\partial^2}{\partial X^2} + \frac{\partial^2}{\partial Y^2}\right) \left(\frac{\partial^2 w}{\partial X^2} + \frac{\partial^2 w}{\partial Y^2}\right) + w = 0 \quad (1)$$

The assumption of full-through cracks, however, is realistic only if the parts of the plate separated by the crack can move freely apart. This is true for a long thermal fracture in an infinite ice plate (Bažant 1992a,b), or for very thin plates. In the present problem, the plate wedge prevents the sides of the crack from moving freely apart. Thus, the relative rotations across the crack cause the top portion of the ice plate to come under horizontal compression [Fig. 1(a)]. The resultant of the in-plane compression force in the planes gets shifted above the midthickness. This engenders a dome effect (Fig. 2), which

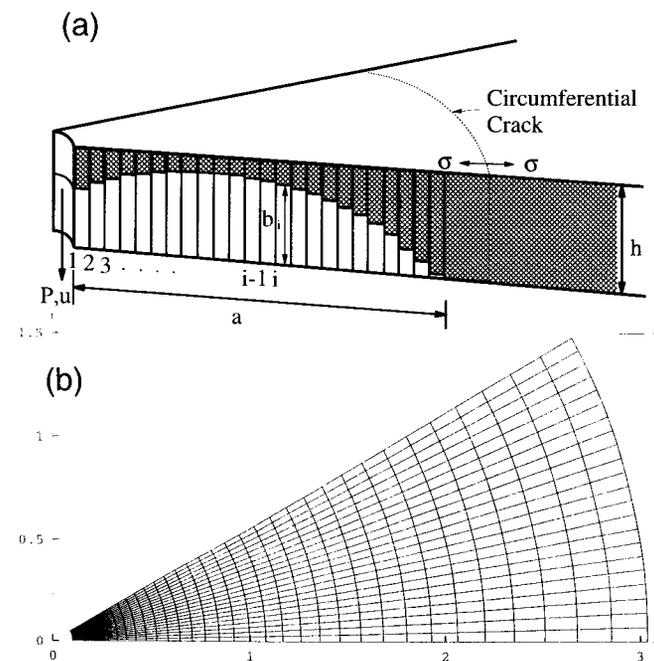


FIG. 2. Diagram of: (a) Subdivision into Vertical Crack Strips for Purpose of Numerical Analysis; (b) Mesh Used for Calculating Compliance Functions of Plate Wedge

helps to carry the vertical load. The previous fracture analyses of this problem, except that of Dempsey et al. (1995a,b), could not capture the dome effect. The compressive stress above the part-through crack can be high enough to cause microcracking damage, which is manifested by a whitening of the ice surface seen from above (Frankenstein 1963). The dome effect is the reason why the cracks reach only through a part of the thickness (for loading up to the maximum load; after the peak, some cracks may penetrate through the whole thickness).

The partial opening of the radial cracks is a three-dimensional phenomenon. A detailed three-dimensional fracture mechanics analysis would be computationally expensive, if not intractable. In the present analysis, the vertical cracked cross section of the plate is subdivided by discrete nodes into narrow vertical strips. In each strip, the crack is assumed to propagate vertically and independently of the crack propagation in the adjacent strips. This simplification will require us to introduce some rule according to which the vertical crack in each strip initiates from the smooth surface of the plate. In reality, the crack front of course propagates horizontally from one vertical strip to the next and no new cracks initiate. This aspect is neglected in the present analysis. The resulting error may be expected to be small if most of the crack front edge is almost horizontal, which is certainly true since the radial crack is much longer than its vertical depth.

The weakening of the plate by a part-through crack will be modeled in terms of distributed nonlinear softening line springs, in the manner proposed by Okamura et al. (1972) and Rice and Levy (1972). In each infinitesimal vertical strip in the crack plane, the crack is assumed to grow vertically as a function of the bending moment and normal force in the same strip only, i.e., independently of the bending moments and normal forces in the adjacent strips. [This approximation is similar to replacing the foundation on an elastic half-space with the Winkler foundation; e.g., Bažant and Cedolin (1991).] By virtue of the line spring concept, the plate with radial cracks can still be analyzed as a two-dimensional problem. The effect of partial cracking in the plate is reflected by increased compliance, characterized by the line springs. In previous applications of the line spring concept (Rice and Levy 1972; Dempsey et al. 1995a,b), the shape of the depth profile of the part-through crack was assumed. In our approach, however, the depth profile will be considered as unknown and will be solved.

Denote by  $\Delta$  and  $\theta$  the additional in-plane circumferential relative displacement and relative rotation (about the radial ray) that is caused by the radial crack and is work-conjugate to  $N$  and  $M$  [Fig. 1(c)];  $\Delta$  and  $\theta$  vary in an unknown way with the applied load and with the radial distance  $r$ . A positive bending moment is that which causes tension at the bottom surface of the plate, and a positive normal force is that which is tensile. According to the line spring concept

$$\Delta = \lambda_{11}N + \lambda_{12}M, \quad \theta = \lambda_{21}N + \lambda_{22}M \quad (2)$$

where  $\lambda_{ij}(i, j = 1, 2)$  = compliances of the line springs

$$\lambda_{ij} = 2 \frac{1 - \nu^2}{E} \int_0^b k_i(t)k_j(t) dt \quad (3)$$

$k_i (i = 1, 2)$ , where  $k_1$  = stress intensity factor in an infinitely long plate with a single-sided crack (or notch) of depth  $b(r)$ , loaded remotely by a unit force  $N$ ; and  $k_2$  = stress intensity factor of the same strip loaded remotely by a unit moment  $M$ . Approximate expressions for  $\lambda_{11}$  and  $\lambda_{22}$  are given by Tada et al. (1985). So one needs to deduce an approximate expression only for  $\lambda_{12}$ . This has been done by means of (3).

The degree of accuracy of the line spring model was clarified by Kuo et al. (1995). They showed that the stress intensity factors obtained by the line spring model closely approach

those calculated by three-dimensional analysis as the ratio of the radial crack length to the vertical crack depth increases. They also showed that the stress intensity factors calculated by the line spring model are very accurate for part-through cracks for which the ratio of the radial crack length to the crack depth is large, which is the case for sea ice.

The additional relative rotation  $\theta$  and horizontal displacement  $\Delta$  calculated from (2) must be equal to the rotation and displacement calculated from the compliances of the plate wedge. This represents a crack compatibility condition that reads as follows:

$$\lambda_{21}(r)N(r) + \lambda_{22}(r)M(r) = C_{MP}(r) \frac{P}{n} - \int_0^a C_{MM}(r, r')M(r') dr' \quad (4)$$

$$\lambda_{11}(r)N(r) + \lambda_{12}(r)M(r) = - \int_0^a C_{NN}(r, r')N(r') dr' \quad (5)$$

where  $n$  = number of radial cracks of length  $a$ ;  $C_{MP}(r)$  = rotation of the plate at  $r$  due to a unit  $P$ ;  $C_{MM}(r, r')$  = rotation of the plate at  $r$  due to a unit moment acting on the crack surfaces at  $r'$ ; and  $C_{NN}(r, r')$  = circumferential displacement at  $r$  due to a unit normal force  $N$  at  $r'$ . The negative sign in front of the integrals is due to the fact that a positive force on the crack surfaces causes the crack to close.

The applied load  $P$  is related to the load-point displacement  $u$  by the equation

$$u = C_{PP}P + \int_0^a C_{PM}(r)M(r) dr \quad (6)$$

where  $C_{PM}(r) = C_{MP}(r)$ ; and compliance  $C_{PP} = u$  caused by loading the plate wedge alone with  $P = 1$ .

Due to symmetry, the compliance functions are calculated for a wedge plate representing one-half of the wedge between two radial cracks [Fig. 1(b)] of length  $a$  and depth  $b(r)$ ;  $b(r) > 0$  for  $a_0 \leq r < a$ , and  $b(r) = 0$  for  $r \geq a$  along the radial line with a crack. The possibility that the radial crack lengths might become unequal is not considered.

The compliance influence functions can be discretely represented by compliance matrices. They are solved numerically, e.g., by the finite-element method or finite-difference method. The integrals in (4) and (5) are then approximated by sums, and thus (4) and (5) yield a system of nonlinear algebraic equations.

## CRITERIA FOR PLASTIC AND LINEAR ELASTIC FRACTURE MECHANICS (LEFM) STAGES OF CRACK GROWTH

In a discrete formulation with  $n$  nodes along the radial ray containing the crack, there are  $3n + 1$  unknown variables—namely, the nodal values of  $M$ ,  $N$ , and  $b$ , and the applied load  $P$ , with the load-point displacement  $u$  being specified. Eqs. (6), (4), and (5) yield  $2n + 1$  discrete compatibility equations. To obtain the same number of discrete equations and unknowns, we need one more equation for each node. The necessary equation is the fracture criterion. Various types of the fracture criterion needed for various stages of the analysis are schematically represented in Fig. 1(d).

The condition for crack propagation in the LEFM stage [stage 3 in Fig. 1(d)] may be written as

$$K_I = K_f, \quad (a_0 \leq r \leq a) \quad (7)$$

in which

$$K_I = k_1[b(r)]N(r) + k_2[b(r)]M(r) \quad (8)$$

and  $K_I$  = stress intensity factor;  $K_f = \sqrt{E'G_f}$  = critical stress intensity factor (fracture toughness) of sea ice;  $G_f$  = fracture energy of ice;  $E$ ,  $\nu$  = Young's modulus and Poisson's ratio of ice; and  $E' = E/(1 - \nu^2)$ .

Initially, the ice plate is elastic [stage 1 in Fig. 1(d)]. The initiation of the vertical crack strips from the plate surface ( $b = 0$ ) cannot be handled by LEFM. In general, one could introduce the cohesive crack model of Hillerborg type for the initiation of cracks (Bažant and Planas 1998). But that would be unnecessarily complicated because, as the computations confirm, the portion of the radial crack length in which the crack is in the cohesive (or plastic) state is very small, and the maximum depth to which a plastic crack reaches is only about  $0.02h$ . But when  $h$  is small ( $h \leq 0.2$  m), the plate fails by a punch-through cone around the circular loaded area.

Therefore, we can adopt a simplified form of the cohesive crack model. We will base it on the assumption that, after reaching the tensile strength limit  $f'_t$  of ice, the crack grows as a plastic crack at yield limit as long as both  $M$  and  $N$  are below the values that correspond, according to LEFM [Fig. 1(d)], to the values of  $\theta$  and  $\Delta$ . Specifically, we assume in this simplified model that, in the plastic limit state, the stress distribution consists of a constant normal stress equal to strength  $f'_t$  over the entire crack length and a linearly distributed normal stress across the remainder of the cross section (the ligament). Thus, after taking into account the horizontal and moment equilibrium conditions, we find that the plastic stage of the crack [stage 2 in Fig. 1(d)] is characterized by

$$\sigma_M(b) + \sigma_N(b) = f'_t \quad (9)$$

in which

$$\sigma_M(b) = \frac{6M}{h(h+2b)}, \quad \sigma_N(b) = \frac{N}{h-b} \quad (10)$$

Strictly adhering to the theory of plasticity, one would have to introduce for the plastic stage 2 a normality rule as the flow rule that determines the relation between the ratio  $\lambda/\Delta$  and the ratio  $M/N$ . However, (9) and (10) imply a different relation between these two ratios, namely

$$\frac{\Delta}{\lambda} = \frac{\lambda_{11} + \lambda_{12}(M/N)}{\lambda_{21} + \lambda_{22}(M/N)} \quad (11)$$

which differs from the normality rule of plasticity. So, with the strict plastic formulation, the transition from the plastic stage 2 to the LEFM stage 3 would be very complicated. It would not occur for  $M$  and  $N$  simultaneously. One would have to distinguish various transitional stages, and if such transitional stages were ignored, the values of  $M$  and  $N$  would change from the plastic to the LEFM stage discontinuously, by jumps. Numerical convergence of the solution would then be difficult to achieve.

To circumvent the aforementioned problem, the following simplifying idea is proposed: The flow rule for a plastic crack is defined by LEFM. This means that the ratio  $\lambda/\Delta$  is assumed to be given by (11), which is a nonassociated flow rule. With this expedient assumption, both  $M$  and  $N$  are guaranteed to transit from the plastic stage 2 to the LEFM stage 3 simultaneously and continuously, without jumps.

When  $b = 0$ , the limit value of the ratio in (11) for  $b \rightarrow 0$  needs to be used because  $\lambda_y = 0$  in that case. However, to avoid calculations of this limit value, the crack depth  $b$  is immediately extended from 0 to depth  $10^{-6}h$  as soon as the elastically calculated stress reaches the tensile strength  $f'_t$  of ice. With this simplifying assumption, (11) never needs to be used, because, for  $b > 0$ , it is embedded in the compliance condition (3). Thus, in computations, it suffices to use the LEFM compatibility equations in (4) and (5) at all times

through stages 1, 2, and 3. In stage 2, these equations are used in conjunction with the plastic limit state criterion in (9), and in stage 3 in conjunction with the LEFM criterion in (8). In this manner, a continuous variation of  $M$  and  $N$  from stage 2 to stage 3 is ensured.

The limit state conditions of plastic state and of brittle crack propagation in (9) and (8) may be jointly expressed as

$$\max[K_I - K_f, f'_t - \sigma_M(b) - \sigma_N(b)] = 0 \quad (12)$$

This criterion may be used for both stages 2 and 3.

The plasticization of a part of the thickness of the ice plate is a simple approximation for damage caused by microcracking. In experiments, the microcracking is visually manifested by whitening of the ice plate. The microcracks develop principally at the interfaces between the columnar ice crystals and at the voids filled by brine.

The problem of crack initiation in the vertical strips is different from that studied by Bažant et al. (1979), Li and Bažant (1994), and Li et al. (1995). The reader is also referred to Bažant and Cedolin (1991, section 12.6). In those studies, the focus was on the initial crack spacing. But here, for the vertical strips, the problem of their spacing does not arise. The crack strips open only along the same radial ray.

Development of a large in-plane compressive force can cause the crack strip to unload. The unloading first causes a reduction of the stress intensity factor  $K_I$  below the critical value  $K_f$ . As long as  $0 < K_I < K_f$ , the crack strip can neither grow nor shorten. This is labeled as stage 4. If the case  $K_I = 0$  is attained,  $K_I$  cannot decrease any more and further unloading causes the crack surfaces to come into contact. Thus, the length of the opened portion of the crack strip diminishes, which is equivalent to negative fracture growth at  $K_I = 0$ . This is labeled as stage 5. Stages 4 and 5 were included in the computer program, but in the present computations they have never been encountered. The crack strips were found to grow all the way to the maximum load. But stages 4 and 5 would no doubt occur in postpeak loading, and of course for unloading (decreasing  $P$ ). Another type of unloading criterion would have to be programmed if the unloading were to begin from the plastic stage (stage 2), but this case has never been encountered.

## NUMERICAL SOLUTION OF ICE PLATE WITH PART-THROUGH RADIAL CRACKS

Before tackling the crack problem, the values of the compliances  $C_{MF}(r)$ ,  $C_{MM}(r, r')$ , and  $C_{NN}(r, r')$  are calculated for the nodes  $i = 1, 2, \dots, n$  along the ray with the crack and are stored as matrices. If the load-point displacement  $u$  is specified, there are  $3n + 1$  unknown quantities to be solved; they are the internal forces  $M(r)$ ,  $N(r)$  and the crack depths  $b(r)$  at the nodes of the radial ray with the crack, and the applied load  $P$ . These unknowns can be solved from  $3n + 1$  nonlinear algebraic equations consisting of (6), (4), (5), and (12). If the solution is found, the radial crack length  $a$  can be obtained from the location of the last node that is not in the elastic state (stage 1).

The aforementioned solution procedure can be used in an incremental loading approach, in which the load-point displacement is incremented in small steps. However, this solution procedure was found to converge very slowly, and the resulting load-deflection curves were not very smooth. The reason was an unsystematic representation of the radial crack length  $a$ , defined as the distance from the origin to the first node that is in the elastic state (stage 1). The main problem was the representation of a radial crack tip that lies somewhere between two nodes. The numerical model cannot capture the difference between the crack with a tip in the middle between

two nodes and the crack with a tip very close to one of the nodes. This causes roughness in the calculated response.

The problem has been remedied by adopting the radial crack length  $a$  (or its dimensionless parameter  $\alpha = a/L_0$ ) as the controlled variable, instead of the displacement  $u$ . This makes it possible to move the tip of the radial crack, lying at distance  $a$ , from one node to the next in each loading step. Thus, the crack length  $a$  is forced to take only values compatible with the mesh. This approach adds the load-point displacement  $u$  as an additional unknown in each loading step. But at the same time the value of  $b$  is prescribed as 0 at one of the nodes (the crack tip node). So the number of unknowns remains  $3n + 1$ . With this modification, the solution converges much faster and the computed response becomes smoother.

The system of  $3n + 1$  equations, with the inequality criteria for the stages of all the crack strips, is highly nonlinear. An efficient way for solving such an equation system is the Levenberg-Marquardt nonlinear optimization algorithm (Levenberg 1944; Marquardt 1963). All the equations are written such that the right-hand sides are zero. If an approximate solution is substituted, the right-hand sides are not exactly zero. The optimization algorithm is used to minimize the sum of the squares of the right-hand sides of all the equations. This sum cannot be negative. Ideally, if this sum could be reduced to zero, the right-hand side of each equation would then be also zero and the solution would be exact. In practice, it suffices to reduce the sum to a sufficiently small positive value satisfying a specified tolerance, which ensures the right-hand sides of all the equations to be sufficiently small.

Attempting to solve this equation system right away for some specified value of  $a$ , one would not obtain a unique and physically correct solution, because the sum of squares of the right-hand sides of nonlinear equations has typically many local minima. The correct solution can be obtained only if a very good initial state, close to the correct solution, can be supplied as the input for the start of the iterations in the next step. Fortunately, the present problem belongs to a special class of problems in which the solution can be traced in small steps from an initial state (in this case,  $a = 0$ ) for which the solution is known. The solution obtained for the crack tip at one node is used as the input of the initial estimate of the solution for the start of the iterations. If the nodal spacing is sufficiently small, the solution for the tip at the previous node provides a very good estimate for starting the iterations. When the convergence is too slow, one needs to diminish the spacing of the nodes, which amounts to reducing the loading steps.

The growth of radial cracks in a star pattern does not lead to a maximum load and postpeak softening. The deflection curve is always rising. As known from small-scale field experiments (Frankenstein 1963) and confirmed for thick plates by the present calculations, the maximum load is determined by the initiation of circumferential cracks [Fig. 1(b)]. We assume these cracks to initiate anywhere along the radial crack, and the initiation to be decided by the strength criterion. Therefore, after the iterations for each step converge, the values of the radial normal stresses  $\sigma_{rr}$  on top of the plate are calculated for each node. This is done on the basis of the deflection curvature  $w_{,rr}$  along the radial ray and twist angle  $w_{,\theta r}$  along the  $\theta$  arc ( $r = \text{constant}$ ). These are calculated approximately by a second-order finite-difference formula from the nodal deflections  $w$ . During the finite-difference calculations of the elastic compliance matrices of an ice plate wedge, the influence matrices of the circumferential bending stresses are obtained as well. They include the curvatures in the radial direction of the wedge,  $F_{,rr}$ ; the curvatures in the  $\theta$  direction,  $F_{,\theta\theta}$ ; and the twist curvatures,  $F_{,\theta r}$ , per unit value of the bending stress  $\sigma_M$  and the normal stress  $\sigma_N$  along the radial ray with the crack and the applied load  $P$ . Labeling the components

corresponding to  $\sigma_M$ ,  $\sigma_N$ , and  $P$  by superscripts  $M$ ,  $N$ , and  $P$ , and grouping  $\sigma_{rr}$ ,  $\sigma_{\theta\theta}$ , and  $\sigma_{\theta r}$  for all the nodes on the radial ray into column matrices, we have

$$\sigma_{rr} = F_{rr}^M \sigma_M + F_{rr}^N \sigma_N + F_{rr}^P P \quad (13)$$

$$\sigma_{\theta\theta} = F_{\theta\theta}^M \sigma_M + F_{\theta\theta}^N \sigma_N + F_{\theta\theta}^P P \quad (14)$$

$$\sigma_{\theta r} = F_{\theta r}^M \sigma_M + F_{\theta r}^N \sigma_N + F_{\theta r}^P P \quad (15)$$

The maximum principal stress in the horizontal plane is  $\sigma_I = [\sigma_{rr} + \sigma_{\theta\theta} + \sqrt{4\sigma_{\theta r}^2 + (\sigma_{rr} - \sigma_{\theta\theta})^2}]/4$ . When the maximum value  $\sigma_{I\max}$  among all the nodes on the crack line reaches or exceeds the strength limit  $f'_I$ , the circumferential crack initiates.

Since the maximum load is decided by the strength criterion, one might think that there should be no size effect. But this is not the case, as the computations confirm. The reason is that the failure occurs only as a consequence of radial crack growth and the strength limit is attained only when the ratio  $\alpha = a/L_0$  reaches a certain value, which tends to a constant as the plate thickness  $h$  is increasing. The attainment of a certain relative crack length  $\alpha$  is decided by the energy release criterion of fracture mechanics. Hence the size effect.

## CLOSING REMARK

In the present paper, the method of numerical fracture analysis of the problem has been formulated. The numerical calculations, their analysis, comparison with test results, the questions of scaling, and formulation of the conclusions will be the subject of the companion paper (Bažant and Kim 1998), which follows in this issue.

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# SIZE EFFECT IN PENETRATION OF SEA ICE PLATE WITH PART-THROUGH CRACKS. I: THEORY. II: RESULTS<sup>a</sup>

Discussion by J. P. Dempsey<sup>3</sup>

A thorough examination of the quasi-static penetration of a floating elastic-brittle plate via a fracture mechanics approach has been presented by Bažant and Kim. Bažant and Kim reach the conclusion that there is a size effect (in terms of the plate thickness,  $h$ ). A few of the assumptions made by these authors will be examined in this discussion.

The formulation presented by Bažant and Kim assumes both that a radial system of part-through cracks is formed and that the appearance of these radial cracks is accompanied by stable crack growth. The analysis proceeds by subdividing each part-through crack into narrow vertical strips (the  $i$ th strip being of length  $b_i$ , with ligament  $h - b_i$ ). In each strip, the crack is assumed to propagate vertically, independently of the crack propagation in the adjacent strips. A simplified form of a cohesive crack model is adopted, with the crack initially growing as a plastic crack.

The assumed stable development of the part-through radial cracks does not match experimental observations, especially for thin to moderately thick ice sheets ( $h < 0.5$  m). The initiation of cracks in ice almost always leads to unstable crack growth (DeFranco and Dempsey 1994). The radial cracking that occurs prior to the formation of circumferential cracks and subsequent penetration is understood to occur suddenly and to be through-the-thickness. In other words, a system of through-the-thickness radial cracks occurs, with rapid radial and through-the-thickness crack propagation. Even though these radial cracks are subjected to the dome or arching effect, crack growth instability in ice is sufficient to allow through-the-thickness cracks to form (in thick ice sheets, it is plausible to assume that the through-the-thickness cracking would be prevented by the arching effect). Dempsey et al. (1995) studied radial cracking with closure for the case of a clamped plate subjected to a concentrated lateral load. By assuming that the closure width was a function of the radial crack length only, Dempsey et al. (1995) obtained an analytical solution that facilitated a thorough examination of the dependencies of the closure width, the nucleated radial crack lengths, the energy release rate, and the penetration load. In particular, the latter analysis made it clear that radial crack growth instability would accompany the nucleation of any radial crack system. A finite-element study of a radially cracked floating plate by Sodhi (1996) confirmed the broad applicability of the conclusions reached by Dempsey et al. (1995).

An implicit requirement underlying the size effect analysis presented by Bažant and Kim is the stable formation of process zones (contiguous to each traction-free crack front) that scale self-similarly with the ice sheet thickness. However, if sudden and unstable radial crack formation takes place, with full through-the-thickness crack-face separation and subsequent compressive closure (unilateral contact, in other words), there is no logical way in which one can simultaneously assume the stable formation of process zones; there are, in fact, no ligaments subjected to bending, but instead pairs of completely separated crack faces subjected to ever-increasing pres-

sure due to the arching action. This pressure grows to be of such magnitude that zones of circumferential microcracking in the plane of the ice sheet have been observed to occur, at variable radial distances away from the load. The radial crack lines have been observed to “whiten” with intense microcracking (Frankenstein 1966), and this is consistent with unilateral contact conditions of the receding type (Dundurs 1995), in which the extent of contact remains invariable with increasing load (in the case of elastic media; creep may alter this behavior, but not significantly). The issue of crack growth stability and whether the radial cracks would form stably or unstably was bypassed by Bažant and Kim, since they adopted the radial crack length  $a$  as the controlled variable. Their formulation, therefore, does not include a condition related to crack growth stability. By controlling the radial crack length numerically, their crack growth simulation is more stable than could be obtained in ice even under closed loop displacement controlled loading. For the majority of situations encountered, the much less stable condition of load control is operative.

For the case of relatively thick ice sheets, it is plausible that a radial crack system could form that would be comprised of part-through cracks. These part-through cracks would still form suddenly and, because of crack growth instability, would immediately partially close, with conditions of  $K = 0$  along the crack front. Even on further loading, the remaining ligaments would be subjected to the compression induced by arching, and only during load-up would the crack fronts experience tension and process zone growth. The stable formation of crack-tip contiguous—but not necessarily self-similar—process zones would be expected to occur, but only for the case of rather thick ice sheets (thick here is estimated to mean  $h \geq 1$  m).

If there is a size effect in ice thickness, it is important that it be determined, especially from the viewpoint of vehicles landing on, or traveling on, the ice. Safety is of primary concern in this case, and breakthrough is to be avoided. However, for the case of submarine surfacing, successful breakthrough is paramount, and a realistic load resistance estimate is all important. Given that the data in Fig. 5 of the authors' paper do not “visually demonstrate the invalidity of Sodhi's claim that there is no size effect,” one would intuitively favor a more conservative approach in the latter instance.

Conclusion: A fundamental requirement of a Bažant-type size effect analysis is the stable and self-similar growth of crack-front contiguous cohesive-type process zones. Such behavior is deemed implausible for the problem at hand. While a size effect may occur for thick ice sheets, it is unlikely to be significant for ice thicknesses less than 1 m.

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## Discussion by Devinder S. Sodhi<sup>4</sup>

In their papers, the authors arrive at the conclusion there is a size effect on the failure load of floating ice sheets for ice thicknesses greater than 0.2 m. However, the results of their analysis are only useful if the assumptions made in their anal-

<sup>a</sup>December 1998, Vol. 124, No. 12, by Zdeněk P. Bažant and Jang Jay H. Kim (Papers 14531 and 17980).

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ysis correspond to the real situation during vertical loading and breakthrough failure of floating ice sheets.

## PART I

The process of a gradually increasing axisymmetric load on a floating ice sheet results in the following sequence of events: (1) elastic deformations; (2) formation of radial cracks; (3) wedging of radially cracked segments of ice sheets; (4) formation of many circumferential cracks; and (5) breakthrough due to large deformation or brittle failure of ice. If the loading rate is low, we also need to consider creep deformation of ice along with elastic deformation. During field tests, it is often difficult to observe their formation because of snow cover. During small-scale tests, the formation of radial cracks is a very short-time event. They propagate to a length of about 2–3 times the characteristic length and arrest. After the formation of radial cracks, compressive stresses in the top part of the ice sheet support the load because of the wedging or dome effect. The compressive stresses cause creep deformation of ice, resulting in further deformation.

The results of linear elastic fracture mechanics analysis are not immediately relevant to the propagation of cracks in a creeping material. The results of Slepyan (1990) and Bazant and Li (1994) are particularly flawed, because the interference between segments during elastic deflections of wedge-shaped beams was ignored. Dempsey et al. (1995) presented a formulation of plates having radial cracks with closure. Bazant et al. (1995) and Bazant and Kim (1998) consider closure of part-through cracks, and the failure criterion is the formation of the first circumferential crack. They did not consider the creep deformation of ice, nor did they consider the formation of multiple circumferential cracks, which have been observed in small-scale and full-scale tests. The authors arrive at a result that the dependence of breakthrough load  $P_f$  is proportional to  $h^{3/2}$  using the results of field tests by Frankenstein (1963, 1968) and Lichtenberger et al. (1974). Those field tests were conducted by loading an ice sheet at a constant rate, and some of these tests lasted for hours. Therefore, it is not reasonable to use the results of those field tests to support the conjecture that fracture, while ignoring creep, gives the size effect  $P_f \propto h^{3/2}$  for ice thickness greater than 0.2 m. Their criterion that an ice sheet fails when the first circumferential crack forms is also not correct, because many circumferential cracks form around the area of load application before final breakthrough takes place.

## PART II

In their analysis, the authors considered a hole of radius equal to 10% of the characteristic length and assumed the load to be distributed at the periphery of the hole. Because there is considerable deformation of material in the area close to the center, the conclusion they have reached may not be totally correct.

On page 1320, they state that “Frankenstein made extensive observations on lake ice, which can be assumed to behave similarly as sea ice.” Yet they criticized Sodhi (1995b, 1998) at the bottom of page 1321 by saying that “a second questionable aspect of Sodhi’s (1995a,b) evaluation of test data is that he correlated in the same diagram the test results from different test series while implying the same ice properties. However, the ice properties were most likely quite different.” Nevertheless, the authors plot the data from tests with freshwater and sea ice in Figs. 5(c and d).

On page 1321, the authors state: “In view of the high scatter and limited size range of the available data, it cannot be claimed, however, that results actually prove the present theory.” Yet the authors state on the bottom of the same page: “Nevertheless, all the plots in Fig. 5 visually demonstrate the invalidity of Sodhi’s claim that there is no size effect.” In Figs. 5(a and b) of the paper, the authors have not really proven the existence of a size effect by fitting curves through three sets of data having high scatter and a narrow range of ice thickness.

In Fig. 6, results of small-scale and full-scale tests are plotted in terms of ice thickness versus failure load. This figure includes the data from ICEX-93 tests, in which ice penetration forces were measured during uplifting and breakthrough of floating ice sheets by two submarines (Dane 1993; Sodhi 1998). A line  $P_f = 1,934 h^2$  (where  $P_f$  is in kN and  $h$  is in m), obtained from the results of small-scale tests, passes through plots of full-scale data, which have considerable scatter. Because this line passes through the middle of the full-scale data, the discussor concluded that there is no size effect for ice thickness up to 2 m (Sodhi 1995b, 1998). Compilation of field data by Gold (1971) also supports failure load being proportional to the square of the ice thickness. Accepting the authors’ conclusion that there is no size effect for ice thickness less than 0.2 m, the discussor has plotted a line representing  $P_f \propto h^{3/2}$  in Fig. 6 from a point on the line ( $P_f = 1,934 h^2$ ), where ice thickness is equal to 0.2 m. This line does not fit the data obtained from full-scale tests on freshwater and sea ice.

The authors raise a point in the paper that the properties of

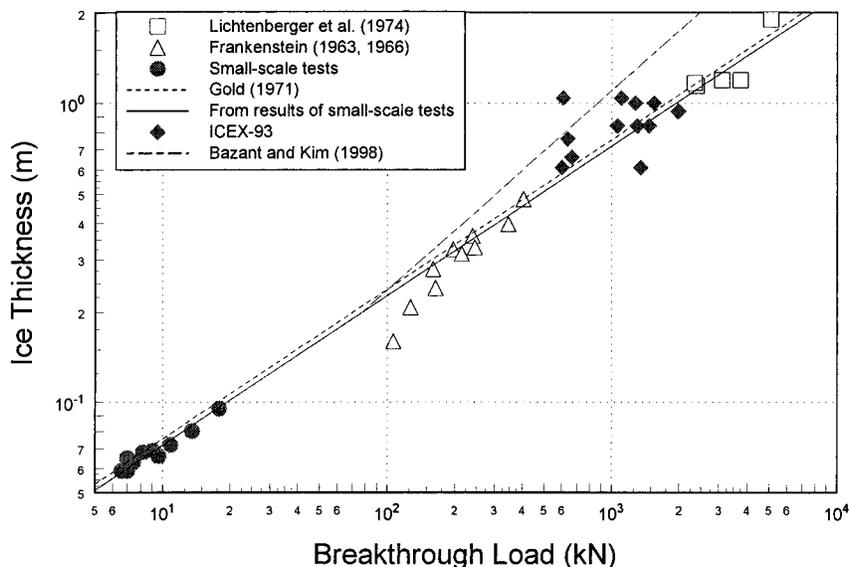


FIG. 6. Ice Thickness versus Breakthrough Load of Floating Ice Sheets

freshwater and sea ice may influence the failure load. However, the discussor considered creep properties of freshwater and saline ice and did not find much deviation between a line ( $P_f \propto h^2$ ) and the estimated failure loads (Sodhi 1995a). The dependence of failure loads on salinity of ice appears to be a secondary effect, but its dependence on  $h^2$  is supported by the strength failure criterion (Bazant 1993) because of creep deformation during wedging action.

On page 1322, the authors state: "Sea ice exhibits creep, and the effective fracture energy as well as the strength depends on the rate of crack growth." Analysis of this problem incorporating creep will require abandoning LEFM, on which they base their present conclusions.

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## Closure by Zdeněk P. Bažant,<sup>5</sup> Fellow, ASCE, and Jang Jay H. Kim<sup>6</sup>

### DEMPSEY'S DISCUSSION

Dempsey's thoughtful and stimulating discussion is deeply appreciated by the writers. Citing certain simplifications made in the paper and revoking his own analytical solution, Dempsey states that dynamic fracture propagation instabilities may cause the size effect to be significant only for rather thick ice plates, thicker than about 1 m. Dempsey et al.'s (1995) elegant analytical solution, however, rested on even stronger simplifications, which render his conclusion about the lack of size effect for not too thick plates unjustified.

Dempsey assumes the cracks to reach through the full ice thickness, which implies the stress intensity factor  $K_I$  at the boundary of the crack closure zone (contact zone) is zero. Consequently, there is no dissipative mechanism at all in Dempsey et al.'s solution. No energy is dissipated by the fracture process as modeled. Despite the possibility of dynamic instabilities described by Dempsey, this seems to be a severe simplification.

Another drastic simplification in Dempsey et al.'s (1995) solution is that the depth profile of the open crack along the radial coordinate is assumed to be uniform from the load point up to the tip of the radial crack, with a discontinuous jump at the tip. The numerical solution in the paper, by contrast, revealed that the depth of the opened crack varies strongly with the radial coordinate and, at the radial crack front, approaches zero continuously.

The solution in the paper has proven that a static loading process cannot produce radial cracks that cut through the full ice thickness. Dempsey argues that full-through cracks are produced by dynamic instabilities, after which the crack partially closes because of arching (or dome) action. To support his view, he cites the fact that, in field experiments, the top surface of ice was seen to whiten along the radial cracks. This observation, however, does not prove Dempsey's point, in the writers' opinion. Cracks actually reaching the surface were not observed in the field. The observed whitening of the top surface of the ice was more likely caused by distributed cracking, which occurs in the fracture process zone of sea ice. The cor-

rect interpretation should be that the fracture process zone reaches close to the top surface. But this is not incompatible with the notion that the equivalent LEFM cracks reach to about 85% of ice thickness, as found in the paper.

Dempsey is not right in stating that "the issue of crack growth stability . . . was bypassed by Bažant and Kim." Because, as shown in the paper, the vertical load increases with an increasing displacement, it is immediately clear that the solution obtained is stable (which means that this is a fracture problem of positive geometry, in fracture mechanics terminology). Contrary to Dempsey's comment, the solution is stable regardless of whether the radial crack length or the load-point displacement the crack length control instead of the displacement control was not to achieve stability of the actual response but merely to improve the convergence of iterations (or ensure stability of the numerical algorithm).

In principle, of course, it should not be ruled out that removal of some simplifying assumptions may lead to a significantly different solution exhibiting dynamic instabilities. There exist two possible sources of the dynamic instabilities emphasized by Dempsey: (1) strong inhomogeneity of sea ice; and (2) three-dimensionality of fracture propagation near the radial crack front, alluded to by Dempsey, which is undescrivable by the assumed vertical propagation along an infinitesimal strip.

At the critical state of the stability limit, a structure is at the limit of static response (equilibrium). When stability is lost, the response becomes dynamic (i.e., there must be inertia forces to satisfy D'Alembert equations of dynamic equilibrium). Since the static solution for a homogeneous ice plate is stable, the only possible cause of unstable crack jumps (inevitably dynamic) is periodic inhomogeneity of ice properties. The value of fracture toughness  $K_{Ic}$ , considered constant in the paper, actually fluctuates randomly along the crack path (with some dominant wavelength  $l_c$ , representing the dominant spectral component of the random process of  $K_{Ic}$  as a function of crack path length).

In crack path segments in which  $K_{Ic}$  is decreasing fast enough, crack propagation may become unstable, dynamic. But it must be a snap-through instability, with a jump to a new stable equilibrium state, which must occur in the next crack path segment in which  $K_{Ic}$  is growing, constant, or not decreasing fast enough. Since every material is inhomogeneous, such instabilities occur in all fracture. They get manifested by acoustic emissions. Yet static LEFM still provides the correct approximation on the macroscale.

One might think that the rate of energy to form the fracture should be equal to the rate of stored energy release minus the rate of the energy radiated by acoustic waves. But the energy of acoustic emissions in ice may surely be considered negligible compared with the total energy needed to form the cracks. In concrete, for example, the acoustic emissions, due to snap-throughs at each fluctuation of fracture toughness caused by aggregate pieces, are as strong as in ice, yet it is generally accepted that the energy they radiate is insignificant compared with the energy required for concrete fracture. Otherwise, static fracture analysis of concrete would be impossible. Besides, it would actually be incorrect to subtract the energy of acoustic emissions, because it is never subtracted during the measurement of fracture energy. So the fracture energy value used in fracture calculations already includes the energy of acoustic emissions.

Dempsey apparently believes that the typical length of the segments of decreasing  $K_{Ic}$  along the crack's path (or the dominant spectral wavelength  $l_c$ , or the length of crack front jumps) is not microscopic, negligibly short compared with the radial crack length, but relatively long. But unless this length were

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comparable to the entire radial crack length (i.e., unless almost the whole radial crack forms dynamically), a static fracture analysis must still provide at least an approximate overall description, correct in the energetic sense.

Static approximations to dynamic instability in the form of a snap-through from one equilibrium state (the initial uncracked state) to another equilibrium state (the full-through crack with partial closure) must generally satisfy Maxwell's condition of energy equivalence (whose classical example is the Maxwell line through the instability in the van der Waals pressure-volume diagram for the vapor-liquid phase transition). But even if a dynamic snap-through from an uncracked state to a full-through crack followed by a partial crack closure were the actual fracture mechanism, Dempsey et al.'s solution does not appear to be energy consistent.

The solution in the paper, on the other hand, is energy consistent. Unlike Dempsey et al.'s solution, it guarantees the rate of release of the stored strain energy and gravitational energy of sea water to be equal to the rate of energy needed to form the radial cracks in ice, corresponding to the given value of the fracture energy of ice. Thus, the condition of overall energy balance is satisfied.

In view of the foregoing considerations, as well as the fact that no solution with a dynamic instability has yet been presented, Dempsey's concern about the dynamic instabilities appears exaggerated. It is clear from the solution in the paper that, under the assumptions made, the load is continuously increasing with the crack length as well as with the load-point displacement. This guarantees continuous stability up to the moment of formation of the circumferential cracks, provided that the ice is treated as homogeneous.

The second suspected source of error, the three-dimensionality, is reflected in Dempsey et al.'s solution to a lesser degree than by the solution in the paper. Dempsey et al.'s assumption that the depth of open crack along the radial crack is uniform, with a sudden jump to zero at the radial crack front (a place where the dynamic crack jumps would have to take place), is a rather severe simplification of a plausible fracture shape. In the paper, the open crack depth is variable and at the radial crack front approaches zero without any discontinuity. The depth variation is found to be quite significant. Therefore, the deviation from the actual three-dimensional behavior is evidently greater for Dempsey et al.'s solution.

It is strange that, while questioning the existence of size effect except in very thick plates, Dempsey ignores the evidence given by Fig. 5 in Part II of the paper. That figure shows the results of three field tests, and each of them clearly shows, despite high scatter, that a strong size effect is present even for a size range beginning with 0.1 m.

In conclusion, the writers remain convinced that the simplifications made in the fracture and size effect analysis of the paper were not unreasonable and that the numerical solution presented, with all its approximations, ought to be more realistic than the analytical solution of Dempsey et al., ingenious and elegant though it may be. In particular, the writers do not agree with Dempsey that a static analysis leading to "stable and self-similar growth" would be implausible. Simplified though the analysis in the paper obviously is, it nevertheless appears to be a reasonable simplification.

## SODHI'S DISCUSSION

Sodhi has made some interesting and thought-provoking points. However, his severe criticism is unconvincing and, in the writers' opinion, invalid.

It is true that the neglect of radial crack closures in Slepian (1990) and Bažant and Li (1994) was an oversimplification, but these early studies, judged as "particularly flawed" by Sodhi, represented necessary steps in the evolution toward a

realistic fracture analysis and clarified some important aspects of the scaling problem. Prior to Dempsey et al. (1995) and Bažant et al. (1995), no fracture studies of ice plate penetration took the crack closures with the inherent dome effect into account (some limit analysis studies did, but to treat ice as a plastic material without softening damage is a much more serious "flaw," in the writers' opinion).

There is no dispute that certain simplifying assumptions were made in the paper, but the writers believe them to be reasonable and sufficiently realistic. One simplification was the neglect of creep, which is repeatedly reproached by Sodhi. However, assuming that creep would not mitigate the size effect is not baseless.

There used to be a widespread intuitive misconception that the influence of creep is like that of plasticity, which tends to increase the process zone size, thereby making the response less brittle and the size effect weaker. But the influences of creep and plasticity are very different.

The influence of creep on scaling of brittle failures of concrete, which is doubtless quite similar from the mechanics viewpoint (albeit different in physical origin), was studied in depth at Northwestern University, along with the effect of the crack propagation velocity; see, e.g., Bažant and Gettu (1992); Bažant et al. (1993); Bažant and Planas (1998); and especially Bažant and Li (1997) and Li and Bažant (1997). The conclusion from these studies, backed by extensive fracture testing of concrete and rock at very different rates, is that, unless creep actually prevents crack formation, creep in the material always makes the size effect due to cracks stronger. In the logarithmic size effect plot of nominal strength versus structure size, it causes a shift to the right, toward the LFM asymptote.

In light of these studies, Sodhi's claim (in his last paragraph) that "incorporating creep will require abandoning an LFM approach" must be seen as erroneous. The opposite is in fact true: The slower the loading (or the longer its duration), the closer to LFM is the size effect in a cracked structure. The physical reason, clarified by numerical solutions of stress profiles with a rate-dependent cohesive crack model (Li and Bažant 1997), is that the highest stresses in the fracture process zone at the crack front get relaxed by creep, which tends to reduce the effective length of the fracture process zone. The shorter the process zone, the higher the brittleness of response is and the shorter the size effect. This explains why experiments on notched concrete specimens consistently show the size effect to be stronger at a slower loading (Bažant and Planas 1998). It is highly probable that the same will be verified for ice, once size effect tests at very different loading rates are carried out.

From the aforementioned studies, it thus transpires that, in order to take the influence of creep on the size effect approximately into account, one does not need to abandon equivalent LFM, as claimed by Sodhi. It suffices, in the case of very slow loading, to reduce the value of fracture energy (or fracture toughness) and decrease the effective length  $c_f$  of the fracture process zone. Even these adjustments, however, are important only when loading durations differing by several orders of magnitude are considered, which is not the case for the ice penetration tests cited by Sodhi.

Sodhi also states that considering the load to be applied along the circumference of a hole of a radius of about 10% of the characteristic length must have caused the results not to be "totally correct," apparently meaning not totally representative of the idealized case of a concentrated load applied at a point. However, the conclusions ought to be essentially correct. Fracture is at a maximum load driven by the global energy release from the ice plate—sea water system and is not very sensitive to local disturbances near the load application point, where reach is limited according to Saint-Venant principle.

Sodhi's comments in the second paragraph of Part II are taken out of context and result from a misunderstanding of the criticism in the original paper of Sodhi's previous way of handling the available data sets. In Figs. 5(c and d) of the paper, cited by Sodhi, the coordinates are not the actual thickness  $D$  and nominal strength  $\sigma_N$  but their relative values, which are normalized by the values of  $\lambda_0 l_0$  and  $Bf'_i$  only *after* these values have already been determined for each data set separately. The two plots were presented in the paper merely for visual demonstration; they were not used for actually identifying the material parameters from the test data. On the other hand, in his previous works cited from the paper, and again in his present discussion, Sodhi plots the data from different data sets in the same plot and actually uses regression in this plot to determine the parameter values. The criticism of such a procedure stated in detail in the paper is valid.

Since the relation of the ice properties in various data sets is not known a priori, an arbitrary vertical or horizontal shift (in  $\log \sigma_N$ ) of the group of data points from one data set against that from another data set is allowed and must be considered. Just by choosing a suitable vertical or horizontal shift of the data groups, any desired conclusion can thus be obtained—the presence of a strong size effect, or the absence of any size effect (in Sodhi's case). Nothing is thus proven by Sodhi's plot. This is the salient point criticized in the paper.

The kind of plot shown in Fig. 6 and discussed in Sodhi's fourth paragraph, Part II, is misleading for two reasons: (1) as known from Buckingham's theorem of dimensional analysis, general physical laws are correct only if they can be written in a dimensionless form; and (2) the breakthrough load  $P_{\max}$  must obviously depend on ice strength  $f'_i$ . To achieve a dimensionless coordinate, the breakthrough load in Fig. 6 must be divided by  $f'_i h^2$ ,  $h$  being the ice thickness (a division by  $f'_i$  amounts to a horizontal shift in the logarithmic scale). But then it is not a priori clear how the  $f'_i$  values for different data sets relate to each other, because they have not been separately identified in advance.

Consequently, the relative horizontal positions of the groups of circles, triangles, diamonds, and squares in Fig. 6 must be considered as undetermined in advance. This implies that Sodhi's plot in Fig. 6 can be valid only for one kind of ice, not for different kinds simultaneously. Arbitrary vertical shifts of one data group against another, due to unknown differences in  $f'_i$ , would have to be considered in Fig. 6 if the breakthrough load were normalized by the ice strength. [Here the shifts are not vertical, as considered in the paper, but rather horizontal, because Sodhi for some reason inverts the coordi-

minates; the ice thickness (normalized) would normally be the coordinate and the breakthrough load (normalized) the ordinate.]

The ice thickness  $h$  in Fig. 6 should of course also be normalized to yield a dimensionless coordinate. One way to do that might be to adopt as the ordinate the dimensionless parameter  $h\rho_w/f'_i$ , where  $\rho_w$  is the specific weight of water (of dimension  $N/m^3$ ). In that case, the vertical and horizontal shifts in Fig. 6 are the same and thus the plot looks the same after the shifts. But  $\rho_w/f'_i$  is not the only possible normalizing factor for  $h$  and is in fact not the most reasonable one.

If fracture plays any role, then either the characteristic length  $l_0$  of the cohesive crack model or the effective length of the fracture process zone in the sense of equivalent LEFM must somehow appear in the solution. So the ice thickness  $h$  should correctly be normalized by  $l_0$ . In other words, the ordinate  $h$  in Fig. 6 should be replaced by the relative thickness  $h/l_0$ . With this reasonable normalization of  $h$ , the arbitrariness of the horizontal shifts pointed out in the previous paragraph remains. Ignoring this kind of normalization of  $h$ , which is implicit to Sodhi's approach, is tantamount to assuming a priori that fracture mechanics plays no role in the problem and that there is no size effect. Given that such a hypothesis is implied, Sodhi's use of Fig. 6 to dismiss the size effect appears to be a circular argument.

Still another noteworthy point, already made in the paper, is that the coordinate of the size effect plots should not be the load  $P$  but the nominal strength  $\sigma_N = P/h^2$ . The case of no size effect then corresponds to a horizontal line. The plot in terms of  $P$  superimposes on the size effect the underlying proportionality of  $P$  to  $h^2$  corresponding to the strength theory, which does not represent a size effect as generally understood. This obscures the size effect, as demonstrated by Figs. 4(b and c) of the paper. Sodhi does not question this demonstration, yet he persists in his discussion in plotting the size effect again in terms of  $P$  rather than  $\sigma_N$ .

## APPENDIX. REFERENCES

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