Continuous Relaxation Spectrum for Concrete Creep and its Incorporation into Microplane Model M4

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Abstract: Efficient numerical finite-element analysis of creeping concrete structures requires the use of Kelvin or Maxwell chain models, which are most conveniently identified from a continuous retardation or relaxation spectrum, the spectrum in turn being determined from the given compliance or relaxation function. The method of doing that within the context of solidification theory for creep with aging was previously worked out by Bažant and Xi in 1995 but only for the case of a continuous retardation spectrum based on the Kelvin chain. The present paper is motivated by the need to incorporate concrete creep into the recently published Microplane Model M4 for nonlinear triaxial behavior of concrete, including tensile fracturing and behavior under compression. In that context, the Maxwell chain is more effective than the Kelvin chain, because of the kinematic constraint of the microplanes used in M4. The paper shows how to determine the continuous relaxation spectrum for the Maxwell chain, based on the solidification theory for aging creep of concrete. An extension to the more recent microprestress-solidification theory is also outlined and numerical examples are presented.

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Introduction

Within service stress levels, concrete creep approximately follows the principle of superposition. The constitutive equation relating stress to strain has the form of a Volterra integral equation whose kernel is the compliance function of the material. Because the use of an integral equation in structural analysis is computationally inefficient, it is preferable to convert the integral-type creep law to a rate-type form based on either the Kelvin or the Maxwell chain models, which can be most conveniently identified from a continuous retardation or relaxation spectrum.

Because the Dirichlet series expansion of a given compliance function directly leads to the Kelvin chain, the solidification theory was formulated in terms of the retardation spectrum (Bažant and Xi 1995). However, when a nonlinear triaxial constitutive model for concrete such as Microplane Model M4 (Bažant et al. 2000a) is to be generalized for creep, the use of the Maxwell chain is more convenient because of the kinematic constraint of the microplanes. Furthermore, with the Maxwell chain, it is easier to trace the evolution of free energy and dissipated energy than with the Kelvin chain (Jirásek 2000).

The purpose of this paper, whose idea was briefly outlined in a recent conference presentation (Zi and Bažant 2001), is to formulate a continuous relaxation spectrum corresponding to the Maxwell chain model for the solidification theory of aging creep of concrete. An additional purpose is to show how the Maxwell chain can be incorporated into Microplane Model M4.

Continuous Relaxation Spectrum

The Dirichlet series expansion of a nonaging relaxation function, which has the form

$$R(\xi)=\sum_{\mu=1}^{N} E_\mu e^{-\xi/\tau_\mu} \tag{1}$$

leads to the Maxwell chain model in which E_μ, τ_μ=elastic moduli and relaxation times of Maxwell chain units; $\xi=t-t'$=time lag; t=current time; and t'=age at loading. Considering the limit case of infinitely many Maxwell units with continuously distributed relaxation times τ_μ, one may write

$$R(\xi)=\int_0^\infty L^*(\tau)e^{-\xi/\tau}d\tau \tag{2}$$

where the time lag $\xi=t-t'$ is the only time variable in the case of nonaging creep, and $L^*(\tau)$ represents a continuous distribution of the elastic moduli.

In the case of a discrete spectrum, the determination of both τ_μ and E_μ from test data is an ill-conditioned problem. In that case, τ_μ must be properly chosen. A uniform distribution of τ_μ in the logarithmic scale of time is a good choice (Bažant and Prasannan 1989). The values of E_μ corresponding to the chosen τ_μ may be determined by minimizing the quadratic norm of the difference between the approximation and the given relaxation function. Because E_μ depends on the choice and spacing of τ_μ, the discrete spectrum of E_μ as a function of the discrete relaxation times τ_μ is not unique.

Setting $L^*(\tau)=L(\tau)/\tau$ and $\zeta=1/\tau$, Eq. (2) becomes

$$R(\xi)=\int_{-\infty}^{\infty} L(\tau)e^{-\xi/\tau}d\ln\tau=\int_0^\infty \zeta^{-1}L(\zeta^{-1})e^{-\xi\zeta}d\zeta \tag{3}$$

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This equation can be regarded as the Laplace transformation of function $\zeta^{-1}L(\zeta^{-1})$ (e.g., Tschoegl 1989). Therefore, the function $L(\tau)$ can be determined by the inverse Laplace transformation of Eq. (3). The same inversion procedure as used in the preceding development of the retardation spectrum (Bažant and Xi 1995) may now be adopted, although several other techniques to determine $L(\tau)$ exist. The inverse Laplace transformation of a function $f(x)$ is defined asymptotically as follows (Widder 1971):

$$f(x)=\int_0^\infty e^{-xt}\phi(t)dt \quad \text{with} \quad \phi(y)=\lim_{k\to\infty}\frac{(-1)^k}{k!}\left(\frac{k}{y}\right)^{k+1}f^{(k)}\left(\frac{k}{y}\right) \tag{4}$$

where $f^{(k)}=k$th derivative of f. So, according to Eqs. (3) and (4), the inverse transformation $L(\tau)$ of a nonaging relaxation function is

$$L(\tau)=\lim_{k\to\infty}\frac{(-k\tau)^k}{(k-1)!}R^{(k)}(k\tau) \tag{5}$$

Therefore, when the relaxation function $R(\xi)$ is given, the kth-order approximate spectrum is obtained by using a finite value of k ($k\geqslant 1$). Same as shown previously, for the retardation spectrum, the third-order approximation of $L(\tau)$ is found to give sufficient accuracy for practical purposes.

For practical implementation, the continuous spectrum must be approximated by discrete values corresponding to a discrete Maxwell chain model for a given relaxation time span (τ_1,τ_N). The moduli of Maxwell units in Eq. (2) for the chosen relaxation times are unambiguously determined by

$$E_\mu=L(\tau_\mu)\ln 10\Delta(\log\tau_\mu)=2.303L(\tau_\mu) \quad \text{when}$$
$$\Delta(\log\tau_\mu)=1.0 \tag{6}$$

The E_μ values corresponding to the given τ_μ values are unique, but they of course depend on the spacing of τ_μ while being independent of the choice of τ_1. To minimize the discretization error, one should add to Eq. (1) one more spring of modulus E_∞ that is not coupled to any serial dashpot. E_∞ corresponds to a Maxwell unit with infinite relaxation time. Note that E_∞ does not affect the transformation relation (5) because $k\geqslant 1$; adding a constant to $R(\xi)$ has no effect on the result. E_∞ is obtained from the minimization of the error

$$E_\infty=\frac{1}{\tau_N-\tau_1}\left\{\int_{\tau_1}^{\tau_N}R(\xi)d\xi+\sum_{\mu=1}^{N}E_\mu\tau_\mu e^{-\xi/\tau_\mu}\Big|_{\tau_1}^{\tau_N}\right\} \tag{7}$$

The final form of the approximation by the Maxwell chain is

$$R(\xi)=\sum_{\mu=1}^{N}E_\mu e^{-\xi/\tau_\mu}+E_\infty \tag{8}$$

Note again that the determination of E_∞ is similar to the determination of inverse modulus A_0 corresponding to a Kelvin unit with zero retardation time; viz.

$$A_0=\frac{1}{\tau_N-\tau_1}\left\{\int_{\tau_1}^{\tau_N}C(\xi)d\xi-\sum_{\mu=1}^{N}A_\mu\tau_\mu e^{-\xi/\tau_\mu}\Big|_{\tau_1}^{\tau_N}\right\} \tag{9}$$

Simplification for Rate Effect in Structural Dynamics

While computations of long-time response require a relaxation spectrum spanning many decades in the logarithm of time, the computation of dynamic response of concrete structure to impact, blast, groundshock, or earthquake can be carried out with a much narrower spectrum. A previous study (Bažant et al. 2000a) showed how the long-range compliance function $J(t,t')$ for concrete creep can be replaced by an approximately equivalent nonaging linear compliance function $J_{ch}(\xi)$ corresponding to a tangentially equivalent single Maxwell unit (with modulus E_{ch} and viscosity η_{ch}) whose relaxation time $\tau_{ch}=E_{ch}/\eta_{ch}$ corresponds to the characteristic duration t_{ch} of the dynamic event. Approximately, $\tau_{ch}=t_{ch}/2$ (Fig. 1). This yields the conditions $J_{ch}(\tau_{ch})=1/E_{ch}+t_{ch}/\tau_{ch}$, $(dJ/d\xi)_{\tau_{ch}}=\dot{J}_{ch}(\tau_{ch})=1/\eta_{ch}$ from which

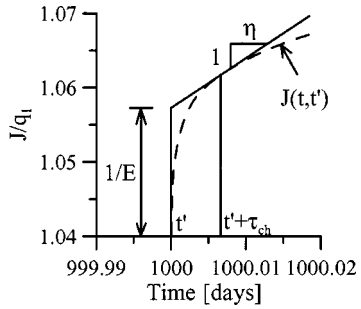

Fig. 1. Example of tangential approximation $J_{ch}(t)$ of compliance function $J(t,t')$ with single Maxwell unit for $t'=1{,}000$ days, and $\tau_{ch}=10$ min

$$\eta_t=\frac{1}{\dot{J}_{ch}}, \quad E_t=\frac{1}{J_{ch}+t_{ch}\dot{J}_{ch}} \tag{10}$$

Amalgamation with Model B3

Partial Moduli of Maxwell Chain in Solidification Theory

For basic creep, the total strain rate in Model B3 for creep of concrete (Bažant and Baweja 1995, 2000) consists of an elastic strain rate, aging viscoelastic strain rate, aging viscous flow rate, and inelastic strain rate (Fig. 2). The aging viscoelastic strain rate is obtained by dividing the nonaging viscoelastic strain rate by the current load-bearing volume fraction $v(t)$.

$$\dot{\varepsilon}=\dot{\varepsilon}_e+\dot{\varepsilon}_v+\dot{\varepsilon}_f+\dot{\varepsilon}_0=q_1\dot{\sigma}+\frac{\dot{\gamma}}{v(t)}+\frac{q_4}{\eta(t)}\sigma(t)+\dot{\varepsilon}_0 \tag{11}$$

where $\dot{\varepsilon}_e$=elastic strain rate; $\dot{\varepsilon}_v$=aging viscoelastic strain rate, with the aging mechanism being caused by the growth of the solid volume fraction due to solidification in the pores of the hardened cement paste in concrete (Bažant and Prasannan 1989); $\dot{\varepsilon}_f$=flow strain rate; $\dot{\varepsilon}_0$=inelastic strain rate; $\dot{\gamma}$=nonaging viscoelastic strain rate of the solidifying material; q_1 and q_4=empirical material parameters; and $\sigma(t)$=current stress. The rate $\dot{\gamma}$ at constant stress σ is defined as

$$\dot{\gamma}=\sigma C(\xi), \quad C(\xi)=q_2\ln[1+(\xi/\lambda_0)^n] \tag{12}$$

where q_2=empirical constant; $n=0.1$; $\lambda_0=1$ day; and $C(\xi)$=nonaging compliance (Bažant and Prasannan 1989; Bažant and Xi 1995; Bažant et al. 1997) which was approximated in previous works (Bažant and Prasannan 1989; Bažant and Xi 1995; Bažant et al. 1997) by the retardation spectrum of a Kelvin chain. In this paper, the approximation is based on the Maxwell chain.

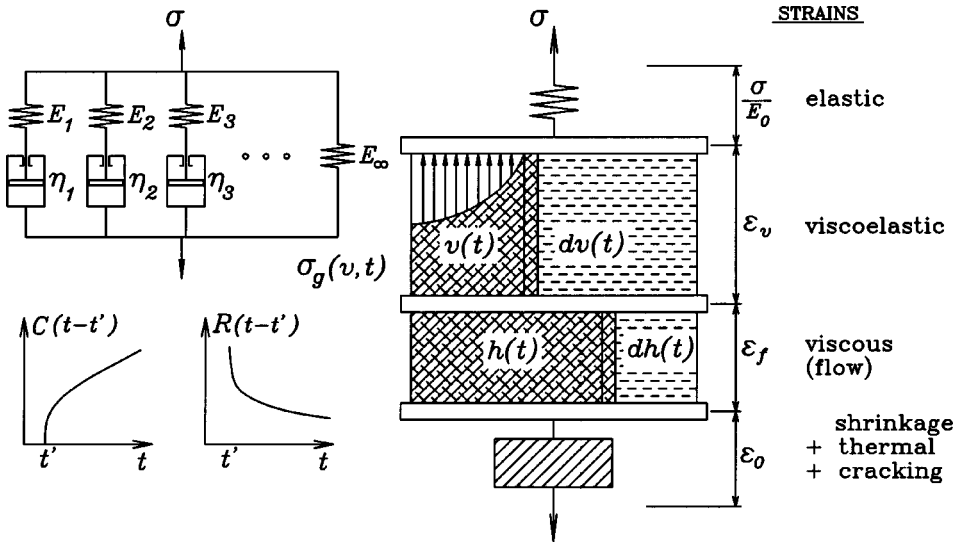

Fig. 2. Solidification theory and Maxwell chain model

The nonaging relaxation spectrum $R(\xi)$, which is the counterpart of the nonaging retardation spectrum $C(\xi)$, is not known. Therefore, in order to determine the partial moduli of the Maxwell chain, one may use the well-known identity

$$R(0)C(t)+\int_0^t C(t-t')\dot{R}(t')dt'=1 \tag{13}$$

The least-squares approximation of the data on $C(t)$ with Eq. (13) (Bažant 1975, 1982; RILEM 1988), or collocation of Eq. (13) at the chosen set of discrete times, leads to a linear matrix equation, from which the partial moduli can be solved. The discrete times must be chosen, or else ill posedness is unavoidable. The partial moduli also depend on the choice of t_{N+1} (Jirásek 2000), which corresponds to E_∞. Depending on this choice, negative values may sometimes result for some partial moduli, which is physically unreasonable (Bažant 1975; 1982; Jirásek 2000).

Once the continuous relaxation spectrum $R(\xi)$ is approximately calculated from $C(\xi)$, the partial moduli are uniquely and efficiently obtained as a discrete approximation of $R(\xi)$. A good approximation of a nonaging relaxation function can be obtained from the nonaging compliance function simply by algebraic inversion (Trost 1967; Bažant and Kim 1979; Tschoegl 1989); viz.

$$R(\xi)=\frac{1}{C(\xi)} \tag{14}$$

Therefore, for the approximation based on the third order of differentiation, the spectrum $L(\tau)$ may be calculated as

$$L(\tau)=-\frac{27\tau^3}{2C(3\tau)^4}[C'''(3\tau)C^2(3\tau) -6C'(3\tau)C''(3\tau)+6C'^3(3\tau)] \tag{15}$$

All the steps are straightforward. In Fig. 3, the spectrum of the optimum partial moduli E_μ obtained from Eq. (13) is compared to the resulting continuous relaxation spectrum, for which the relaxation times are chosen as $\tau_1=0.001$ day, $\tau_2=0.01$ day, . . . , $\tau_6=100$ days. Both spectra seem to be close to each other, but the latter is much smoother.

The analytical nonaging compliance and relaxation functions are compared to their Maxwell chain approximations in Fig. 4. Both show a good match within the given range of the relaxation times, i.e., from 0.001 day to 100 days. The initial unbounded spike in the relaxation function is not reproduced because of the finiteness of the approximation, although one might question the acceptability of an infinite spike. Since Model B3 represents an addition (or serial coupling) of serial strain components, the overall relaxation function is always finite even though the analytical relaxation function of the solidifying part is unbounded.

Not surprisingly, Eqs. (5) and (6) are identical to Eqs. (9) and (15) for the continuous retardation spectrum (Bažant and Xi 1995) if $R(\xi)$ is replaced by $-C(\xi)$ and E_μ by A_μ.

Quasi-Elastic Incremental Stress-Strain Relation for Creep

Assuming the nonaging viscoelastic strain rate $\dot{\gamma}$ to be constant within each time step, one can exactly integrate the differential equation of each Maxwell unit to obtain the partial stress at $t=i+1$

$$\sigma_\mu^{i+1}=\eta_\mu\dot{\gamma}+(\sigma_\mu^i-\eta_\mu\dot{\gamma})e^{-E_\mu\Delta t/\eta_\mu} \tag{16}$$

Expressing $\dot{\gamma}$ from this equation and substituting it into Eq. (11), one obtains a quasi-elastic incremental approximation of the constitutive law (Bažant 1971; 1975);

$$\Delta\sigma=E''(\Delta\varepsilon-\Delta\varepsilon'') \tag{17}$$

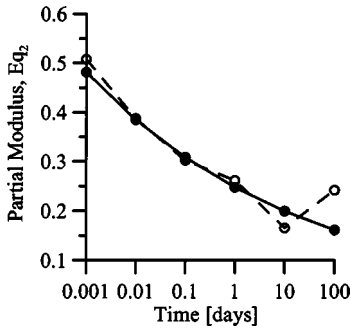

Fig. 3. Comparison of partial moduli E_μ determined from same compliance function using classical least-squares method (dashed line) and present continuous spectrum method (solid line)

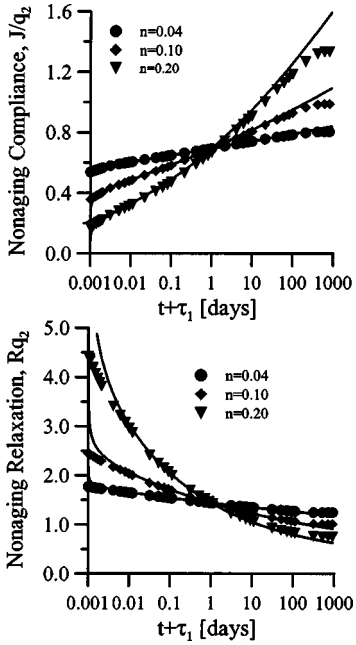

Fig. 4. Comparison of curves of analytical relaxation and compliance functions (solid lines) with curves of their Maxwell chain approximations (data points) determined from relaxation function (14). Top: For nonaging relaxation function; bottom: For nonaging compliance function.

$$E''=\left(q_1+\frac{1}{v^{i+1/2}D}+q_4\frac{\Delta t}{2\eta^{i+1}}\right)^{-1}$$

$$\Delta\varepsilon''=\frac{\Delta\gamma''}{v^{i+1/2}}+q_4\frac{\Delta t}{\eta^{i+1/2}}+\Delta\varepsilon_0 \tag{18}$$

where

$$D=\sum_{\mu=1}^{N}E_\mu\lambda_\mu+E_\infty \quad \text{and} \quad \Delta\gamma''=\frac{1}{D}\sum_{\mu=1}^{N}\sigma_\mu^i\Delta y_\mu\lambda_\mu \tag{19}$$

with

$$\Delta y_\mu=\frac{E_\mu}{\eta_\mu}\Delta t=\frac{\Delta t}{\tau_\mu};\quad \lambda_\mu=\frac{1-e^{-\Delta y_\mu}}{\Delta y_\mu} \tag{20}$$

The changes of the internal variables are then obtained from

$$\Delta\sigma_\mu=E_\mu\lambda_\mu\left(\frac{\Delta\sigma}{D}+\gamma''\right)-\sigma_\mu^i\Delta y_\mu\lambda_\mu \tag{21}$$

Eqs. (17)–(21) are the same as those obtained in various other ways by Bažant and Prasannan (1989) and Carol and Bažant (1993). Setting $\Delta\sigma=1$ only for the first step and $\Delta\sigma=0$ for the subsequent steps, one can compute from Eq. (17) the aging compliances. They are compared to the analytical compliances in Fig. 5, which demonstrates a good match.

Generalization to Microprestress-Solidification Theory

As a further refinement of the original solidification theory, the microprestress-solidification theory has been formulated (Bažant et al. 1997; Bažant 2001). In this theory, the following replacement is made in Eq. (11):

$$\frac{q_4}{\eta(t)}\sigma(t)\rightarrow\frac{\sigma(t)}{\eta(S)} \tag{22}$$

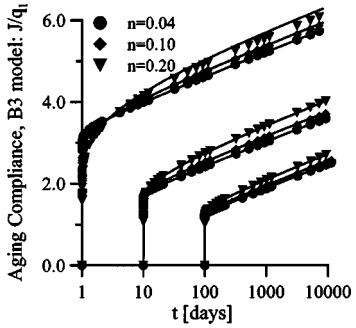

Fig. 5. Curves of analytical aging compliance function of model B3 (solid lines) and their Maxwell chain approximations (data points)

where $\eta(S)=1/cpS^{p-1}$; S=microprestress; and c=material parameter. The evolution of microprestress is obtained from the equation

$$\dot{S}+c_0S^p=\dot{\mu} \tag{23}$$

where $c_0=2c/q_4$ and μ=chemical potential of evaporable pore water$=-k_1[T\ln h+(\mu_1/k_1)]$. Here, k_1,μ_1=material parameters; T=absolute temperature; and h=humidity. Eq. (23) represents the relaxation of microprestress. Since S decreases very rapidly near $t=t_0$, where t_0=initial time, an explicit Eulerian integration scheme does not work. Iteration is necessary to obtain the next microprestress S^{i+1} with desired accuracy. This iteration, which is not cheap, can easily be avoided. Assuming $\dot{\mu}(h,T)$ to be constant within each time step, one can obtain an exact solution of Eq. (23) within the time step. With the initial conditions $S=S^i$ at $t=t^i$ (always $p>1$, and in the theory $p=2$)

$$S^{i+1}=\frac{1}{c_0\Delta t+(S^i)^{1-p}}+\left(\frac{\dot{\mu}^i}{c_0}\right)^{1/p} \tag{24}$$

The adjustments that need to be made to the aforementioned algorithm are relatively simple and may be stated as follows: (1) calculate or read the change of the chemical potential μ; (2) calculate S^{i+1} using (24); and (3) calculate the corresponding viscosity $\eta^{i+1}=1/(2cS^{i+1})$.

Incorporation into Microplane Model

In Microplane Model M4, which has recently been developed at Northwestern University (Bažant et al. 2000a; Caner and Bažant 2000), the constitutive law is formulated in terms of the stress and strain vectors on each microplane. The strain vectors are kinematically constrained to the strain tensor. Because of this constraint, the Maxwell chain is more convenient for creep calculations than the Kelvin chain (if a static constraint were used, the Kelvin chain model would, of course, be more convenient). A brief summary of the basic relations of the microplane model is attached in the Appendix (for the details, see Bažant et al. 2000a).

On each microplane, the strain vector is decomposed into the volumetric strain, deviatoric strain, and shear strain. The trial stress of each component is computed from the uniaxial elastic constitutive law (33). The Maxwell chain is implemented simply by replacing the elastic constitutive law (33) for each microplane by the viscoelastic constitutive law given by Eq. (17). The increments of the volumetric, deviatoric and shear stress components may be written as

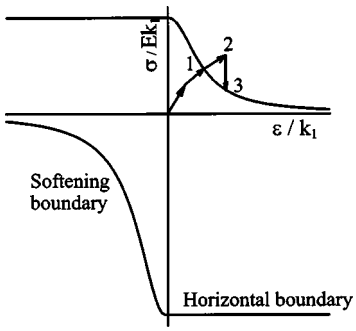

Fig. 6. Stress-strain boundary of microplane model M4 and stress drop to boundary when trial stress exceeds boundary (where k_1 is a dimensionless scaling parameter)

$$\Delta\sigma_V = E_V''(\Delta\varepsilon_V - \Delta\varepsilon_V''), \quad \Delta\sigma_D = E_D''(\Delta\varepsilon_D - \Delta\varepsilon_D'') \quad (25a)$$

$$\Delta\sigma_T = E_T''(\Delta\varepsilon_T - \Delta\varepsilon_T'') \quad (25b)$$

The inelastic behavior of concrete is, in Model M4, characterized by the so-called stress-strain boundaries (strain-softening yield limits) for the volumetric, deviatoric, and shear behaviors. Inside the boundaries (up to Point 1 in Fig. 6), the stresses change according to aging viscoelasticity. If the value of a stress exceeds the associated boundary, the stress value in each time step is made to drop at a constant strain to the value of the boundary.

The algorithm of computation in each time step Δt may be summarized as follows: (1) first compute the trial stresses from Eq. (25) (Point 2 in Fig. 6); (2) compare all the stress values to the corresponding boundaries; if they exceed the boundaries, correct them (Point 3 in Fig. 6); (3) obtain the final stress increment, $\Delta\sigma = \sigma_3 - \sigma_1$ (Fig. 6); and (4) update all internal variables (partial stresses) in Eq. (21) using the final $\Delta\sigma$.

Finally, it must be emphasized that the aging viscoelastic behavior modeled in this paper is only one of two mechanisms of the time (or rate) dependence. The other is the time (or rate) dependence of the opening of cracks or microcracks. That mechanism can be described by a rate-dependent formulation of the softening stress-separation curve of the cohesive crack model (Bažant et al. 2000b), which is highly nonlinear and was derived from the activation energy theory for the rupture of bonds. Alternatively, the crack band model or nonlocal model may be used for this purpose, as shown in Bažant et al. (2000b).

The crack opening rate mechanism is very important for short-time creep and dominates the nonlinear fracturing behavior in the range of dynamic loading. But it might be unimportant for long-time creep, although experimental evidence in this regard is lacking. It is, of course, irrelevant for all the time-dependent response in the absence of fracturing, which occurs at low-stress levels if no drying takes place.

Conclusions

1. For the nonaging constituent in the solidification theory, the Maxwell chain model is equally convenient as the Kelvin chain model since the nonaging relaxation function can be obtained from the nonaging compliance function easily, and with good accuracy;
2. The continuous relaxation spectrum of the nonaging constituent in the solidification theory is unique and is readily determined on the basis of the inverse Laplace transform. The modulus of each unit in a discrete Maxwell model is easily and uniquely determined from the continuous relaxation spectrum when the relaxation times are suitably chosen. The Maxwell chain model may be generalized for aging according to the solidification theory to yield a complete incremental quasi-elastic constitutive law;
3. The present model can also be introduced into the microprestress-solidification theory, which provides for aging and drying creep a simpler and more fundamental characterization than the original solidification theory (Bažant et al. 1997). An efficient algorithm to calculate the evolution of microprestress is presented. The incremental relations are based on the exact solution of the evolution of microprestress under the assumption that the rate of chemical potential of evaporable pore water is constant within each time step; and
4. The one-dimensional creep model may be naturally extended to three dimensions. To extend the creep formulation to nonlinear triaxial behavior, the model may be combined with the microplane model. The kinematic constraint used in Microplane Model M4 makes the Maxwell chain model more convenient than the Kelvin chain model. In the viscoelastic generalization, the elastic constitutive law for each microplane is simply replaced by the incremental quasi-elastic constitutive law characterizing creep increments.

Acknowledgments

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Appendix: Basic Relations of Microplane Model

Formulation of Kinematic Constraint

In the formulation of kinematic constraint, the normal ε_N and the shear microstrains ε_L and ε_M are the resolved components of the macroscopic strain tensor ε_{ij} (where the subscripts, $i=1, 2, 3$, refer to Cartesian coordinates x_i). The orientation of a microplane is characterized by its unit normal n_k. The component of the strain vector ε_j^n on any microplane is $\varepsilon_j^n = \varepsilon_{jk} n_k$ (Bažant and Prat 1988). The normal strain vector and its magnitude are

$$\varepsilon_{N_i} = n_i n_j n_k \varepsilon_{jk} \quad (26)$$

$$\varepsilon_N = n_j \varepsilon_j^n = n_j n_k \varepsilon_{jk} = N_{ij} \varepsilon_{ij} \quad (27)$$

where $N_{ij} = n_i n_j$ (the repeated Latin lowercase subscripts indicate summation over 1, 2, 3).

The shear strain components in two orthogonal (suitably chosen) directions m_i and l_i tangential to the microplane (normal to n_i) are $\varepsilon_M = m_i(\varepsilon_{ij} n_j)$ and $\varepsilon_L = l_i(\varepsilon_{ij} n_j)$. Because of the symmetry of ε_{ij}

$$\varepsilon_M = M_{ij} \varepsilon_{ij}, \quad \varepsilon_L = L_{ij} \varepsilon_{ij} \quad (28)$$

where $M_{ij} = (m_i n_j + m_j n_i)/2$ and $L_{ij} = (l_i n_j + l_j n_i)/2$ (Bažant and Prat 1988).

Static Equivalence

Since the foregoing kinematic constraint relates the strains on the microplanes to the macroscopic strain tensor, the static equiva-

lence can be enforced only approximately. This is done by the virtual work theorem which is written for the surface of a unit hemisphere (Bažant 1984).

$$\frac{2\pi}{3}\sigma_{ij}\delta\varepsilon_{ij}=\int_{\Omega}(\sigma_N\delta\varepsilon_N+\sigma_L\delta\varepsilon_L+\sigma_M\delta\varepsilon_M)d\Omega$$

$$=\int_{\Omega}(\sigma_N N_{ij}+\sigma_L L_{ij}+\sigma_M M_{ij})\delta\varepsilon_{ij}d\Omega \quad (29)$$

The normal stress and strain are split into their volumetric and deviatoric parts, and if Eq. (29) is written separately for the volumetric and deviatoric components, one has

$$\sigma_{ij}=\sigma_V\delta_{ij}+\sigma_{ij}^D \quad (30)$$

$$\sigma_{ij}^D=\frac{3}{2\pi}\int_{\Omega}\left[\sigma_D\left(N_{ij}-\frac{\delta_{ij}}{3}\right)+\sigma_L L_{ij}+\sigma_M M_{ij}\right]d\Omega \quad (31)$$

The elastic increments of the stresses in each microplane over the time step (or load step) are written as

$$\Delta\sigma_V=E_V\Delta\varepsilon_V,\quad \Delta\sigma_D=E_D\Delta\varepsilon_D,\quad \text{and}\ \Delta\sigma_T=E_T\Delta\varepsilon_T \quad (32)$$

where $E_V=E/(1-2\nu)$, $E_D=5E/(2+3\eta)(1+\nu)$, $E_T=\eta E_D$; and η=parameter that can be chosen, the best choice being $\eta=1$ (Carol et al. 1991; Bažant et al. 1996; Carol and Bažant 1997).

The volumetric-deviatoric split makes it possible to reproduce the full range of Poisson's ratio $-1\leqslant\nu\leqslant 0.5$ in elastic analysis. The term $-\delta_{ij}/3$ in Eq. (32) ensures $\sigma_{kk}^D=0$ even when $\int_{\Omega}\sigma_D d\Omega\neq 0$. The integration is conducted numerically according to an optimal Gaussian integration formula for a spherical surface, characterized by discrete directions $\mu=1,\ldots,N_m$ and the corresponding weights w_μ

$$\sigma_{ij}=\frac{3}{2\pi}s_{ij}\approx 6\sum_{\mu=1}^{N_m}w_\mu s_{ij}^{(\mu)} \quad (33)$$

where $s_{ij}=\int_{\Omega}(\sigma_N N_{ij}+\sigma_L L_{ij}+\sigma_M M_{ij})\delta\varepsilon_{ij}d\Omega$ and N_m=the number of the microplanes.

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