International Journal of Solids and Structures 41 (2004) 511–557

INTERNATIONAL JOURNAL OF SOLIDS and STRUCTURES

www.elsevier.com/locate/ijsolstr

# A framework for microplane models at large strain, with application to hyperelasticity

Ignacio Carol [a,*], Milan Jirásek [b], Zdeněk P. Bažant [c]

[a] School of Civil Engineering, Esc Tec Sup d'Enginyers de Camins Canals i Ports de Barcelona (ETSECCPB), Technical University of Catalonia (UPC), Jordi Girona 1-3, Edif. D2, E-08034 Barcelona, Spain

[b] Laboratory of Structural and Continuum Mechanics, Swiss Federal Institute of Technology at Lausanne (EPFL), CH-1015 Lausanne, Switzerland

[c] Departments of Civil Engineering and Material Science, Northwestern University, Evanston, IL 60208, USA

Received 28 September 2002; received in revised form 6 June 2003

**Abstract**

A general framework is proposed for the formulation of microplane models at large strain. It is based on the thermodynamic approach to microplane formulation recently presented by the authors, which defines the macroscopic free energy of the material as an integral of a microplane free-energy potential over all possible orientations. By simple differentiation with respect to strain, it is possible to obtain the consistent definition of microplane stresses and integral expressions for evaluation of the macroscopic stress tensor. To apply this approach to large strains, new microplane strain measures need to be defined, including volume change, stretch of fibers, "thickening" of planes, deviatoric parts of the stretch and thickening, and distortion (shear) angles. Based on these, various microplane formulations are developed. Each formulation starts with the definition of microplane stresses and the derivation of the integral expressions which are valid for the general case of dissipative materials. Then, these expressions are particularized to specific forms of hyperelastic potentials leading to various hyperelastic models. The simplest model, with a quadratic microplane potential in terms of the fiber stretch, corresponds to the classical Gaussian statistical theory of long-chain molecules and leads to the neo-Hookean type of macroscopic free-energy potential. Many other, more complex forms of the microplane potential are investigated and their relation to existing models for rubber elasticity is analyzed. It is shown that, in the small-strain limit, they collapse into well-known small-strain microplane formulations, either with restricted or with unrestricted values of Poisson's ratio.


*Keywords:* Constitutive model; Microplane model; Large strain; Large deformations; Finite strain; Finite elasticity; Rubber elasticity; Hyperelasticity; Anisotropy

* Corresponding author. Tel.: +34-93-401-6509; fax: +34-93-71-7251.
*E-mail address:* ignacio.carol@upc.es (I. Carol).

0020-7683/$ - see front matter © 2003 Elsevier Ltd. All rights reserved.
doi:10.1016/S0020-7683(03)00416-5

## 1. Introduction

Since the first microplane formulations were developed in the 1980s (Bažant, 1984; Bažant and Gambarova, 1984; Bažant and Oh, 1985), this approach has proven to be a powerful tool for constitutive modeling. The basic idea, namely that the constitutive material behavior as a relation between stress and strain tensors can be "assembled" from the behavior of planes with different orientations within the material such as slip planes, microcracks, particle contacts, etc., dates back to the failure envelopes of Mohr (1900) and the "slip theory of plasticity" of Taylor (1938) and Batdorf and Budiansky (1949). Related ideas are also the basis of some modern multi-surface plasticity models, for instance those that represent multiple cracking (Carol and Prat, 1995).

However, the slip theory of plasticity and its extensions are based on the *static constraint*, that is, on the assumption that the stress traction on each of those planes is equal to the projection to the macroscopic stress tensor. One of the distinctive assumptions of the microplane model is the *kinematic constraint*, according to which normal and tangential (shear) strains on each plane are assumed to be equal to the corresponding projections of the strain tensor. This alternative assumption is much better suited for quasi-brittle and softening behavior of materials such as concrete or rock, to which most of the microplane models proposed in recent years have been devoted (Bažant, 1984; Bažant and Gambarova, 1984; Bažant and Oh, 1985; Bažant and Prat, 1988a,b; Carol et al., 1992; Bažant et al., 1996a,b, 2000b; Caner and Bažant, 2000; Ožbolt et al., 2001). Perhaps the main advantage of microplane models over more classical tensor-based plasticity or damage formulations is the implicit representation of load-induced anisotropic behavior, which is naturally included since each plane in the system is subject to a different load history and may exhibit a different strain, stiffness, etc. For many materials this representation is close to physical reality and results into very good fits of experimental data under a variety of situations (Bažant and Prat, 1988b; Bažant et al., 1996a,b, 2000b). The microplane models proposed so far have been mostly in the small-strain regime, except for specific extensions of small-strain concrete models to include behavior under high confinement (for instance under impact situations), which have been constructed intuitively and without a solid theoretical framework (Bažant et al., 1996a,b, 2000a).

A requirement for the development of general formulations at large strain is full thermodynamic consistency, and the first goal should be to reproduce the well-known hyperelastic behavior. In microplane formulations, the kinematic micro–macro constraint and the fundamental constitutive laws at the plane level imply the need to find a static relation between stresses on each microplane and the macroscopic stress tensor. Since it is in general not possible to satisfy the "double constraint" (i.e., the kinematic constraint and the static constraint at the same time), these relations must be of a weak nature and are written in an integral form. In classical microplane formulations these relations were developed by application of the Principle of Virtual Work (PVW) linking the microplane and macroscopic (tensor) levels, which necessarily requires a dose of intuition (definition of microplane stress components, conjugate pairs, etc.). This was the procedure adopted by the majority of microplane models for concrete (Bažant and Gambarova, 1984; Bažant and Prat, 1988a; Carol et al., 1992; Bažant et al., 1996a,b), which exhibit excellent data fitting capabilities, but thermodynamic consistency cannot be guaranteed in all situations, and spurious energy generation or dissipation cannot be avoided under certain loading paths.

These deficiencies have been detected recently, when the thermodynamic basis for microplane formulations was established by the authors (Carol et al., 2001). The main idea is that a free-energy potential is introduced for each microplane, and its integral over all possible orientations leads to the standard macroscopic free energy. This concept, combined with the kinematic constraint, provides a thermodynamically consistent procedure to define conjugate microplane stresses, and to develop the correct integral static relations needed. The procedure has already been exploited to establish a first series of plasticity and damage models in small strain (Kuhl et al., 2001; Kuhl and Ramm, 2000).

The thermodynamic approach for microplane models also opens the door to the consistent and rational development of large-strain microplane formulations, which is the main purpose of this paper. The structure of the paper is as follows: The thermodynamic approach to microplane formulations is briefly reviewed in Section 2, and in Section 3 it is applied to small strains, considering all the microplane strain variables normally used, and systematically developing the corresponding integral stress-evaluation formulae and the relations between microplane elastic constants and their macroscopic counterparts.

In Section 4, the general approach for large-strain microplane formulations is described, including all fundamental kinematics at the macroscopic and microplane levels. In particular, all microplane strain measures that are potential candidates to be included as arguments of the microplane free-energy function are defined, and some useful relations are derived.

The first incursion into specific large-strain formulations is given in Section 5, with the simplest formulation based on the stretch of a generic fiber within the material. This formulation, which is also developed via the principle of virtual work for comparison, encompasses the Gaussian theory of long-chain molecules developed half a century ago for rubber materials. With a simple quadratic potential, this model turns out to be equivalent to the incompressible neo-Hookean material. In spite of its instructive value, however, the model exhibits some questionable features such as non-zero microplane stress in the undeformed state. These limitations are overcome with the improved formulations of Section 6, also based on fiber stretch only, but using more complex microplane energy functions. However, in the small-strain regime these formulations are still limited to a fixed Poisson ratio of 1/4.

More elaborate formulations involving additional microplane strains are presented in Sections 7–9. These are developed for general dissipative behavior, and then are particularized for specific forms of hyperelastic microplane potential. In this way, they turn out equivalent to progressively more general forms of neo-Hookean (Section 7) and Mooney-Rivlin (Sections 8 and 9) material, and in the small-strain regime they collapse into well-known microplane formulations with partially restricted (Section 8) and unrestricted (Sections 7 and 9) Poisson ratios.

Finally, in Section 10 the main developments and the new possibilities open by the paper, are summarized.

**Notation.** In this paper we use a compact tensorial notation. Tensors are denoted by bold face letters and, whenever possible, upper-case letters are used for Lagrangian quantities (referred to the initial, undeformed configuration) while lower-case letters are used for Eulerian quantities (referred to the current, deformed configuration). Simple contraction of two tensors is denoted by a dot "·", double contraction by a colon ":", and a direct (outer) product by the symbol $\otimes$. Superscript $(\ )^{\mathrm{t}}$ denotes transposition, and $|\mathbf{u}|$ is the norm of a vector (first-order tensor) $\mathbf{u}$, defined as $|\mathbf{u}| = \sqrt{\mathbf{u}\cdot\mathbf{u}}$. It is convenient to introduce the second-order unit tensor $\mathbf{I}$ with components $\delta_{ij}$ (Kronecker delta), the fourth-order symmetric unit tensor $\mathscr{I}^{\mathrm{sym}}$ with components $I^{\mathrm{sym}}_{ijkl} = \frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, the fourth-order volumetric projection tensor $\mathscr{I}^{\mathrm{vol}} = \frac{1}{3}\mathbf{I}\otimes\mathbf{I}$, and the fourth-order deviatoric projection tensor $\mathscr{I}^{\mathrm{dev}} = \mathscr{I}^{\mathrm{sym}} - \mathscr{I}^{\mathrm{vol}}$. The square brackets "$[\cdots]$" enclose the arguments of a function.

## 2. Thermodynamic approach to microplane formulations

The thermodynamic approach to microplane-based constitutive modeling has been introduced in Carol et al. (1998) and formalized in Carol et al. (2001). It has already been used as the basis for some prototype models in Kuhl et al. (2001) and Kuhl and Ramm (2000). The fundamental ideas of this approach are summarized in this section.

The first, standard assumption is the existence of a "macroscopic" potential $\Psi[\boldsymbol{\varepsilon}, \boldsymbol{\xi}]$, representing the free energy density per unit mass under isothermal conditions, which depends on some tensor characterizing strain, $\boldsymbol{\varepsilon}$, and on a given set of internal variables, $\boldsymbol{\xi}$. Multiplying $\Psi$ by the mass density in the initial (undeformed) configuration, $\rho_0$, gives the free energy per unit initial volume.

An additional assumption, specific to microplane formulations, is that the macroscopic free energy per unit initial volume can be obtained by collecting the contributions of elementary units called the microplanes, which all reside in the same macroscopic material point but have different spatial orientations. Each microplane is characterized by its unit normal **N** in the initial configuration. The physical entity represented by the microplane could be a plane of weakness normal to **N**, a fiber parallel to **N**, or a more complex, oriented microstructural unit. The end points of unit vectors corresponding to all the microplane orientations fill the unit sphere, but since microplanes with normals **N** and −**N** are physically identical, it is sufficient to consider the unit hemisphere, which is denoted as $\Omega$. According to the foregoing assumption, the free energy per unit initial volume can be written as

$$\rho_0 \Psi = \frac{3}{2\pi} \int_\Omega \Psi_\Omega[\boldsymbol{\varepsilon}^{(\mathbf{N})}, \xi, \mathbf{N}] \, \mathrm{d}\Omega \tag{1}$$

Here, $3/2\pi$ is a scaling factor that will later simplify certain expressions, $\Psi_\Omega[\boldsymbol{\varepsilon}^{(\mathbf{N})}, \xi, \mathbf{N}]$ is the microplane free energy with units of energy per unit volume and unit solid angle, and

$$\boldsymbol{\varepsilon}^{(\mathbf{N})} = \mathscr{F}_\varepsilon[\boldsymbol{\varepsilon}, \mathbf{N}] \tag{2}$$

is a suitable microplane strain measure, which is assumed to be uniquely defined by the macroscopic strain tensor $\boldsymbol{\varepsilon}$ and the microplane normal **N**. Eq. (2) is the general form of the *kinematic constraint*, characteristic of microplane models. If the material is initially isotropic, all the microplanes have the same properties and the microplane free energy $\Psi_\Omega$ does not depend on **N** explicitly but only through $\boldsymbol{\varepsilon}^{(\mathbf{N})}$. In that case, the argument **N** can be dropped from the integrand in (1).

Following the standard Coleman method (Coleman and Gurtin, 1967), stress is obtained as the derivative of the volumetric free energy with respect to strain. Differentiating (1) and applying the chain rule, one obtains

$$\boldsymbol{\sigma} = \frac{\partial(\rho_0 \Psi)}{\partial \boldsymbol{\varepsilon}} = \frac{3}{2\pi} \int_\Omega \frac{\partial \Psi_\Omega}{\partial \boldsymbol{\varepsilon}^{(\mathbf{N})}} \bullet \frac{\partial \boldsymbol{\varepsilon}^{(\mathbf{N})}}{\partial \boldsymbol{\varepsilon}} \, \mathrm{d}\Omega = \frac{3}{2\pi} \int_\Omega \boldsymbol{\sigma}^{(\mathbf{N})} \bullet \frac{\partial \boldsymbol{\varepsilon}^{(\mathbf{N})}}{\partial \boldsymbol{\varepsilon}} \, \mathrm{d}\Omega \tag{3}$$

where

$$\boldsymbol{\sigma}^{(\mathbf{N})} = \frac{\partial \Psi_\Omega}{\partial \boldsymbol{\varepsilon}^{(\mathbf{N})}} \tag{4}$$

can be identified as the microplane stress measure work-conjugate to the microplane strain measure $\boldsymbol{\varepsilon}^{(\mathbf{N})}$, the term $\partial \boldsymbol{\varepsilon}^{(\mathbf{N})}/\partial \boldsymbol{\varepsilon}$ is obtained by differentiating the kinematic constraint (2), and the symbol $\bullet$ stands for a general scalar product, because the microplane strain measure $\boldsymbol{\varepsilon}^{(\mathbf{N})}$ can be a scalar, vector, or a set of scalars and vectors.

Note that the foregoing expressions are valid for general dissipative materials. In the particular case of hyperelasticity, as later considered in the paper, the internal variables $\xi$ may be omitted from the free energy arguments. However, when integral expressions are derived in this and subsequent sections, internal variables are always retained for the sake of generality of such expressions.

## 3. Framework for small-strain microplane theory

### 3.1. Stress evaluation formulae

Practically all the microplane formulations developed in the 1980s and 1990s were limited to small strain. The simplest model, proposed by Bažant and Oh (1985) and nowadays called M1, characterized the strain on the microplane level by the strain component normal to the microplane, defined as

$$\varepsilon_N = \mathbf{n}\cdot\boldsymbol{\varepsilon}\cdot\mathbf{n} = (\mathbf{n}\otimes\mathbf{n}) : \boldsymbol{\varepsilon} = \mathscr{N} : \boldsymbol{\varepsilon} \tag{5}$$

where $\mathscr{N} = \mathbf{n}\otimes\mathbf{n}$ is a second-order tensor that helps to simplify the notation. In this case, the microplane strain measure $\boldsymbol{\varepsilon}^{(\mathbf{N})}$ is represented by a single scalar, $\varepsilon_N$. Subsequent refinements enriched $\boldsymbol{\varepsilon}^{(\mathbf{N})}$ by the microplane shear strain, $\boldsymbol{\varepsilon}_T$, and divided the normal strain $\varepsilon_N$ into the volumetric strain, $\varepsilon_V$, which is the same for all the microplanes, and the deviatoric strain, $\varepsilon_D$, which can also be understood as the normal projection of the deviatoric part of the strain tensor. These strain measures can be evaluated as

$$\boldsymbol{\varepsilon}_T = \mathbf{n}\cdot\boldsymbol{\varepsilon} - \mathbf{n}\varepsilon_N = \mathscr{T} : \boldsymbol{\varepsilon} \tag{6}$$

$$\varepsilon_V = \tfrac{1}{3}\mathbf{I} : \boldsymbol{\varepsilon} = \mathscr{V} : \boldsymbol{\varepsilon} \tag{7}$$

$$\varepsilon_D = \varepsilon_N - \varepsilon_V = \left(\mathbf{n}\otimes\mathbf{n} - \tfrac{1}{3}\mathbf{I}\right) : \boldsymbol{\varepsilon} = (\mathscr{N} - \mathscr{V}) : \boldsymbol{\varepsilon} = \mathscr{D} : \boldsymbol{\varepsilon} \tag{8}$$

where

$$\mathscr{T} = \mathbf{n}\cdot\mathscr{I}^{\mathrm{sym}} - \mathbf{n}\otimes\mathbf{n}\otimes\mathbf{n}, \quad \mathscr{V} = \tfrac{1}{3}\mathbf{I}, \quad \mathscr{D} = \mathbf{n}\otimes\mathbf{n} - \tfrac{1}{3}\mathbf{I} \tag{9}$$

are auxiliary tensors introduced for convenience. Note that $\boldsymbol{\varepsilon}_T$ is a first-order tensor while $\varepsilon_N$, $\varepsilon_V$ and $\varepsilon_D$ are scalars, and that $\mathscr{T}$ is a third-order tensor while $\mathscr{N}$, $\mathscr{V}$ and $\mathscr{D}$ are second-order tensors.

A quite general framework for small-strain microplane models is obtained if all of the foregoing microplane strain measures are considered as arguments of the free energy $\Psi_\Omega[\varepsilon_N, \boldsymbol{\varepsilon}_T, \varepsilon_V, \varepsilon_D, \boldsymbol{\xi}]$. This is always possible because nothing prevents us from using arguments of a function which are not independent, provided that all the existing dependencies are taken into account while taking the derivatives such as those implied in (3). In this case, the derivatives of the microplane strains with respect to the strain tensor follow from (5)–(8): $\partial\varepsilon_N/\partial\boldsymbol{\varepsilon} = \mathscr{N}$, $\partial\boldsymbol{\varepsilon}_T/\partial\boldsymbol{\varepsilon} = \mathscr{T}$, etc. The specific stress evaluation formula obtained from (3) therefore reads

$$\boldsymbol{\sigma} = \frac{3}{2\pi}\int_\Omega (\sigma_N\mathscr{N} + \boldsymbol{\sigma}_T\cdot\mathscr{T} + \sigma_V\mathscr{V} + \sigma_D\mathscr{D})\,\mathrm{d}\Omega \tag{10}$$

where

$$\sigma_N = \frac{\partial\Psi_\Omega}{\partial\varepsilon_N}, \quad \boldsymbol{\sigma}_T = \frac{\partial\Psi_\Omega}{\partial\boldsymbol{\varepsilon}_T}, \quad \sigma_V = \frac{\partial\Psi_\Omega}{\partial\varepsilon_V}, \quad \sigma_D = \frac{\partial\Psi_\Omega}{\partial\varepsilon_D} \tag{11}$$

are the microplane stresses work-conjugate with $\varepsilon_N$, $\boldsymbol{\varepsilon}_T$, $\varepsilon_V$ and $\varepsilon_D$, respectively. Eqs. (11) are the so-called microplane constitutive laws.

By keeping only certain selected terms, (10) can be particularized to various microplane formulations from the literature. For instance, model M1 (Bažant and Oh, 1985) considered the normal strain $\varepsilon_N$ as the only microplane strain measure, and the corresponding stress evaluation formula

$$\boldsymbol{\sigma} = \frac{3}{2\pi}\int_\Omega \sigma_N\mathscr{N}\,\mathrm{d}\Omega = \frac{3}{2\pi}\int_\Omega \sigma_N\mathbf{n}\otimes\mathbf{n}\,\mathrm{d}\Omega \tag{12}$$

was derived from the principle of virtual work. Clearly, (12) is a special case of (10).

One of the early microplane models, nowadays called model $\mathrm{M1}^0$ (Bažant, 1984), was based on microplane strain measures $\varepsilon_N$ and $\boldsymbol{\varepsilon}_T$ and the corresponding microplane stresses $\sigma_N$ and $\boldsymbol{\sigma}_T$. Since $\boldsymbol{\varepsilon}_T$ is always perpendicular to $\mathbf{n}$, $\boldsymbol{\sigma}_T$ has the same property. It is convenient to write $\boldsymbol{\sigma}_T = \sigma_T\mathbf{t}$ where $\sigma_T = |\boldsymbol{\sigma}_T|$ is the magnitude of the shear microplane stress and $\mathbf{t} = \boldsymbol{\sigma}_T/|\boldsymbol{\sigma}_T|$ is its direction. Using the relations $\mathbf{t}\cdot\mathbf{n} = 0$ and $\mathbf{t}\cdot(\mathbf{n}\cdot\mathscr{I}^{\mathrm{sym}}) = (\mathbf{t}\otimes\mathbf{n}) : \mathscr{I}^{\mathrm{sym}} = \frac{1}{2}(\mathbf{t}\otimes\mathbf{n} + \mathbf{n}\otimes\mathbf{t})$, it is possible to show that the stress evaluation formula for model $\mathrm{M1}^0$ reads

$$\boldsymbol{\sigma} = \frac{3}{2\pi}\int_\Omega (\sigma_N \mathcal{N} + \boldsymbol{\sigma}_T \cdot \mathcal{T})\,d\Omega = \frac{3}{2\pi}\int_\Omega \left(\sigma_N \mathbf{n}\otimes\mathbf{n} + \sigma_T \frac{1}{2}(\mathbf{n}\otimes\mathbf{t} + \mathbf{t}\otimes\mathbf{n})\right)d\Omega \tag{13}$$

In model M2 (Bažant and Prat, 1988a,b), the normal strain was divided into the volumetric and deviatoric parts, and the stress was evaluated simply by substituting $\sigma_N = \sigma_V + \sigma_D$ into (13). However, this does not correspond to the stress evaluation formula derived consistently from the free-energy potential as a special case of (10), which reads

$$\begin{aligned}\boldsymbol{\sigma} &= \frac{3}{2\pi}\int_\Omega (\sigma_V \mathcal{V} + \sigma_D \mathcal{D} + \boldsymbol{\sigma}_T \cdot \mathcal{T})\,d\Omega \\ &= \frac{3}{2\pi}\int_\Omega \left(\sigma_V \frac{\mathbf{I}}{3} + \sigma_D\left(\mathbf{n}\otimes\mathbf{n} - \frac{\mathbf{I}}{3}\right) + \sigma_T \frac{1}{2}(\mathbf{n}\otimes\mathbf{t} + \mathbf{t}\otimes\mathbf{n})\right)d\Omega\end{aligned} \tag{14}$$

The differences have been analyzed and interpreted in Carol et al. (2001). Both formulae give the same results in the case of isotropic linear elasticity (which explains why this discrepancy was not detected earlier), but they differ significantly for non-linear behavior, especially if the microplane constitutive law for the deviatoric microplane stress $\sigma_D$ strongly depends on the sign of the deviatoric microplane strain $\varepsilon_D$ (different behaviors in tension and compression). The original formula of model M2 may lead to spurious energy dissipation or generation under certain load cycles, as demonstrated by an example in Carol et al. (2001). The most recent implementation of the microplane model for concrete, called M4 (Bažant et al., 2000b; Caner and Bažant, 2000), uses the consistent stress evaluation formula that follows from (10) by dropping the term with $\sigma_N$. Nevertheless, model M4 does not fully fit into the present framework because the microplane constitutive laws that link the strains $\boldsymbol{\varepsilon}_T$, $\varepsilon_V$ and $\varepsilon_D$ to the stresses $\boldsymbol{\sigma}_T$, $\sigma_V$ and $\sigma_D$ are quite complicated and can hardly be presented in the form of (11).

### 3.2. Linear elasticity

The special case of linear elasticity is obtained with a quadratic potential

$$\Psi_\Omega[\varepsilon_N, \boldsymbol{\varepsilon}_T, \varepsilon_V, \varepsilon_D] = \tfrac{1}{2}(E_N\varepsilon_N^2 + E_T\boldsymbol{\varepsilon}_T\cdot\boldsymbol{\varepsilon}_T + E_V\varepsilon_V^2 + E_D\varepsilon_D^2) \tag{15}$$

which yields linear microplane constitutive laws

$$\sigma_N = E_N\varepsilon_N, \quad \boldsymbol{\sigma}_T = E_T\boldsymbol{\varepsilon}_T, \quad \sigma_V = E_V\varepsilon_V, \quad \sigma_D = E_D\varepsilon_D \tag{16}$$

Substituting (16) and (5)–(8) into (10) provides the macroscopic stress–strain law $\boldsymbol{\sigma} = \mathcal{E} : \boldsymbol{\varepsilon}$ where

$$\mathcal{E} = \frac{3}{2\pi}\int_\Omega (E_N \mathcal{N}\otimes\mathcal{N} + E_T \mathcal{T}^t \cdot \mathcal{T} + E_V \mathcal{V}\otimes\mathcal{V} + E_D \mathcal{D}\otimes\mathcal{D})\,d\Omega \tag{17}$$

is the elastic stiffness tensor.

In general, the microplane elastic moduli $E_N$, $E_T$, $E_V$ and $E_D$ can depend on the microplane orientation $\mathbf{n}$, which provides a very natural framework for anisotropic elasticity. In the isotropic case, the microplane moduli are constant and the integral in (17) can be evaluated in closed form. Using the formulae from Bažant and Oh (1985) and Lubarda and Krajcinovic (1993), it is easy to show that

$$\frac{3}{2\pi}\int_\Omega \mathcal{N}\,d\Omega = \mathbf{I} \tag{18}$$

$$\frac{3}{2\pi}\int_\Omega \mathcal{N}\otimes\mathcal{N}\,d\Omega = \mathcal{I}^{vol} + \frac{2}{5}\mathcal{I}^{dev} \tag{19}$$

$$\frac{3}{2\pi}\int_{\Omega}\mathscr{T}^{t}\cdot\mathscr{T}\,d\Omega=\frac{3}{5}\mathscr{I}^{dev} \tag{20}$$

$$\frac{3}{2\pi}\int_{\Omega}\mathscr{V}\otimes\mathscr{V}\,d\Omega=\mathscr{I}^{vol} \tag{21}$$

$$\frac{3}{2\pi}\int_{\Omega}\mathscr{D}\otimes\mathscr{D}\,d\Omega=\frac{2}{5}\mathscr{I}^{dev} \tag{22}$$

Substituting (19)–(22) into (17) yields the macroscopic stiffness tensor in the form

$$\mathscr{E}=(E_N+E_V)\mathscr{I}^{vol}+\tfrac{1}{5}(2E_N+3E_T+2E_D)\mathscr{I}^{dev} \tag{23}$$

Comparing this with the standard expression in terms of the bulk modulus $K$ and shear modulus $G$, we obtain

$$3K=E_N+E_V,\quad 10G=2E_N+3E_T+2E_D \tag{24}$$

Since there are only two independent macroscopic elastic moduli, it should be possible to reproduce linear isotropic elasticity using only two non-zero microscopic elastic moduli. However, one must be careful to make the right choice, otherwise it is impossible to cover the full range of thermodynamically admissible macroscopic parameters with non-negative values of the microscopic moduli. This becomes clear from the expression for the Poisson ratio,

$$\nu=\frac{3K-2G}{6K+2G}=\frac{5E_V+3E_N-2E_D-3E_T}{10E_V+12E_N+2E_D+3E_T} \tag{25}$$

For a model with a single microplane strain, the value of Poisson's ratio cannot be controled. A model with $\varepsilon_V$ only, would give $\nu=0.5$, which corresponds to a fluid that transmits only hydrostatic pressure but no deviatoric stress. Model M1 uses $\varepsilon_N$ only, and the resulting Poisson ratio is $\nu=0.25$, which is known to be the value characteristic of a homogenized random lattice in three dimensions (spatial truss). Hypothetical models using only $\varepsilon_D$ or $\boldsymbol{\varepsilon}_T$ or both, but no other microplane strain measures, would give $\nu=-1$; they represent a strange material in which the mean hydrostatic pressure always remains zero but a purely deviatoric stress can be transmitted.

For a model with two or more microplane strain components, the value of Poisson's ratio can be controled, but only within the range bounded by the values that correspond to the special cases with a single microplane strain component. So the model $M1^0$ with $\varepsilon_N$ and $\boldsymbol{\varepsilon}_T$ can cover only the range $-1\leqslant\nu\leqslant 0.25$, and a hypothetical model with $\varepsilon_N$ and $\boldsymbol{\varepsilon}_V$ could cover only the range $0.25\leqslant\nu\leqslant 0.5$. The full range $-1\leqslant\nu\leqslant 0.5$ can be covered only by models that work with $\varepsilon_V$ and with at least one of $\varepsilon_D$ or $\boldsymbol{\varepsilon}_T$. This condition is satisfied by models M2 and M4 (Bažant and Prat, 1988a; Bažant et al., 2000b).

## 4. General framework for large-strain microplane theory

### 4.1. General stress evaluation formula

One major advantage of the thermodynamic approach to microplane-based constitutive modeling is that the structure of Eqs. (1)–(4) is general (not restricted to small strain), and therefore the previous formulation of Section 3 can be naturally extended to large deformation without the need for any major additional assumptions.

On the macroscopic level, the deformation must be characterized by some large-strain tensor, and differentiation of the free energy gives its corresponding conjugate stress tensor. In this study, the Green's

Lagrangian (GL) strain tensor $\mathbf{E}$ and its conjugate second Piola–Kirchhoff (sPK) stress tensor $\boldsymbol{\Sigma}$ are chosen for this purpose. The GL and other Lagrangian (or material) strain tensors are constructed in the usual way: if $\mathbf{x}[\mathbf{X}]$ is the function that maps the initial coordinate $\mathbf{X}$ onto the current coordinate $\mathbf{x}$ of the same material particle, then the deformation gradient $\mathbf{F}$, the rotation $\mathbf{R}$, the stretch $\mathbf{U}$, the right Cauchy–Green (rCG) tensor $\mathbf{C}$ and the GL strain tensor $\mathbf{E}$ are defined as

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{X}} = \mathbf{R} \cdot \mathbf{U}, \quad \mathbf{C} = \mathbf{F}^{\mathrm{t}} \cdot \mathbf{F} = \mathbf{U}^2, \quad \mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{I}) \tag{26}$$

We recall also the polar decomposition

$$\mathbf{F} = \mathbf{R} \cdot \mathbf{U} = \mathbf{V} \cdot \mathbf{R} \tag{27}$$

where the rotation tensor $\mathbf{R}$ is orthogonal and the right stretch tensor $\mathbf{U}$ and left stretch tensor $\mathbf{V}$ are symmetric and positive definite. Tensor $\mathbf{U}$ is the tensorial square root of the rCG tensor, because

$$\mathbf{C} = \mathbf{F}^{\mathrm{t}} \cdot \mathbf{F} = (\mathbf{R} \cdot \mathbf{U})^{\mathrm{t}} \cdot (\mathbf{R} \cdot \mathbf{U}) = \mathbf{U} \cdot \mathbf{R}^{\mathrm{t}} \cdot \mathbf{R} \cdot \mathbf{U} = \mathbf{U}^2 \tag{28}$$

while tensor $\mathbf{V} = \mathbf{R} \cdot \mathbf{U} \cdot \mathbf{R}^{\mathrm{t}}$ is the tensorial square root of the left Cauchy–Green tensor $\mathbf{b}$, defined as

$$\mathbf{b} = \mathbf{F} \cdot \mathbf{F}^{\mathrm{t}} = \mathbf{V}^2 = \mathbf{R} \cdot \mathbf{C} \cdot \mathbf{R}^{\mathrm{t}} \tag{29}$$

The inverse of $\mathbf{b}$ is also known as the Finger tensor, $\mathbf{b}^{-1}$.

The choice of the GL tensor as the basic strain tensor for the microplane formulation is of course only one of many possible alternatives, but it seems the most advantageous, for two reasons. First, if induced anisotropy is to be captured, it is necessary to use a strain measure defined with respect to the initial configuration. Second, among all possible Lagrangian strain tensors, the GL tensor (or the linearly related rCG tensor) leads to the simplest expressions for the microplane strain measures with a direct physical meaning, which are proposed in the next section. At the same time, its conjugate sPK stress tensor is also convenient because of its direct push-forward relation with the Cauchy stress tensor $\boldsymbol{\sigma}$, which facilitates the physical interpretation of the resulting equations.

The next step will be to select and define the microplane strains. This is an essential aspect for which there are several options with various degrees of generality and complexity, as described in Section 4.2 and in the following sections of the paper. But before doing that, a general stress evaluation formula along the line of (3) can be written for the large-strain microplane formulations based on the GL strain tensor:

$$\boldsymbol{\Sigma} = \frac{\partial(\rho_0 \Psi)}{\partial \mathbf{E}} = \frac{3}{2\pi} \int_{\Omega} \frac{\partial \Psi_{\Omega}}{\partial \mathbf{E}} \, \mathrm{d}\Omega = \frac{3}{2\pi} \int_{\Omega} \frac{\partial \Psi_{\Omega}}{\partial \mathbf{E}^{(\mathbf{N})}} \bullet \frac{\partial \mathbf{E}^{(\mathbf{N})}}{\partial \mathbf{E}} \, \mathrm{d}\Omega = \frac{3}{2\pi} \int_{\Omega} \boldsymbol{\Sigma}^{(\mathbf{N})} \bullet \frac{\partial \mathbf{E}^{(\mathbf{N})}}{\partial \mathbf{E}} \, \mathrm{d}\Omega \tag{30}$$

Here, $\mathbf{E}^{(\mathbf{N})}$ is some (still unspecified) set of quantities characterizing strain on the microplane level and linked to $\mathbf{E}$ by the kinematic constraint

$$\mathbf{E}^{(\mathbf{N})} = \mathscr{F}_{\mathbf{E}}[\mathbf{E}, \mathbf{N}] \tag{31}$$

which is now in general non-linear, and $\boldsymbol{\Sigma}^{(\mathbf{N})}$ is the set of work-conjugate microplane stresses, evaluated from the microplane constitutive laws

$$\boldsymbol{\Sigma}^{(\mathbf{N})} = \frac{\partial \Psi_{\Omega}}{\partial \mathbf{E}^{(\mathbf{N})}} \tag{32}$$

Note that the microplane normal in the initial configuration is now denoted by capital $\mathbf{N}$, while lower-case $\mathbf{n}$ is reserved for a certain unit vector in the deformed configuration, to be defined in the next subsection.

### 4.2. Microplane strain measures

To proceed further, we need to specify what microplane strains are considered, i.e. what are the components of $\mathbf{E}^{(\mathbf{N})}$. In order to keep track of the same material microplanes through the deformation process, all expressions will be written in terms of material quantities. This is why it is convenient to work on the macroscopic level with a Lagrangian strain measure. However, in contrast to the small-strain theory, the microplane strains $\mathbf{E}^{(\mathbf{N})}$ *will not be obtained by linear projection of* $\mathbf{E}$ *on the microplane*.

The choice of microplane strain measures is to a certain extent arbitrary, and it should be guided by their physical sense, according to the previous experience with microplane formulations in the small-strain range. Along this line, it seems logical to try to capture normal and tangential mechanisms (in the spirit of $\mathrm{M1}^0$), or volumetric, deviatoric and tangential mechanisms (in the spirit of M2). However, in the large-strain range, there are multiple possibilities for characterizing each of these kinematic mechanisms. As the first step, the most natural candidates for microplane strain measures are listed here:

(a) **Normal strain** can be characterized in two ways:

(a.1) By the *microplane stretch* of the fiber initially aligned with the microplane normal $\mathbf{N}$,

$$\lambda_{\mathrm{N}} = |\mathbf{F}\cdot\mathbf{N}| = \sqrt{\mathbf{N}\cdot\mathbf{F}^{\mathrm{t}}\cdot\mathbf{F}\cdot\mathbf{N}} = \sqrt{\mathbf{N}\cdot\mathbf{C}\cdot\mathbf{N}} = \sqrt{\mathbf{N}\cdot(\mathbf{I}+2\mathbf{E})\cdot\mathbf{N}} = \sqrt{1+2\mathbf{N}\cdot\mathbf{E}\cdot\mathbf{N}} \tag{33}$$

Note that the microplane strain measure $\lambda_{\mathrm{N}}$ is easily expressed in terms of the projected component of the GL tensor, $E_{\mathrm{NN}} = \mathbf{N}\cdot\mathbf{E}\cdot\mathbf{N} = \mathcal{N} : \mathbf{E}$. This is rather an exception, as we will soon see.

(a.2) By the *microplane thickening* of the layer of material lying in the microplane with initial normal $\mathbf{N}$,

$$\bar{\lambda}_{\mathrm{N}} = \frac{1}{|\mathbf{F}^{-\mathrm{t}}\cdot\mathbf{N}|} = \frac{1}{\sqrt{\mathbf{N}\cdot\mathbf{C}^{-1}\cdot\mathbf{N}}} \tag{34}$$

(b) **Shear strain** can be characterized by the *angular distortion*, $\gamma_{\mathrm{N}}$, of the layer of material lying in the microplane with initial normal $\mathbf{N}$, which satisfies the relation

$$\tan\gamma_{\mathrm{N}} = \sqrt{(\mathbf{N}\cdot\mathbf{C}\cdot\mathbf{N})(\mathbf{N}\cdot\mathbf{C}^{-1}\cdot\mathbf{N}) - 1} = \sqrt{\frac{\lambda_{\mathrm{N}}^2}{\bar{\lambda}_{\mathrm{N}}^2} - 1} \tag{35}$$

In the small-strain limit, $\tan\gamma_{\mathrm{N}}$ reduces to the magnitude of the microplane shear vector $\boldsymbol{\varepsilon}_{\mathrm{T}}$. If each microplane responds isotropically in its own plane (which is of course a much weaker hypothesis than macroscopic isotropy), the direction of the shear vector plays no role, and it is sufficient to characterize the shear strain by a scalar.

(c) In the large-strain theory, the relative **change of volume** is characterized by the Jacobian $J = \det\mathbf{F}$. The classical small-strain volumetric–deviatoric split is now more appropriately called volumetric-distortional split and is based on the multiplicative decomposition of the deformation gradient (Flory, 1961),

$$\mathbf{F} = \mathbf{F}_{\mathrm{V}}\cdot\mathbf{F}_{\mathrm{D}} = \mathbf{F}_{\mathrm{D}}\cdot\mathbf{F}_{\mathrm{V}} \tag{36}$$

where $\mathbf{F}_{\mathrm{V}} = J^{1/3}\mathbf{I}$ is an isotropic tensor describing a pure volume change, and

$$\mathbf{F}_{\mathrm{D}} = J^{-1/3}\mathbf{F} \tag{37}$$

is a tensor with unit determinant describing an isochoric change of shape of "pure distortion". The **volumetric part**, $\mathbf{F}_{\mathrm{V}}$, corresponds to an isotropic stretch

$$\lambda_{\mathrm{J}} = (\det\mathbf{F})^{1/3} = J^{1/3} \tag{38}$$

in all directions. So it is natural to use $\lambda_J$ as the microplane strain measure characterizing the volumetric part of the deformation. In the small-strain limit, $\lambda_J$ reduces to $1+\varepsilon_V$ where $\varepsilon_V$ is the mean normal strain.

(d) The **distortional strain** microplane measure (large-strain counterpart to the deviatoric small-strain measure $\varepsilon_D$) is derived from $\mathbf{F}_D$ in the same way as the normal strain measure was derived from $\mathbf{F}$. This means that the distortional deformation is characterized on the microplane level by one of the following variables:

(d.1) The *distortional microplane stretch* of the fiber initially aligned with the microplane normal $\mathbf{N}$ is evaluated as

$$\lambda_D = |\mathbf{F}_D \cdot \mathbf{N}| = \sqrt{\mathbf{N}\cdot\mathbf{F}_D^t\cdot\mathbf{F}_D\cdot\mathbf{N}} = \sqrt{\mathbf{N}\cdot\mathbf{C}_D\cdot\mathbf{N}} \tag{39}$$

where $\mathbf{C}_D = \mathbf{F}_D^t \cdot \mathbf{F}_D = J^{-2/3}\mathbf{F}^t\cdot\mathbf{F} = J^{-2/3}\mathbf{C}$.

(d.2) The *distortional microplane thickening* of the layer of material lying in the microplane with initial normal $\mathbf{N}$ is evaluated as

$$\bar{\lambda}_D = \frac{1}{|\mathbf{F}_D^{-t}\cdot\mathbf{N}|} = \frac{1}{\sqrt{\mathbf{N}\cdot\mathbf{C}_D^{-1}\cdot\mathbf{N}}} \tag{40}$$

Note that, with previous definitions, the following relations are satisfied:

$$\lambda_N = |\mathbf{F}\cdot\mathbf{N}| = |J^{1/3}\mathbf{F}_D\cdot\mathbf{N}| = J^{1/3}|\mathbf{F}_D\cdot\mathbf{N}| = \lambda_J\lambda_D \tag{41}$$

$$\bar{\lambda}_N = \frac{1}{|\mathbf{F}^{-t}\cdot\mathbf{N}|} = \frac{1}{|J^{-1/3}\mathbf{F}_D^{-t}\cdot\mathbf{N}|} = J^{1/3}\frac{1}{|\mathbf{F}_D^{-t}\cdot\mathbf{N}|} = \lambda_J\bar{\lambda}_D \tag{42}$$

This means that the multiplicative nature of the volumetric-distortional split is reflected on the microplane level. Consequently, the shear measure $\gamma_N$ defined in (35) satisfies the relation

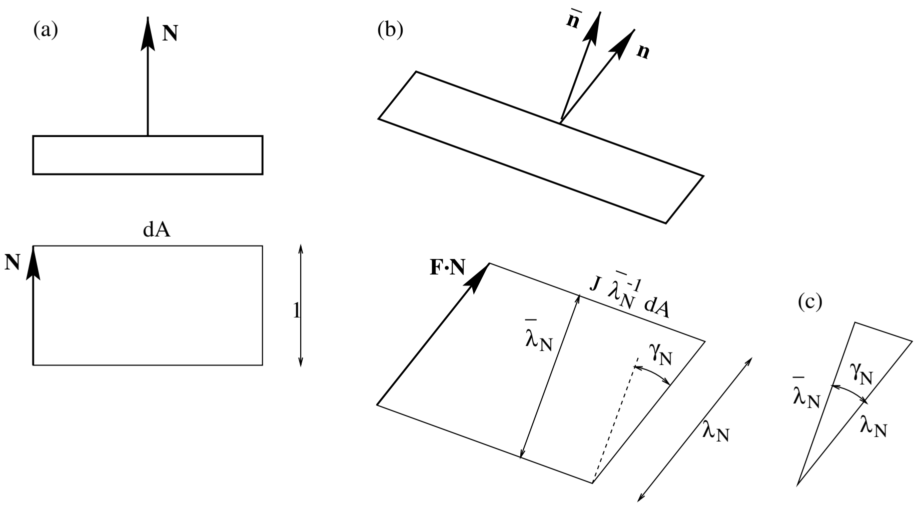


Fig. 1. (a) Microplane in initial configuration, (b) microplane in deformed configuration, (c) relationship between normal and shear microplane strain measures.

$$\tan \gamma_N = \sqrt{\frac{\lambda_N^2}{\bar{\lambda}_N^2} - 1} = \sqrt{\frac{\lambda_D^2}{\bar{\lambda}_D^2} - 1} \tag{43}$$

Exploiting the formula $\cos \gamma_N = (1 + \tan^2 \gamma_N)^{-1/2}$, we can derive useful relations

$$\bar{\lambda}_N = \lambda_N \cos \gamma_N, \quad \bar{\lambda}_D = \lambda_D \cos \gamma_N \tag{44}$$

The geometrical meaning of vectors $\mathbf{N}$, $\mathbf{n}$ and $\bar{\mathbf{n}}$ and of microplane strain measures $\lambda_N$, $\bar{\lambda}_N$ and $\gamma_N$ is illustrated in Fig. 1.

## 5. Simple formulation based on microplane stretch $\lambda_N$

To illustrate the basic concepts, let us first present a simple microplane formulation that characterizes the strain on each microplane by a single scalar—the microplane stretch $\lambda_N$—and exhibits one of the simplest forms of the free-energy function. Models with other microplane strain measures will be developed in Sections 7–9.

### 5.1. Stress evaluation formula

In the present simple model, the microplane strain measure $\mathbf{E}^{(N)}$ is just the scalar $\lambda_N$. Differentiating the microplane free-energy function

$$\Psi_\Omega = \Psi_\Omega[\lambda_N, \xi] \tag{45}$$

with respect to $\lambda_N$, we obtain the corresponding microplane stress measure

$$\Sigma_N = \frac{\partial \Psi_\Omega}{\partial \lambda_N} \tag{46}$$

To set up a specific form of the stress evaluation formula (30), we compute the partial derivative $\partial\lambda_N/\partial\mathbf{E} = \lambda_N^{-1}\mathbf{N} \otimes \mathbf{N}$ (the details of this derivation are given in Appendix A) and substitute it into (30) instead of $\partial\mathbf{E}^{(N)}/\partial\mathbf{E}$. The macroscopic sPK stress tensor is then expressed as

$$\boldsymbol{\Sigma} = \frac{3}{2\pi} \int_\Omega \Sigma_N \lambda_N^{-1} \mathbf{N} \otimes \mathbf{N} \, \mathrm{d}\Omega \tag{47}$$

and contravariant push-forward divided by $J$ leads to the Cauchy stress tensor

$$\boldsymbol{\sigma} = \frac{1}{J} \mathbf{F} \cdot \boldsymbol{\Sigma} \cdot \mathbf{F}^t = \frac{3}{2\pi J} \int_\Omega \Sigma_N \lambda_N^{-1} \mathbf{F} \cdot \mathbf{N} \otimes \mathbf{N} \cdot \mathbf{F}^t \, \mathrm{d}\Omega \tag{48}$$

Note that $\mathbf{F} \cdot \mathbf{N}$ is a vector aligned with the fiber in the deformed configuration, and its norm is equal to the microplane stretch $\lambda_N$. Thus, (48) can be rewritten as

$$\boldsymbol{\sigma} = \frac{3}{2\pi J} \int_\Omega \Sigma_N \lambda_N \mathbf{n} \otimes \mathbf{n} \, \mathrm{d}\Omega \tag{49}$$

where

$$\mathbf{n} = \frac{\mathbf{F} \cdot \mathbf{N}}{|\mathbf{F} \cdot \mathbf{N}|} = \frac{\mathbf{F} \cdot \mathbf{N}}{\lambda_N} \tag{50}$$

is the unit vector characterizing the fiber direction in the deformed configuration.

In a large-strain theory, most equations can be obtained in both Lagrangian and Eulerian settings. Eq. (47) directly represents the Lagrangian version of the stress evaluation formulae in the microplane model, since every term in it is referred to the "material" configuration. In order to identify its spatial counterpart, Eq. (49) still needs the replacement of the solid angle differential, which will change from $\mathrm{d}\Omega$ to $\mathrm{d}\omega$ during deformation. The relation between the solid angle differentials in the initial and deformed configurations, derived in Appendix B, is

$$\mathrm{d}\Omega = \frac{\lambda_\mathrm{N}^3}{J}\,\mathrm{d}\omega \tag{51}$$

Using this in (49), one obtains

$$\boldsymbol{\sigma} = \frac{3}{2\pi}\int_\Omega \sigma_\mathrm{n}\mathbf{n}\otimes\mathbf{n}\,\mathrm{d}\omega \tag{52}$$

where

$$\sigma_\mathrm{n} = \frac{\lambda_\mathrm{N}^4\Sigma_\mathrm{N}}{J^2} \tag{53}$$

Eq. (52) has been written in the same format as its small-strain counterpart (12). This exactly corresponds to the philosophy of an Eulerian formulation of the microplane model in which the microplane orientations are assigned in deformed configuration. The new stresses $\sigma_\mathrm{n}$ have the meaning of the normal "Cauchy" stresses on these Eulerian microplanes (note the lowercase subscript). However, other quantities in (53) such as $\lambda_\mathrm{N}$ and $\Sigma_\mathrm{N}$ still carry uppercase subscript 'N', because they correspond to the stretch and material stress of the fiber with original orientation **N** *which then has become* **n**. Strictly speaking, though, a fully Eulerian expression should only refer to the deformed configuration, and therefore only contain lowercase subscripts. This can be achieved by considering the deformed configuration as the current state, and from it "looking back" to the initial state. According to this the Eulerian deformation gradient and fiber stretch may be defined as

$$\mathbf{f} = \mathbf{F}^{-1}, \quad \mathbf{f}^\mathrm{t}\cdot\mathbf{f} = \mathbf{b}^{-1}, \quad \lambda_\mathrm{n} = \sqrt{\mathbf{n}\cdot\mathbf{b}^{-1}\cdot\mathbf{n}} = \frac{1}{\lambda_\mathrm{N}} \tag{54}$$

Using these, (53) may be rewritten as

$$\sigma_\mathrm{n} = \frac{\Sigma_\mathrm{n}}{\lambda_\mathrm{n}^4 J^2} \tag{55}$$

(note that we could also define an inverse Jacobian $j = 1/J$, which is however not used for obvious notation reasons). Since in this model the same physical fiber is referred to by 'n' or 'N', all this distinction may seem superfluous in this case. However, it may be useful in more complex models involving the layer thickening and distortion angle. The integral in (52) is taken over all microplane orientations in the deformed configuration. Formally, the domain of integration could be denoted as $\omega$, but since the deformed microplane orientations again fill a unit hemisphere, we keep the original symbol $\Omega$.

### 5.2. Derivation from principle of virtual work

To get some insight into the physical meaning of the foregoing formulae, let us explore an alternative derivation based on the principle of virtual work. We shall consider a special type of material microstructure consisting of randomly oriented fibers. Each fiber is assumed to be under uniaxial stress. Consider a fiber of initial cross-sectional area $A$ and length $L$. After deformation, the fiber has an area $a$, length $l$, and it transmits an axial force $S$. The ratio $S/A$ corresponds to the first Piola–Kirchhoff stress in the fiber, and so we denote it as $\sigma_{1PK}$. On the microlevel, the constitutive behavior of the material is characterized by the dependence of $\sigma_{1PK}$ on the fiber stretch, $\lambda = l/L$, and on some internal variables (which we do not list explicitly).

Suppose that the directional distribution of individual fibers is uniform and that $\Theta$ is the initial relative volume of all the fibers. If we divide all spatial directions (represented by points on the surface of a unit hemisphere) into elementary sectors (represented by infinitesimal facets $\mathrm{d}\Omega$), each fiber can be assigned to one of these sectors. The relative volume of fibers belonging to an elementary sector is $\Theta\,\mathrm{d}\Omega/2\pi$. As mentioned in the previous section, the sector with initial direction $\mathbf{N}$ and of size $\mathrm{d}\Omega$ is, after deformation, transformed into a sector with direction $\mathbf{n} = \mathbf{F}\cdot\mathbf{N}/\lambda_{\mathrm{N}}$ and of size $\mathrm{d}\omega = (J/\lambda_{\mathrm{N}}^3)\,\mathrm{d}\Omega$.

Consider an elementary volume $\mathrm{d}V$ in the initial configuration, transformed into $\mathrm{d}v = J\,\mathrm{d}V$ in the deformed configuration. Virtual power expressed in terms of the macroscopic quantities is

$$P_{\mathrm{mac}} = \boldsymbol{\sigma} : \mathbf{D}\,\mathrm{d}v = \boldsymbol{\sigma} : \mathbf{D}J\,\mathrm{d}V \tag{56}$$

where $\mathbf{D}$ is the rate-of-deformation tensor (symmetric part of the spatial velocity gradient $\mathbf{L} = \dot{\mathbf{F}}\cdot\mathbf{F}^{-1}$). Now we have to express this power in terms of the microscopic quantities. Virtual work in one fiber,

$$S\dot{l} = (\sigma_{1PK}A)\times(\dot{\lambda}L) = \sigma_{1PK}\dot{\lambda}AL \tag{57}$$

is proportional to the initial volume of the fiber, $AL$. The intersection of fibers from an elementary sector $\mathrm{d}\Omega$ with the elementary volume has an initial volume $\Theta\,\mathrm{d}V\,\mathrm{d}\Omega/2\pi$, and so the contribution of fibers from this sector to the virtual power is

$$\mathrm{d}P_{\mathrm{mic}} = \frac{\sigma_{1PK}\dot{\lambda}\Theta}{2\pi}\,\mathrm{d}\Omega\,\mathrm{d}V \tag{58}$$

Summing the contribution of all sectors, we obtain

$$P_{\mathrm{mic}} = \int_{\Omega}\frac{\sigma_{1PK}\dot{\lambda}\Theta}{2\pi}\,\mathrm{d}\Omega\,\mathrm{d}V = \frac{\Theta\,\mathrm{d}V}{2\pi}\int_{\Omega}\sigma_{1PK}\dot{\lambda}\,\mathrm{d}\Omega \tag{59}$$

The instantaneous rate of stretching in a certain direction $\mathbf{n}$ is given by the projection of $\mathbf{D}$ onto that direction, $\mathbf{n}\cdot\mathbf{D}\cdot\mathbf{n}$, and so the time derivative of the stretch $\lambda$ is

$$\dot{\lambda} = \lambda\mathbf{n}\cdot\mathbf{D}\cdot\mathbf{n} = \lambda(\mathbf{n}\otimes\mathbf{n}) : \mathbf{D} \tag{60}$$

Substituting this into (59) and setting $P_{\mathrm{mic}}$ equal to $P_{\mathrm{mac}}$ from (56), we obtain

$$\frac{\Theta\,\mathrm{d}V}{2\pi}\int_{\Omega}\sigma_{1PK}\lambda\mathbf{n}\otimes\mathbf{n}\,\mathrm{d}\Omega : \mathbf{D} = \boldsymbol{\sigma} : \mathbf{D}\,\mathrm{d}v \tag{61}$$

from which

$$\boldsymbol{\sigma} = \frac{\Theta}{2\pi J}\int_{\Omega}\sigma_{1PK}\lambda\mathbf{n}\otimes\mathbf{n}\,\mathrm{d}\Omega \tag{62}$$

Comparing this result to (49) we realize that $\Theta\sigma_{1PK}$ corresponds to $3\Sigma_{\mathrm{N}}$, i.e.,

$$\Sigma_{\mathrm{N}} = \tfrac{1}{3}\Theta\sigma_{1PK} \tag{63}$$

So, for this idealized material with a fibrous microstructure, the thermodynamic force $\Sigma_{\mathrm{N}}$ can be identified as the first Piola–Kirchhoff microstress in the fibers multiplied by one third of the relative volume of fibers. Of course, the same result could have been obtained directly by writing the microplane free energy in terms of the free energy of a fiber.

### 5.3. Gaussian statistical theory of long-chain molecules

Rubber-like solids are formed by long flexible chain-like molecules joined together by chemical cross-links into a three-dimensional network. The molecules assume random configurations that are isotropically distributed in a stress-free state, but under applied deviatoric stress they become oriented. Suppose that the molecule consists of $n$ links, each of them having a length $l$. The total "length" of the molecule, defined as the distance between the ends of the chain (end-to-end distance) would be $nl$ only in the ideal, completely stretched state. The actual end-to-end distance, $r$, is randomly distributed, and its mean-square value in the unstressed state is $r_0 = l\sqrt{n}$. The ratio $r/r_0$ can be considered as a microscopic measure of strain, closely related to the microplane stretch $\lambda_N$. In the literature on rubber elasticity, the assumption that the lengths of individual chains are changed in the same proportion as the dimensions of the bulk rubber is called the *affine deformation assumption*. In the present microplane context, it is equivalent to the kinematic constraint.

According to the classical Gaussian statistical theory of long-chain molecules (summarized, e.g., by Treloar (1975) or Flory (1969)), the force transmitted by the chain is proportional to the end-to-end distance, with proportionality factor $3kT/r_0^2$ where $k$ is the Boltzmann constant and $T$ is the absolute temperature. The energy stored in the chain is then proportional to the square of the ratio $r/r_0$. This motivates a microplane model with microplane free energy

$$\Psi_\Omega[\lambda_N] = \tfrac{1}{2} E_\lambda \lambda_N^2 \tag{64}$$

where $E_\lambda$ is a material parameter that can be related to the basic microstructural properties.

The free-energy function (64) is a special case of (45). The corresponding microplane stress is

$$\Sigma_N = \frac{\partial \Psi_\Omega}{\partial \lambda_N} = E_\lambda \lambda_N \tag{65}$$

and the stress evaluation formula (49) can be written as

$$\boldsymbol{\sigma} = \frac{3}{2\pi J} \int_\Omega \Sigma_N \lambda_N \mathbf{n} \otimes \mathbf{n} \, d\Omega = \frac{3E_\lambda}{2\pi J} \int_\Omega \lambda_N^2 \mathbf{n} \otimes \mathbf{n} \, d\Omega = \frac{3}{2\pi} \int_\Omega \sigma_n \mathbf{n} \otimes \mathbf{n} \, d\omega \tag{66}$$

with Cauchy microplane stresses

$$\sigma_n = E_\lambda \frac{\lambda_N^5}{J^2} \tag{67}$$

A peculiar feature of this model is that the microplane stresses do not vanish in the initial state, i.e., at $\lambda_N = 1$. This is inherent to the molecular theory, because the forces transmitted by the chains are not the only type of interaction on the microstructural level. Rubber-like solids are usually almost incompressible, and their constant volume is kept by fluid-like interactions among the atoms (Flory, 1961). The simplest way of describing that is to impose the incompressibility constraint, $J = 1$, equivalently written as

$$1 - J = 0 \tag{68}$$

Since this constraint has a macroscopic character, it cannot be enforced on the microplane level. In the presence of a constraint on the deformation field, the stress is obtained by differentiating the sum of the free-energy potential and the left-hand side of the constraint equation multiplied by a Lagrange multiplier, $p$. Differentiating with respect to the GL strain, we obtain the sPK stress

$$\boldsymbol{\Sigma} = \frac{\partial}{\partial \mathbf{E}}(\rho_0 \Psi[\mathbf{E}] + p(1 - J[\mathbf{E}])) = \frac{\partial(\rho_0 \Psi[\mathbf{E}])}{\partial \mathbf{E}} - p\frac{\partial J}{\partial \mathbf{E}} = \boldsymbol{\Sigma}^* - pJ\mathbf{C}^{-1} \tag{69}$$

where $\mathbf{\Sigma}^* = \partial(\rho_0 \Psi[\mathbf{E}])/\partial\mathbf{E}$ is the part of the sPK stress tensor obtained in the usual way from the free-energy potential, and $-pJ\mathbf{C}^{-1}$ is a correction due to the incompressibility constraint. Note that, in the last step of (69), we have used the relation $\partial J/\partial\mathbf{E} = J\mathbf{C}^{-1}$, proven in Appendix A as Eq. (A.2).

The expression for Cauchy stress tensor valid in the incompressible case is obtained by the transformation

$$\boldsymbol{\sigma} = \frac{1}{J}\mathbf{F}\cdot\mathbf{\Sigma}\cdot\mathbf{F}^{t} = \boldsymbol{\sigma}^* - p\mathbf{I} \tag{70}$$

where $\boldsymbol{\sigma}^* = J^{-1}\mathbf{F}\cdot\mathbf{\Sigma}^*\cdot\mathbf{F}^{t}$. Eq. (70) shows that the Cauchy stress is determined up to an arbitrary multiple of the unit tensor, i.e., up to an arbitrary superimposed hydrostatic pressure. This suggests that only the deviatoric part of $\boldsymbol{\sigma}$ is uniquely determined while the volumetric part is arbitrary. Indeed, the deviatoric Cauchy stress

$$\boldsymbol{\sigma}^{\mathrm{dev}} = \mathscr{I}^{\mathrm{dev}} : \boldsymbol{\sigma} = \mathscr{I}^{\mathrm{dev}} : \boldsymbol{\sigma}^* - p\mathscr{I}^{\mathrm{dev}} : \mathbf{I} = \mathscr{I}^{\mathrm{dev}} : \boldsymbol{\sigma}^* \tag{71}$$

is independent of the Lagrange multiplier $p$. Note that the volumetric–deviatoric split has a multiplicative form only for the deformation gradient or the rCG deformation tensor, but the Cauchy stress tensor is still decomposed in the additive way, even under large strain, because the hydrostatic pressure is related to its trace and not to its determinant.

Returning to our specific example, we realize that Eq. (66) should be rewritten as

$$\boldsymbol{\sigma}^{\mathrm{dev}} = \mathscr{I}^{\mathrm{dev}} : \frac{3}{2\pi}\int_{\Omega}\sigma_{\mathrm{n}}\mathbf{n}\otimes\mathbf{n}\,\mathrm{d}\omega = \frac{3}{2\pi}\int_{\Omega}\sigma_{\mathrm{n}}\left(\mathbf{n}\otimes\mathbf{n} - \frac{1}{3}\mathbf{I}\right)\mathrm{d}\omega \tag{72}$$

The expression in the parentheses, $\mathbf{n}\otimes\mathbf{n} - \frac{1}{3}\mathbf{I}$, has a similar structure as tensor $\mathscr{D}$ defined for the small-strain theory in (9). In the small-strain evaluation formula (10), tensor $\mathscr{D}$ is multiplied by the deviatoric microplane stress, $\sigma_{\mathrm{D}}$. The present simple large-strain model could be directly transcribed in terms of $\lambda_{\mathrm{D}}$, $\Sigma_{\mathrm{D}}$ and $\sigma_{\mathrm{D}}$ because, due to the incompressibility constraint, there is no difference between $\lambda_{\mathrm{N}}$ and $\lambda_{\mathrm{D}} = \lambda_{\mathrm{N}}\lambda_{\mathrm{J}} = \lambda_{\mathrm{N}}\cdot 1$.

Another interesting point is that the present microplane model is exactly equivalent to an invariant-based tensorial model, because the integral of $\lambda_{\mathrm{N}}^2$ over the unit hemisphere can be expressed in the closed form; see Eq. (C.3) derived in Appendix C. Interestingly, the average of $\lambda_{\mathrm{N}}^2$ over the unit hemisphere is equal to one third of the invariant

$$I_1 = \mathrm{tr}\,\mathbf{C} = \mathbf{C} : \mathbf{I} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \tag{73}$$

defined as the trace of the rCG tensor or, equivalently, as the sum of squares of principal stretches $\lambda_1$, $\lambda_2$ and $\lambda_3$. The principal stretches are eigenvalues of the right stretch tensor $\mathbf{U} = \mathbf{C}^{1/2}$, and their squares are eigenvalues of the rCG tensor $\mathbf{C}$.

The macroscopic free energy

$$\rho_0\Psi = \frac{3}{2\pi}\int_{\Omega}\Psi_{\Omega}[\lambda_{\mathrm{N}}]\,\mathrm{d}\Omega = \frac{3}{2\pi}\int_{\Omega}\frac{1}{2}E_{\lambda}\lambda_{\mathrm{N}}^2\,\mathrm{d}\Omega = \frac{1}{2}E_{\lambda}I_1 = \frac{1}{2}E_{\lambda}\mathrm{tr}\,\mathbf{C} \tag{74}$$

obtained by closed-form integration is recognized as the elastic potential of the well-known neo-Hookean material model (up to a missing constant term that makes the energy in the undeformed state vanish but does not affect the stresses). Substituting into (69) and (70) and taking into account that $J = 1$, we get the sPK stress tensor

$$\mathbf{\Sigma} = \frac{\partial(\rho_0\Psi[\mathbf{E}])}{\partial\mathbf{E}} - p\frac{\partial J}{\partial\mathbf{E}} = \frac{1}{2}E_{\lambda}\frac{\partial(\mathrm{tr}\,\mathbf{C})}{\partial\mathbf{E}} - pJ\mathbf{C}^{-1} = E_{\lambda}\mathbf{I} - p\mathbf{C}^{-1} \tag{75}$$

and the Cauchy stress tensor

$$\boldsymbol{\sigma} = \frac{1}{J}\mathbf{F}\cdot\boldsymbol{\Sigma}\cdot\mathbf{F}^{t} = \mathbf{F}\cdot\left(E_{\lambda}\mathbf{I} - p\mathbf{C}^{-1}\right)\cdot\mathbf{F}^{t} = E_{\lambda}\mathbf{b} - p\mathbf{I} \tag{76}$$

Note that the deviatoric part of the Cauchy stress,

$$\boldsymbol{\sigma}^{\text{dev}} = E_{\lambda}\mathbf{b}^{\text{dev}} \tag{77}$$

is, as expected, independent of $p$. Here, $\mathbf{b}^{\text{dev}} = \mathscr{I}^{\text{dev}} : \mathbf{b}$ is the deviatoric part in the additive sense.

Eqs. (75) and (76) could alternatively be obtained from the corresponding microplane stress evaluation formulae using the closed-form integrals derived in Appendix C.

## 6. Refined formulations based on microplane stretch $\lambda_N$

### 6.1. Non-Gaussian statistical theory of long-chain molecules: Treloar model

The simple neo-Hookean model derived from the Gaussian statistical theory of long molecular chains combined with the incompressibility constraint captures the basic mechanical behavior of rubber-like solids, but it is appropriate only as long as the extension of the molecules remains well below their maximum possible extension, i.e., only at small and moderate strains. At large extensions, the relationship between the end-to-end distance and the force transmitted by the chain deviates from linearity. Improved, non-Gaussian treatment of a single chain was developed by Kuhn and Grün (1942) and James and Guth (1943). They showed that the mean tensile force on a randomly joined chain is given by

$$f = \frac{kT}{l}\mathscr{L}^{-1}\left[\frac{r}{nl}\right] \tag{78}$$

where $\mathscr{L}^{-1}$ denotes the inverse Langevin function, approximated by the infinite series

$$\mathscr{L}^{-1}[z] = 3z + \frac{9}{5}z^3 + \frac{297}{175}z^5 + \frac{1539}{875}z^7 + \cdots \tag{79}$$

If the non-linear terms are neglected, the Gaussian theory is recovered as a special case.

James and Guth (1943) also developed a model for a network of randomly joined chains, based on the concept of affine deformation (kinematic constraint) and on the simplifying assumption that the network of $N$ chains is equivalent to three sets of $N/3$ chains oriented along the principal axes. In terms of the microplane theory, this assumption means that the integral over the unit hemisphere is replaced by the sum over three mutually orthogonal directions (with a proper scaling factor). However, such a three-point numerical integration scheme is exact only for special integrands, and in a general case it can be considered only as a rough approximation. This deficiency was removed by Treloar (1954), who took full account of the angular distribution of the individual chains and performed the integration graphically for the special case of simple extension. Treloar and Riding (1979) extended this treatment to the case of biaxial strain, using a numerical quadrature method.

The Treloar model is equivalent to the microplane model with microplane free energy given by

$$\Psi_{\Omega}[\lambda_N] = \tfrac{1}{3}E_{\lambda}\Lambda[\lambda_N] \tag{80}$$

where

$$\Lambda[\lambda_N] = \int_0^{\lambda_N}\mathscr{L}^{-1}[z]\,\mathrm{d}z = \frac{3}{2}\lambda_N^2 + \frac{9}{20}\lambda_N^4 + \frac{99}{350}\lambda_N^6 + \frac{1539}{7000}\lambda_N^8 + \cdots \tag{81}$$

is the primitive function of the inverse Langevin function. The quadratic term corresponds to the Gaussian theory and the subsequent terms are higher-order corrections that cannot be integrated in a closed form over the unit hemisphere.

### 6.2. Model with vanishing initial microstresses

The Treloar model based on the Langevin function provides a correction to the simple neo-Hookean model at very large strains. However, certain discrepancies between theoretical predictions and experimental results are observed already at relatively moderate strains. To alleviate them, Thomas (1955) proposed to enrich the free energy derived from Gaussian statistical theory by a term inversely proportional to the square of the chain end-to-end distance. In terms of the microplane model, this means to replace the potential (64) by

$$\Psi_\Omega[\lambda_N] = \tfrac{1}{2}E_{\lambda 1}\lambda_N^2 + \tfrac{1}{2}E_{\lambda 2}\lambda_N^{-2} \tag{82}$$

where $E_{\lambda 1}$ and $E_{\lambda 2}$ are constant parameters. The material microplane stresses is then given by

$$\Sigma_N = \frac{\partial \Psi_\Omega}{\partial \lambda_N} = E_{\lambda 1}\lambda_N - E_{\lambda 2}\lambda_N^{-3} \tag{83}$$

and, if the material is incompressible, the Cauchy microplane stress is

$$\sigma_n = \lambda_N^4 \Sigma_N = E_{\lambda 1}\lambda_N^5 - E_{\lambda 2}\lambda_N \tag{84}$$

If the model parameters $E_{\lambda 1}$ and $E_{\lambda 2}$ are equal, the microplane stresses vanish in the undeformed state. This can be a desirable property for applications to materials in which the microprestress in the initial configuration does not have a physical justification.

If the exponent −2 in the additional energy term is replaced by −3, one can construct a model with vanishing initial microplane stresses that is integrable in a closed form and, under the assumption of incompressibility, is equivalent to the neo-Hookean material. Indeed, in Appendix C it is shown that the average value of $\lambda_N^{-3}$ over all the microplanes is equal to the inverse of the Jacobian; see Eq. (C.11). Therefore, in the incompressible case, the total contribution of the term with $\lambda_N^{-3}$ to the macroscopic free energy is constant and has no effect on the macroscopic stress tensor. So this term can be used to redistribute the microplane stresses such that they vanish in the undeformed state.

Starting from the potential

$$\Psi_\Omega[\lambda_N] = E_\lambda\left(\frac{\lambda_N^2}{2} + \frac{\lambda_N^{-3}}{3} - \frac{5}{6}\right) \tag{85}$$

we obtain microplane material stresses

$$\Sigma_N = \frac{\partial \Psi_\Omega}{\partial \lambda_N} = E_\lambda(\lambda_N - \lambda_N^{-4}) \tag{86}$$

and, still assuming $J = 1$, microplane spatial stresses

$$\sigma_n = \lambda_N^4 \Sigma_N = E_\lambda(\lambda_N^5 - 1) \tag{87}$$

Note that these microplane stresses indeed vanish in the undeformed state, as intended. Their evolution with $\lambda_N$ is represented in Fig. 2. The resulting macroscopic potential in the incompressible case,

$$\rho_0\Psi = \frac{3}{2\pi}\int_\Omega \Psi_\Omega[\lambda_N]\,d\Omega = \frac{1}{2}E_\lambda\frac{3}{2\pi}\int_\Omega \lambda_N^2\,d\Omega + \frac{1}{3}E_\lambda\frac{3}{2\pi}\int_\Omega \lambda_N^{-3}\,d\Omega - \frac{5}{6}E_\lambda\frac{3}{2\pi}\int_\Omega d\Omega = \frac{1}{2}E_\lambda(I_1 - 3) \tag{88}$$

is the standard elastic potential of the neo-Hookean material.

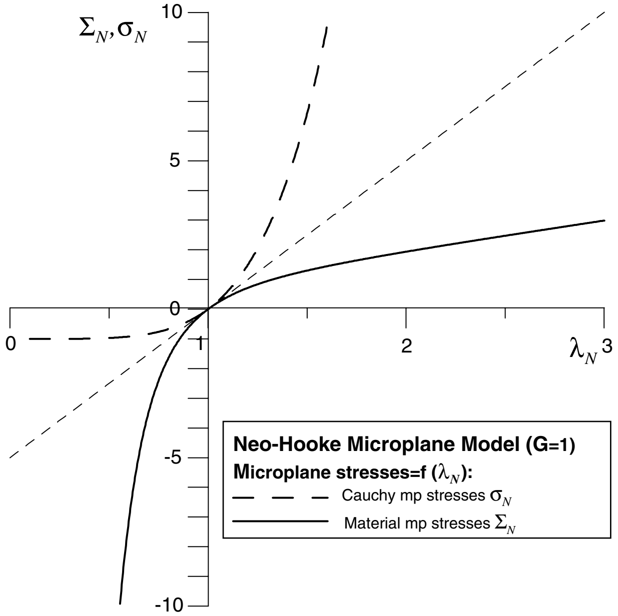


Fig. 2. Evolution of material and Cauchy microplane stresses with $\lambda_N$ (at fixed $J = 1$) for the improved model based on $\lambda_N$.

### 6.3. Small-strain limit

If the strains are sufficiently small, all model equations should collapse into the corresponding equations of the small-strain theory. The transition to the small-strain limit is straightforward for models with vanishing microplane stresses in the undeformed state. In that case, we can set $J \approx 1$, $\lambda_N \approx 1$ and $\mathbf{n} \approx \mathbf{N}$ and substitute these approximations into (49). The small-strain formula (12) is then recovered. However, special care is needed in the presence of initial microplane stresses. In this case, the leading term in the expansion of $\Sigma_N$ around $\lambda_N = 1$ is a strain-independent constant, and the corresponding integral contributes only to the volumetric part of $\boldsymbol{\sigma}$, which is undetermined. To get the deviatoric part correctly, it is not sufficient to consider the second term in the expansion of $\Sigma_N$ but also the second terms in the expansions of $J$, $\lambda_N$ and $\mathbf{n}$.

To get insight into the origin and role of individual terms, let us write the microplane stress in the form

$$\Sigma_N(\lambda_N) = \Sigma_{N0} + \Sigma_{N1}(\lambda_N) \tag{89}$$

where $\Sigma_{N0} = \Sigma_N(1)$ is the value of the microplane stress in the "unstretched state" and $\Sigma_{N1}(\lambda_N) = \Sigma_N(\lambda_N) - \Sigma_N(1)$ is the increment of the microplane stress due to stretching. Note that $\Sigma_{N0}$ is a constant and $\Sigma_{N1}$ is small if $\lambda_N$ is close to 1. The small-strain theory is based on the assumption that the components of the displacement gradient are small compared to 1. Using the notation and the approximations from Appendix D, we can write the expansion of formula (49) around the undeformed state as follows:

$$\begin{aligned}\boldsymbol{\sigma} &= \frac{3(1-3\varepsilon_V)}{2\pi}\int_\Omega (\Sigma_{N0}+\Sigma_{N1})(1+\varepsilon_N)(\mathcal{N}+\mathcal{N}\cdot\boldsymbol{\varepsilon}+\boldsymbol{\varepsilon}\cdot\mathcal{N}-\mathcal{N}\cdot\boldsymbol{\omega}+\boldsymbol{\omega}\cdot\mathcal{N}-2\varepsilon_N\mathcal{N})\,\mathrm{d}\Omega + \mathrm{O}[\epsilon^2] \\ &= \frac{3\Sigma_{N0}}{2\pi}\int_\Omega ((1-3\varepsilon_V-\varepsilon_N)\mathcal{N}+\mathcal{N}\cdot\boldsymbol{\varepsilon}+\boldsymbol{\varepsilon}\cdot\mathcal{N}-\mathcal{N}\cdot\boldsymbol{\omega}+\boldsymbol{\omega}\cdot\mathcal{N})\,\mathrm{d}\Omega + \frac{3}{2\pi}\int_\Omega \Sigma_{N1}\mathcal{N}\,\mathrm{d}\Omega + \mathrm{O}[\epsilon^2]\end{aligned} \tag{90}$$

If $\Sigma_{N0} = 0$, we get the formula

$$\boldsymbol{\sigma} = \frac{3}{2\pi}\int_{\Omega} \Sigma_N \mathbf{N} \otimes \mathbf{N}\, d\Omega + O[\epsilon^2] \tag{91}$$

which has the same structure as formula (12) used in the small-strain theory. In the general case, the right-hand side must be augmented by a term proportional to $\Sigma_{N0}$, which reflects the influence of the micro-prestress. Substituting $\varepsilon_N = \mathcal{N} : \boldsymbol{\varepsilon}$ and exploiting formulae (18) and (19), we can evaluate the factor multiplying $\Sigma_{N0}$ in (90) as

$$\begin{aligned}
&\frac{3}{2\pi}\int_{\Omega}((1 - 3\varepsilon_V - \varepsilon_N)\mathcal{N} + \mathcal{N} \cdot \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} \cdot \mathcal{N} - \mathcal{N} \cdot \boldsymbol{\omega} + \boldsymbol{\omega} \cdot \mathcal{N})\, d\Omega \\
&\quad = (1 - 3\varepsilon_V)\mathbf{I} - \frac{3}{5}\varepsilon_V \mathbf{I} - \frac{2}{5}\boldsymbol{\varepsilon} + 2\boldsymbol{\varepsilon} \\
&\quad = (1 - 2\varepsilon_V)\mathbf{I} + \frac{8}{5}\boldsymbol{\varepsilon}_D
\end{aligned} \tag{92}$$

where $\boldsymbol{\varepsilon}_D = \boldsymbol{\varepsilon} - \varepsilon_V \mathbf{I}$ is the deviatoric part of the small-strain tensor. The final form of the stress evaluation formula is then

$$\boldsymbol{\sigma} = (1 - 2\varepsilon_V)\Sigma_{N0}\mathbf{I} + \frac{8}{5}\Sigma_{N0}\boldsymbol{\varepsilon}_D + \frac{3}{2\pi}\int_{\Omega} \Sigma_{N1}\mathcal{N}\, d\Omega \tag{93}$$

Interestingly, even though the microprestress $\Sigma_{N0}$ corresponds to a hydrostatic stress state, it contributes to the deviatoric part of the macroscopic stress whenever the strain has a non-zero deviatoric part. This contribution can even be quite substantial, as illustrated by our example of a microplane version of the neo-Hookean material with $\Sigma_N$ given by (65), for which $\Sigma_{N0} = E_\lambda$ and $\Sigma_{N1} = E_\lambda \varepsilon_N + O[\epsilon^2]$. In this case, the contribution of $\Sigma_{N0}$ to the deviatoric stress is $(8/5)E_\lambda \boldsymbol{\varepsilon}_D$ while the contribution of $\Sigma_{N1}$ is only $(2/5)E_\lambda \boldsymbol{\varepsilon}_D$. On the other hand, for the microprestress-free version of the neo-Hookean material, with $\Sigma_N$ given by (83), we have $\Sigma_{N0} = 0$ and $\Sigma_{N1} = 5E_\lambda \varepsilon_N + O[\epsilon^2]$. In both cases, the small-strain macroscopic elastic law reads $\boldsymbol{\sigma} = 2E_\lambda \boldsymbol{\varepsilon}_D$ (plus an undetermined hydrostatic term), and the model parameter $E_\lambda$ therefore has the physical meaning of the macroscopic shear modulus, $G$.

### *6.4. Compressible extension*

So far, we have considered models based on the microplane stretch as being subjected to the incompressibility constraint. For models with non-zero microstresses in the initial state, this constraint is essential because it generates a hydrostatic pressure that ensures that the macroscopic stress can vanish. However, the model with vanishing initial microstresses presented in Section 6.2 may also be used without imposing incompressibility, as described here.

Consider again the microplane energy function $\Psi_\Omega[\lambda_N]$ defined by Eq. (85) of Section 6.2. Material microplane stresses $\Sigma_N$, obtained by derivative with respect to $\lambda_N$, are not affected by the incompressibility assumption, and therefore are still given by formula (86). The Cauchy microplane stresses $\sigma_n$ are, however, different because they depend on $J$ according to (53). So, in the compressible case, (87) must be replaced by the more general formula

$$\sigma_N = \frac{\lambda_N^4 \Sigma_N}{J^2} = E_\lambda \frac{1}{J^2}(\lambda_N^5 - 1) \tag{94}$$

In the small-strain limit, the model collapses into the small-strain microplane model with $\sigma_N = E_N \varepsilon_N$ as the only microplane stress. From the previous equations and plots of $\Sigma_N$ and $\sigma_N$, it follows that in the small-strain regime

$$\varepsilon_{\rm N} \approx \lambda_{\rm N} - 1, \quad \sigma_{\rm N} \approx \Sigma_{\rm N}, \quad E_{\rm N} = \left.\frac{\partial \Sigma_{\rm N}}{\partial \lambda_{\rm N}}\right|_{\lambda_{\rm N}=1} = \left.\frac{\partial \sigma_{\rm N}}{\partial \lambda_{\rm N}}\right|_{\lambda_{\rm N}=1, J=1} = 5E_{\lambda} \tag{95}$$

This value of $E_{\rm N}$ can be substituted into (24) with all the remaining microplane moduli equal to zero, to get the macroscopic moduli

$$G = \frac{E_{\rm N}}{5} = E_{\lambda}, \quad K = \frac{E_{\rm N}}{3} = \frac{5}{3}G \tag{96}$$

Same as before, the coefficient $E_{\lambda}$ has the meaning of the (initial) shear modulus, $G$. In the compressible case we also obtain an initial bulk modulus $K$, but it is not independent of $G$, and the resulting Poisson ratio is fixed to $\nu = 0.25$, same as for the small-strain model with $\sigma_{\rm N}$ only.

By inserting microplane potential (85) into integral formula (1) and applying the expressions derived in Appendix C for the integrals over the hemisphere of $\lambda_{\rm N}^2$ and $\lambda_{\rm N}^{-3}$ without the incompressibility constraint, one finally obtains the macroscopic free energy

$$\rho_0 \Psi = \frac{3}{2\pi}\int_{\Omega} \Psi_{\Omega}\, {\rm d}\Omega = \frac{1}{2}G\left({\rm tr}\,\mathbf{C} + \frac{2}{J} - 5\right) \tag{97}$$

which also vanishes in the undeformed state (for $\mathbf{C} = \mathbf{I}$ and $J = 1$) and corresponds to a basic form of compressible neo-Hookean material (Ogden, 1984).

Macroscopic sPK stress tensor can be obtained either from the microplane material stresses (86) integrated over the hemisphere in the initial configuration using (47), or by partial differentiation of the macroscopic potential (97) with respect to the GL strain $\mathbf{E}$. Either way, the resulting expression is

$$\mathbf{\Sigma} = G\left(\mathbf{I} - \frac{1}{J}\mathbf{C}^{-1}\right) \tag{98}$$

The Cauchy stress tensor may be evaluated either by push-forward of the sPK stress and scaling by $1/J$, or from the Cauchy microplane stresses (94) integrated over the hemisphere in the deformed configuration using (52). Either way, the final expression is

$$\boldsymbol{\sigma} = G\frac{1}{J}\left(\mathbf{b} - \frac{1}{J}\mathbf{I}\right) \tag{99}$$

Note that, if the constraint $J \equiv 1$ is assumed, this compressible model coincides with the incompressible one developed in previous Section 6.2. On the other hand, the deviatoric part of the Cauchy stresses is $\boldsymbol{\sigma}^{\rm dev} = \mathbf{b}^{\rm dev}/J$, and therefore it only differs from its incompressible counterpart (77) by a factor $1/J$. The hydrostatic behavior of the model is obtained by specifying $\mathbf{b} = J^{2/3}\mathbf{I}$ in previous equation, which leads to the Cauchy volumetric stress

$$\sigma_{\rm v} = \frac{{\rm tr}\,\boldsymbol{\sigma}}{3} = G(J^{-1/3} - J^{-2}) \tag{100}$$

This is a function with positive slope $3K = 5G$ at $J = 1$, which exhibits a plunge to minus infinity for $J \to 0$, and a decreasing slope in tension with a peak about $J = 3$ and a slightly decreasing slope beyond that. This kind of tensile behavior in terms of Cauchy stresses is also observed in other models in further sections and it will be discussed there. In any case, it is clear that once the shear constant $G$ is determined, the volumetric behavior of the model is fixed, in agreement with the initial fixed Poisson ratio of 0.25. To overcome this limitation and to be able to adjust the volumetric behavior independently of the behavior in shear, it is necessary to separate the free energy functions in two parts, one of them depending only on the volumetric deformation measure $\lambda_{\rm J}$ (or of its third power, $J = \lambda_{\rm J}^3$), and the other one on the distortional stretch $\lambda_{\rm D}$, as will be done in the following section.

## 7. Microplane formulation based on $\lambda_J$, $\lambda_D$: compressible neo-Hookean material with unrestricted $\nu$

In materials such as rubber, the behavior is controled not only by long polymer chains, but also by the bulk matrix in which these chains are embedded. As a first approximation, the matrix can be considered as a fluid providing some volumetric resistance to the overall deformation. This would motivate the development of microplane formulations based on $J$ (or $\lambda_J = J^{1/3}$) and $\lambda_N$. However, it is more convenient to exploit the relation $\lambda_N = J^{1/3}\lambda_D$ and consider $\lambda_D$ and $\lambda_J$ as arguments of a potential. This leads to simpler expressions with uncoupled additive terms.

### 7.1. General stress evaluation formulae

Consider the microplane free energy in the form

$$\Psi_\Omega = \Psi_\Omega[\lambda_J, \lambda_D, \xi] \tag{101}$$

where $\xi$ is a set of internal variables, present only if the material exhibits dissipative behavior. Using (30), the macroscopic stress evaluation formula for the sPK stress may be expressed as

$$\mathbf{\Sigma} = \frac{3}{2\pi}\int_\Omega \left(\frac{\partial \Psi_\Omega}{\partial \lambda_J}\frac{\partial \lambda_J}{\partial \mathbf{E}} + \frac{\partial \Psi_\Omega}{\partial \lambda_D}\frac{\partial \lambda_D}{\partial \mathbf{E}}\right) \mathrm{d}\Omega \tag{102}$$

After appropriate substitution of derivatives developed in Appendix A, this can be rewritten as

$$\mathbf{\Sigma} = \frac{1}{2\pi}\int_\Omega \Sigma_J \lambda_J \mathbf{C}^{-1}\,\mathrm{d}\Omega + \frac{3}{2\pi}\int_\Omega \Sigma_D\left(\lambda_J^{-2}\lambda_D^{-1}\mathbf{N}\otimes\mathbf{N} - \frac{\lambda_D}{3}\mathbf{C}^{-1}\right)\mathrm{d}\Omega \tag{103}$$

where

$$\Sigma_J = \frac{\partial \Psi_\Omega}{\partial \lambda_J}, \quad \Sigma_D = \frac{\partial \Psi_\Omega}{\partial \lambda_D} \tag{104}$$

are the material microplane stresses.

Suppose that the volumetric and distortional effects are decoupled in the sense that the mixed derivative of the microplane free-energy potential with respect to $\lambda_J$ and $\lambda_D$ vanishes. In this case, the microstress $\Sigma_J$ does not depend on $\lambda_D$ (and therefore on the microplane orientation), and it can be taken out of the first integral, which gives

$$\mathbf{\Sigma} = \Sigma_J \lambda_J \mathbf{C}^{-1} + \frac{3}{2\pi}\int_\Omega \Sigma_D\left(\lambda_J^{-2}\lambda_D^{-1}\mathbf{N}\otimes\mathbf{N} - \frac{\lambda_D}{3}\mathbf{C}^{-1}\right)\mathrm{d}\Omega \tag{105}$$

Contravariant push-forward of (105) and scaling by $1/J$ leads to the Cauchy stress tensor

$$\boldsymbol{\sigma} = \Sigma_J \lambda_J J^{-1}\mathbf{F}\cdot\mathbf{C}^{-1}\cdot\mathbf{F}^{t} + \frac{3}{2\pi J}\int_\Omega \Sigma_D\left(\lambda_J^{-2}\lambda_D^{-1}\mathbf{F}\cdot\mathbf{N}\otimes\mathbf{N}\cdot\mathbf{F}^{t} - \frac{\lambda_D}{3}\mathbf{F}\cdot\mathbf{C}^{-1}\cdot\mathbf{F}^{t}\right)\mathrm{d}\Omega \tag{106}$$

which, after substitutions and rearrangements, may be expressed as

$$\boldsymbol{\sigma} = \Sigma_J \lambda_J^{-2}\mathbf{I} + \frac{3}{2\pi}\int_\Omega \frac{\Sigma_D \lambda_D}{J}\left(\mathbf{n}\otimes\mathbf{n} - \frac{\mathbf{I}}{3}\right)\mathrm{d}\Omega \tag{107}$$

Finally, substitution of expression (51) for the solid angle differential in the initial configuration in terms of the solid angle differential in the deformed configuration leads to the fully Eulerian version of the stress evaluation formula,

$$\boldsymbol{\sigma} = \sigma_{\rm v}\mathbf{I} + \frac{3}{2\pi}\int_{\Omega}\sigma_{\rm d}\left(\mathbf{n}\otimes\mathbf{n} - \frac{\mathbf{I}}{3}\right)\mathrm{d}\omega \tag{108}$$

with Cauchy microplane stresses

$$\sigma_{\rm v} = \lambda_{\rm J}^{-2}\Sigma_{\rm J}, \quad \sigma_{\rm d} = \frac{\lambda_{\rm D}^4}{J}\Sigma_{\rm D} \tag{109}$$

As it could be expected, formula (108) looks exactly the same as its small-strain counterpart; see (10) with $\sigma_{\rm N} = 0$, $\boldsymbol{\sigma}_{\rm T} = \mathbf{0}$ and $\sigma_{\rm V}$ = same for all microplanes.

### 7.2. Example: compressible neo-Hookean material

We start from the microplane free energy (85) with $\lambda_{\rm N}$ replaced by $\lambda_{\rm D}$ and the constant parameter $E_\lambda$ denoted as $G$, and we add a volumetric function $g[J]/3$ satisfying the conditions

$$g[1] = 0, \quad g'[1] = \left.\frac{\mathrm{d}g}{\mathrm{d}J}\right|_{J=1} = 0 \tag{110}$$

Note that, with these assumptions, the free energy

$$\Psi_{\Omega}[\lambda_{\rm J}, \lambda_{\rm D}] = G(\tfrac{1}{2}\lambda_{\rm D}^2 + \tfrac{1}{3}\lambda_{\rm D}^{-3} - \tfrac{5}{6}) + \tfrac{1}{3}g[J] \tag{111}$$

vanishes in the undeformed state (for $\lambda_{\rm D} = 1$ and $J = 1$).

Using (104), the material microplane stresses may be obtained as

$$\Sigma_{\rm J} = \lambda_{\rm J}^2 g'[J], \quad \Sigma_{\rm D} = G(\lambda_{\rm D} - \lambda_{\rm D}^{-4}) \tag{112}$$

and, using (109), the spatial or Cauchy microplane stresses follow as

$$\sigma_{\rm v} = g'[J], \quad \sigma_{\rm d} = \frac{G}{J}(\lambda_{\rm D}^5 - 1) \tag{113}$$

All these microplane stresses vanish in the undeformed state. The evolution of $\Sigma_{\rm D}$ and $\sigma_{\rm d}$ with $\lambda_{\rm D}$ at constant $\lambda_{\rm J} = J = 1$ is similar to that of $\Sigma_{\rm N}$ and $\sigma_{\rm n}$ with $\lambda_{\rm N}$ in Fig. 2.

### 7.3. Equivalent macroscopic model of the compressible neo-Hookean type

By inserting microplane potential (111) into integral formula (1) and applying expressions (C.3) and (C.12) derived in Appendix C for the integrals over the initial hemisphere of $\lambda_{\rm N}^2$ (in which $\lambda_{\rm N} = J^{1/3}\lambda_{\rm D}$ may be substituted and $J^{1/3}$ extracted out of integral) and $\lambda_{\rm D}^{-3}$, one obtains a tensorial expression for the macroscopic free energy in the form

$$\rho_0\Psi = \frac{3}{2\pi}\int_{\Omega}\Psi_{\Omega}\,\mathrm{d}\Omega = \frac{1}{2}G(\mathrm{tr}\,\mathbf{C}_{\rm D} - 3) + g[J] \tag{114}$$

which corresponds to a specific type of compressible neo-Hookean material with an uncoupled energy potential consisting of a sum of the distortional strain energy and volumetric strain energy function of the general form $g[J]$ (Ogden, 1984). Note that this potential vanishes in the undeformed state (for $\mathbf{C}_{\rm D} = \mathbf{I}$ and $J = 1$).

The macroscopic sPK stress tensor may now be obtained either from the microplane material stresses (112) integrated over the hemisphere in the initial configuration using (105), or by partial differentiation of the macroscopic potential (114) with respect to the GL strain $\mathbf{E}$. Either way, the resulting expression is

$$\boldsymbol{\Sigma} = J^{1/3}g'[J]\mathbf{C}_{\mathrm{D}}^{-1} + GJ^{-2/3}\left(\mathbf{I} - \frac{\operatorname{tr}\mathbf{C}_{\mathrm{D}}}{3}\mathbf{C}_{\mathrm{D}}^{-1}\right) \tag{115}$$

Macroscopic Cauchy stresses may also be evaluated either by push-forward of the sPK stress, or from the Cauchy microplane stresses (113) integrated over the hemisphere in the deformed configuration using (108). Either way, the final expression obtained is

$$\boldsymbol{\sigma} = g'[J]\mathbf{I} + GJ^{-5/3}\left(\mathbf{b} - \frac{\operatorname{tr}\mathbf{b}}{3}\mathbf{I}\right) \tag{116}$$

or, by introducing the distortional part of the left Cauchy–Green tensor $\mathbf{b}_{\mathrm{D}} = J^{-2/3}\mathbf{b}$,

$$\boldsymbol{\sigma} = g'[J]\mathbf{I} + \frac{G}{J}\left(\mathbf{b}_{\mathrm{D}} - \frac{\operatorname{tr}\mathbf{b}_{\mathrm{D}}}{3}\mathbf{I}\right) \tag{117}$$

Note that the right-hand side is additively decomposed into volumetric and deviatoric parts and, therefore, the volumetric and deviatoric Cauchy stresses can be expressed as

$$\sigma_{\mathrm{V}} = \frac{\operatorname{tr}\boldsymbol{\sigma}}{3} = g'[J], \quad \boldsymbol{\sigma}^{\mathrm{dev}} = \frac{G}{J}(\mathbf{b}_{\mathrm{D}})^{\mathrm{dev}} \tag{118}$$

It is a convenient feature that a clean additive decomposition is obtained in the deformed configuration, in spite of the fact that the fundamental volumetric–deviatoric decomposition in material configuration has been established in a product form.

### 7.4. Small-strain limit

Since all the microplane stresses vanish in the initial configuration, the small-strain limit of the present model is easy to construct. In the stress evaluation formula (107), $J$, $\lambda_{\mathrm{J}}$ and $\lambda_{\mathrm{D}}$ can be replaced by 1 and $\mathbf{n}$ by $\mathbf{N}$. Alternatively, one could simply replace $\mathrm{d}\omega$ by $\mathrm{d}\Omega$ in (108). In any case, the model collapses into the small-strain microplane model with microplane stresses $\sigma_{\mathrm{V}}$ and $\sigma_{\mathrm{D}}$, which are equal, up to terms of a higher order, to $\Sigma_{\mathrm{J}}$ and $\Sigma_{\mathrm{D}}$ (and also to $\sigma_{\mathrm{v}}$ and $\sigma_{\mathrm{d}}$, which differ from $\Sigma_{\mathrm{J}}$ and $\Sigma_{\mathrm{D}}$ by terms of the order $\mathrm{O}[\epsilon^2]$).

The initial microplane moduli can be evaluated as

$$E_{\mathrm{D}} = \left.\frac{\partial\Sigma_{\mathrm{D}}}{\partial\lambda_{\mathrm{D}}}\right|_{\lambda=1} = \left.\frac{\partial\sigma_{\mathrm{d}}}{\partial\lambda_{\mathrm{D}}}\right|_{\lambda=1} = 5G \tag{119}$$

$$E_{\mathrm{V}} = \left.\frac{\partial\Sigma_{\mathrm{J}}}{\partial\lambda_{\mathrm{J}}}\right|_{\lambda=1} = \left.\frac{\partial\sigma_{\mathrm{v}}}{\partial\lambda_{\mathrm{J}}}\right|_{\lambda=1} = 3g''[1] \tag{120}$$

where the "subscript" $\lambda = 1$ after a vertical line means in simplified notation that the partial derivatives are evaluated in the undeformed state, i.e., at $\lambda_{\mathrm{J}} = 1$ and $\lambda_{\mathrm{D}} = 1$. Substituting these values of $E_{\mathrm{V}}$ and $E_{\mathrm{D}}$ into (24), with all the remaining microplane moduli set to zero, we can verify that the physical meaning of the material parameter $G$ is indeed the shear modulus of elasticity, and we find out that the second derivative of the potential function $g$ at $J = 1$ has the meaning of the macroscopic bulk modulus $K$. For the present model, the Poisson ratio can take any value between $-1$ (when $K \ll G$) and 0.5 (when $K \gg G$).

## 8. Formulation based on $\lambda_{\mathrm{N}}$ and $\bar{\lambda}_{\mathrm{N}}$: compressible Mooney-Rivlin material with restricted $\nu$

Previous microplane formulations based on $\lambda_{\mathrm{D}}$ and $\lambda_{\mathrm{J}}$ (or $\lambda_{\mathrm{N}}$ and $\lambda_{\mathrm{J}}$) were very useful to establish the basic aspects of finite strain microplane models, but only achieved to reproduce macroscopic hyperelastic potentials of the neo-Hookean type, i.e. involving $J$ and the first invariant of $\mathbf{C}$. However, it is well established that for materials undergoing large elastic deformations such as rubber, this is too simplistic, and realistic models must at least involve the second invariant of $\mathbf{C}$ or, equivalently, the first invariant of $\mathbf{C}^{-1}$ (Ogden, 1982, 1984). In this section, a general formulation involving $\lambda_N$, $\bar{\lambda}_N$ and $\tan\gamma_N$ is presented which, for particular choices of potential, will lead to macroscopic models of the compressible Mooney-Rivlin type.

As explained in Section 4.2, variables $\lambda_N$ and $\bar{\lambda}_N$ are simply related by $\bar{\lambda}_N = \lambda_N \cos\gamma_N$ where $\gamma_N$ is the shear distortion angle (35). This means that the new variable $\bar{\lambda}_N$ could as well be replaced by the distortion angle, and end up with a function of $\lambda_N$ and $\tan\gamma_N$ only. In the same way, one could replace $\tan\gamma_N$ in terms of $\lambda_N$ and $\bar{\lambda}_N$, and end up with a function of these two variables only. In practice, the various forms are fully equivalent and each may be more convenient for certain purposes, although some of them might show singularities at the undeformed state, as it will be shown in the following subsections. For these reasons, and for the sake of generality, in the general derivations of Section 8.1 all three variables $\lambda_N$, $\bar{\lambda}_N$ and $\gamma_N$ will be considered. Although redundant, this is not incorrect (one of the arguments can always be expressed in terms of the other two), and allows us to see what form is obtained for each of the terms in the equations that follow. The choice between them can be done later on the basis of convenience.

### 8.1. General stress evaluation formulae

We consider a microplane free energy of the general form

$$\Psi_\Omega = \Psi_\Omega[\lambda_N, \bar{\lambda}_N, \tan\gamma_N, \boldsymbol{\xi}] \tag{121}$$

where $\boldsymbol{\xi}$ is an appropriate set of internal variables, needed only if the material exhibits dissipative behavior.

Using (30), the macroscopic stress evaluation formula for the sPK stress may be expressed as

$$\boldsymbol{\Sigma} = \frac{3}{2\pi}\int_\Omega \left(\Sigma_N \frac{\partial\lambda_N}{\partial\mathbf{E}} + \bar{\Sigma}_N \frac{\partial\bar{\lambda}_N}{\partial\mathbf{E}} + \Sigma_T \frac{\partial(\tan\gamma_N)}{\partial\mathbf{E}}\right) d\Omega \tag{122}$$

where

$$\Sigma_N = \frac{\partial\Psi_\Omega}{\partial\lambda_N}, \quad \bar{\Sigma}_N = \frac{\partial\Psi_\Omega}{\partial\bar{\lambda}_N}, \quad \Sigma_T = \frac{\partial\Psi_\Omega}{\partial \tan\gamma_N} \tag{123}$$

are the material microplane stresses. After appropriate substitution of derivatives (A.5), (A.12) and (A.15) developed in Appendix A, the formula for the sPK stress tensor can be rewritten as

$$\boldsymbol{\Sigma} = \frac{3}{2\pi}\int_\Omega \left(\frac{\Sigma_N}{\lambda_N}\mathbf{N}\otimes\mathbf{N} + \bar{\Sigma}_N\bar{\lambda}_N^3\mathbf{C}^{-1}\cdot\mathbf{N}\otimes\mathbf{N}\cdot\mathbf{C}^{-1} + \frac{\Sigma_T}{\tan\gamma_N}\left(\frac{1}{\bar{\lambda}_N^2}\mathbf{N}\otimes\mathbf{N} - \lambda_N^2\mathbf{C}^{-1}\cdot\mathbf{N}\otimes\mathbf{N}\cdot\mathbf{C}^{-1}\right)\right) d\Omega \tag{124}$$

Contravariant push-forward of (124) and scaling by $1/J$ leads to the Cauchy stress tensor

$$\boldsymbol{\sigma} = \frac{3}{2\pi J}\int_\Omega \left(\frac{\Sigma_N}{\lambda_N}\mathbf{F}\cdot\mathbf{N}\otimes\mathbf{N}\cdot\mathbf{F}^t + \bar{\Sigma}_N\bar{\lambda}_N^3\mathbf{F}^{-t}\cdot\mathbf{N}\otimes\mathbf{N}\cdot\mathbf{F}^{-1} + \frac{\Sigma_T}{\tan\gamma_N}\left(\frac{1}{\bar{\lambda}_N^2}\mathbf{F}\cdot\mathbf{N}\otimes\mathbf{N}\cdot\mathbf{F}^t - \lambda_N^2\mathbf{F}^{-t}\cdot\mathbf{N}\otimes\mathbf{N}\cdot\mathbf{F}^{-1}\right)\right) d\Omega \tag{125}$$

where use has been made of the relation $\mathbf{F}\cdot\mathbf{C}^{-1} = \mathbf{F}^{-t}$. We can further substitute $\mathbf{F}\cdot\mathbf{N} = \lambda_N\mathbf{n}$, with $\mathbf{n}$ being the unit vector in the deformed configuration along the fiber initially aligned with $\mathbf{N}$. In analogy to that, we can write $\mathbf{F}^{-t}\cdot\mathbf{N} = \bar{\lambda}_N^{-1}\bar{\mathbf{n}}$, where $\bar{\mathbf{n}}$ is a unit vector normal to the deformed microplane. This substitution leads to

$$\boldsymbol{\sigma} = \frac{3}{2\pi J}\int_{\Omega}\left(\lambda_{\mathrm{N}}\Sigma_{\mathrm{N}}\mathbf{n}\otimes\mathbf{n} + \bar{\lambda}_{\mathrm{N}}\bar{\Sigma}_{\mathrm{N}}\bar{\mathbf{n}}\otimes\bar{\mathbf{n}} + \frac{\Sigma_{\mathrm{T}}}{\cos\gamma_{\mathrm{N}}\sin\gamma_{\mathrm{N}}}(\mathbf{n}\otimes\mathbf{n} - \bar{\mathbf{n}}\otimes\bar{\mathbf{n}})\right)\mathrm{d}\Omega \tag{126}$$

A final manipulation is necessary to provide the tensor structure of the type $\mathbf{n}\otimes\mathbf{t}+\mathbf{t}\otimes\mathbf{n}$ present in the tangential term of the small-strain formula (13), which is expected to be recovered in the Eulerian form of the integral equation. A term of such type may be obtained from the third term in the integrand, by introducing the unit vectors

$$\bar{\bar{\mathbf{n}}} = \frac{\mathbf{n}+\bar{\mathbf{n}}}{|\mathbf{n}+\bar{\mathbf{n}}|} = \frac{\mathbf{n}+\bar{\mathbf{n}}}{2\cos[\gamma_{\mathrm{N}}/2]}, \quad \bar{\bar{\mathbf{t}}} = \frac{\mathbf{n}-\bar{\mathbf{n}}}{|\mathbf{n}-\bar{\mathbf{n}}|} = \frac{\mathbf{n}-\bar{\mathbf{n}}}{2\sin[\gamma_{\mathrm{N}}/2]} \tag{127}$$

Vector $\bar{\bar{\mathbf{n}}}$ bisects the angle between $\mathbf{n}$ and $\bar{\mathbf{n}}$, and vector $\bar{\bar{\mathbf{t}}}$ is perpendicular to $\bar{\bar{\mathbf{n}}}$ and at the same time indicates the direction of shearing; see Fig. 3. A possible physical interpretation of the three normals $\mathbf{n}$, $\bar{\mathbf{n}}$ and $\bar{\bar{\mathbf{n}}}$ is based on three types of microstructures or micromechanisms within the heterogeneous material: "fibers", "platelets" and "shear boxes". If these elements are initially aligned with the same normal $\mathbf{N}$, their orientation in the deformed configuration is described by three vectors that are in general different: the new direction of the fiber is given by $\mathbf{n}$, the new normal of the platelet by $\bar{\mathbf{n}}$, and the new alignment of the shear box by $\bar{\bar{\mathbf{n}}}$, which bisects $\mathbf{n}$ and $\bar{\mathbf{n}}$.

If $\mathbf{n}$ is replaced by $\bar{\bar{\mathbf{n}}}\cos[\gamma_{\mathrm{N}}/2] + \bar{\bar{\mathbf{t}}}\sin[\gamma_{\mathrm{N}}/2]$ and $\bar{\mathbf{n}}$ is replaced by $\bar{\bar{\mathbf{n}}}\cos[\gamma_{\mathrm{N}}/2] - \bar{\bar{\mathbf{t}}}\sin[\gamma_{\mathrm{N}}/2]$, the difference between the direct products $\mathbf{n}\otimes\mathbf{n}$ and $\bar{\mathbf{n}}\otimes\bar{\mathbf{n}}$ that appears in (126) can be transformed into

$$\mathbf{n}\otimes\mathbf{n} - \bar{\mathbf{n}}\otimes\bar{\mathbf{n}} = \sin\gamma_{\mathrm{N}}(\bar{\bar{\mathbf{n}}}\otimes\bar{\bar{\mathbf{t}}} + \bar{\bar{\mathbf{t}}}\otimes\bar{\bar{\mathbf{n}}}) \tag{128}$$

Substituting this relation into (126), one obtains an alternative useful form of the Lagrangian integral expression for the Cauchy stresses,

$$\boldsymbol{\sigma} = \frac{3}{2\pi J}\int_{\Omega}\left(\lambda_{\mathrm{N}}\Sigma_{\mathrm{N}}\mathbf{n}\otimes\mathbf{n} + \bar{\lambda}_{\mathrm{N}}\bar{\Sigma}_{\mathrm{N}}\bar{\mathbf{n}}\otimes\bar{\mathbf{n}} + \frac{\Sigma_{\mathrm{T}}}{\cos\gamma_{\mathrm{N}}}(\bar{\bar{\mathbf{n}}}\otimes\bar{\bar{\mathbf{t}}} + \bar{\bar{\mathbf{t}}}\otimes\bar{\bar{\mathbf{n}}})\right)\mathrm{d}\Omega \tag{129}$$

in which the desired "shear-like" structure has appeared in the tangential term, and the factor $\sin\gamma_{\mathrm{N}}$ has disappeared from the denominator. Note that, in the limit of small strain, $\mathbf{n}$ and $\bar{\mathbf{n}}$ tend to coincide and $\sin\gamma_{\mathrm{N}}$ vanishes, reasons for which the previous integral expression (126) could lead to indetermination in that limit case while the new expression (129) remains well-defined.

At this point of similar derivations in previous sections, the solid angle $\mathrm{d}\Omega$ corresponding to original microplane direction $\mathbf{N}$ was expressed in terms of the current solid angle $\mathrm{d}\omega$ corresponding to current microplane direction $\mathbf{n}$. In the present case, however, for the same original orientation $\mathbf{N}$ there are three different deformed microplane directions: $\mathbf{n}$, $\bar{\mathbf{n}}$ and $\bar{\bar{\mathbf{n}}}$. This adds some complexity to the derivation of a fully Eulerian stress-evaluation formula. In order to achieve this result, the single integral of Eq. (129) may be split into three integrals in initial configuration, and for each of them $\mathrm{d}\Omega$ is substituted by a different deformed solid angle: $\mathrm{d}\omega$ defined in (51), $\mathrm{d}\bar{\omega} = (\bar{\lambda}_{\mathrm{N}}^3/J)\,\mathrm{d}\Omega$ which corresponds to the transformation of plane normals around $\bar{\mathbf{n}}$, and finally $\mathrm{d}\bar{\bar{\omega}} = (\bar{\bar{J}}_{\mathrm{N}}/8\cos^3[\gamma_{\mathrm{N}/2}])\,\mathrm{d}\Omega$ where $\bar{\bar{J}}_{\mathrm{N}} = \det[\mathbf{U}/\lambda_{\mathrm{N}} + \mathbf{U}^{-1}\bar{\lambda}_{\mathrm{N}}]$. These relations

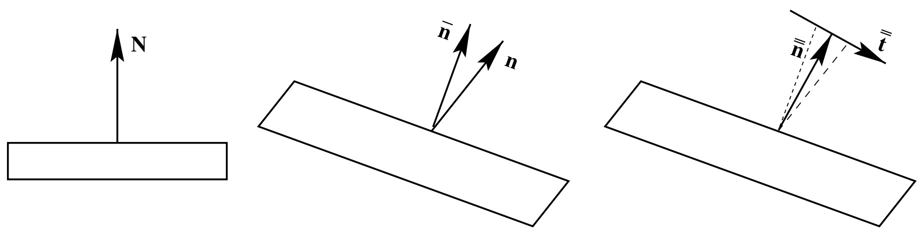


Fig. 3. Definition of vectors $\bar{\bar{\mathbf{n}}}$ and $\bar{\bar{\mathbf{t}}}$.

are derived in Appendix B. By splitting the integral and introducing all previous equations into (129), one obtains

$$\boldsymbol{\sigma} = \frac{3}{2\pi}\int_{\Omega}\frac{\lambda_{\rm N}^4\Sigma_{\rm N}}{J^2}\mathbf{n}\otimes\mathbf{n}\,{\rm d}\omega + \frac{3}{2\pi}\int_{\Omega}\frac{\bar{\Sigma}_{\rm N}}{\bar{\lambda}_{\rm N}^2}\bar{\mathbf{n}}\otimes\bar{\mathbf{n}}\,{\rm d}\bar{\omega} + \frac{3}{2\pi}\int_{\Omega}\frac{\Sigma_{\rm T}8\cos^3[\gamma_{\rm N}/2]}{J\bar{\bar{J}}_{\rm N}\cos\gamma_{\rm N}}(\bar{\bar{\mathbf{n}}}\otimes\bar{\bar{\mathbf{t}}} + \bar{\bar{\mathbf{t}}}\otimes\bar{\bar{\mathbf{n}}})\,{\rm d}\bar{\bar{\omega}} \tag{130}$$

Recovery of the small-strain format still requires one more step. Lagrangian integral formulae such as (129) have the meaning of summing all the terms involving different directions $\mathbf{n}$, $\bar{\mathbf{n}}$ and $\bar{\bar{\mathbf{n}}}$, which were contributed by each microplane of initial direction $\mathbf{N}$. Eulerian integral formulae may be interpreted in the opposite (dual) way: as collecting the contributions of microplanes with different initial orientations $\mathbf{N}$, $\bar{\mathbf{N}}$ and $\bar{\bar{\mathbf{N}}}$, which produce terms involving the same deformed direction $\mathbf{n}$.

In Eq. (130), each integral can be interpreted as a sum, and the aforementioned collection corresponds to regrouping of terms from the three integrals with the same transformed direction $\mathbf{n}$. This leads to

$$\boldsymbol{\sigma} = \frac{3}{2\pi}\int_{\Omega}\sigma_{\rm n}\mathbf{n}\otimes\mathbf{n}\,{\rm d}\omega + \frac{3}{2\pi}\int_{\Omega}\frac{\sigma_{\rm t}}{2}(\mathbf{n}\otimes\mathbf{t} + \mathbf{t}\otimes\mathbf{n})\,{\rm d}\omega \tag{131}$$

where

$$\sigma_{\rm n} = \frac{\lambda_{\rm N}^4\Sigma_{\rm N}}{J^2} + \frac{\bar{\Sigma}_{\bar{\rm N}}}{\bar{\lambda}_{\bar{\rm N}}^2}, \quad \sigma_{\rm t} = \frac{16\Sigma_{\bar{\bar{\rm T}}}\cos^3[\gamma_{\bar{\bar{\rm N}}}/2]}{J\bar{\bar{J}}_{\rm N}\cos\gamma_{\bar{\bar{\rm N}}}} \tag{132}$$

are the Eulerian variables introduced at end of Section 5.1. In (132), $\bar{\lambda}_{\bar{\rm N}}$ is the thickening of the microplane which, transformed as a plane normal has become $\mathbf{n}$ with corresponding material stresses $\bar{\Sigma}_{\bar{\rm N}}$; and $\gamma_{\bar{\bar{\rm N}}}$ is the shear distortion of the microplane which, transformed as a "shear box" has become $\mathbf{n}$ with material shear stresses $\Sigma_{\bar{\bar{\rm T}}}$.

### *8.2. Hyperelastic microplane model based on $\lambda_N$ and $\bar{\lambda}_N$ and equivalent compressible Mooney-Rivlin model*

Consider the microplane energy function

$$\Psi_{\Omega}[\lambda_{\rm N}, \bar{\lambda}_{\rm N}] = A\left(\frac{\lambda_{\rm N}^2}{2} + \frac{\lambda_{\rm N}^{-3}}{3} - \frac{5}{6}\right) + B\left(\frac{\bar{\lambda}_{\rm N}^{-2}}{2} + \frac{\bar{\lambda}_{\rm N}^3}{3} - \frac{5}{6}\right) \tag{133}$$

the exponents of which are motivated by the availability of closed-form solutions for the integrals of such terms over the hemisphere as developed in Appendix C, and the coefficients are adjusted so that the potential vanishes in the undeformed state.

The material microplane stresses are obtained by taking partial derivatives of the potential,

$$\Sigma_{\rm N} = \frac{\partial\Psi_{\Omega}}{\partial\lambda_{\rm N}} = A\left(\lambda_{\rm N} - \lambda_{\rm N}^{-4}\right), \quad \bar{\Sigma}_{\rm N} = \frac{\partial\Psi_{\Omega}}{\partial\bar{\lambda}_{\rm N}} = B\left(\bar{\lambda}_{\rm N}^2 - \bar{\lambda}_{\rm N}^{-3}\right) \tag{134}$$

These microplane stresses vanish in the undeformed state ($\lambda_{\rm N} = \bar{\lambda}_{\rm N} = 1$). The evolution of $\Sigma_{\rm N}$ with $\lambda_{\rm N}$ is the same already shown in Fig. 2 for the previous model of Section 6.2, and the evolution of $\Sigma_{\bar{\rm N}}$ with $\bar{\lambda}_{\rm N}$ exhibits a similar intuitive shape, as represented in Fig. 4.

Cauchy microplane stresses are obtained applying (132), yielding

$$\sigma_{\rm n} = \frac{A}{J^2}(\lambda_{\rm N}^5 - 1) + B(1 - \bar{\lambda}_{\bar{\rm N}}^{-5}), \quad \sigma_{\rm t} = 0 \tag{135}$$

where $\bar{\lambda}_{\bar{\rm N}}$ is the thickening of the plane with initial normal $\bar{\mathbf{N}}$, i.e. the plane whose normal after deformation became $\mathbf{n}$. Note that tangential Cauchy stresses are zero, in spite of having the indirect involvement of the

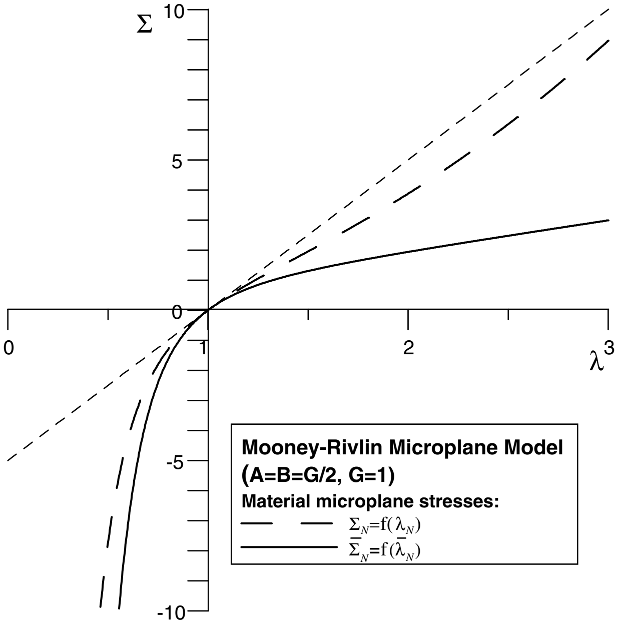


Fig. 4. Evolution of material microplane stresses $\Sigma_N$ and $\bar{\Sigma}_N$ for the model based on $\lambda_N$, $\bar{\lambda}_N$.

shear angle through $\bar{\lambda}_N = \lambda_N \cos\gamma_n$. This result is in accordance with the results of the small strain reduction of the model in the following section.

The macroscopic hyperelastic model equivalent to this microplane formulation is obtained by inserting the microplane potential (133) into integral formula (1), and applying the expressions derived in Appendix C for the integrals over the hemisphere of $\lambda_N^2$, $\lambda_N^{-3}$, $\bar{\lambda}_N^{-2}$ and $\bar{\lambda}_N^3$. This leads to the expression for the macroscopic free energy

$$\rho_0 \Psi = \frac{A}{2}(\operatorname{tr}\mathbf{C} - 3) + \frac{B}{2}(\operatorname{tr}\mathbf{C}^{-1} - 3) + A\left(\frac{1}{J} - 1\right) + B(J - 1) \tag{136}$$

This potential corresponds to a compressible formulation of the Mooney-Rivlin type. Note that $\operatorname{tr}\mathbf{C}^{-1} = I_2/J^2$ where $I_2$ is the standard second invariant of $\mathbf{C}$ expressed in terms of principal values as $I_2 = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2$, therefore for no volume change $J = 1$ one recovers the standard expression $\rho_0\Psi = (A/2)(I_1 - 3) + (B/2)(I_2 - 3)$ (Ogden, 1984). Note also that this potential vanishes in the undeformed state ($\mathbf{C} = \mathbf{I}$, $J = 1$).

Same as in previous sections, the macroscopic sPK stress tensor may be obtained either from the microplane material stresses (134) integrated over the hemisphere in original configuration using (124), or by partial derivative of the macroscopic potential (136) with respect to the GL strain $\mathbf{E}$. Either way, the resulting sPK stress tensor is

$$\mathbf{\Sigma} = A\mathbf{I} + \left(BJ - \frac{A}{J}\right)\mathbf{C}^{-1} - B\mathbf{C}^{-2} \tag{137}$$

which vanishes in the undeformed state.

Macroscopic Cauchy stresses may be evaluated by push-forward of the previous expression and multiplication by factor $1/J$, or by integrating the Cauchy microplane stresses (135) using (131), which leads to

$$\boldsymbol{\sigma} = \frac{1}{J}\left(A\mathbf{b} + \left(BJ - \frac{A}{J}\right)\mathbf{I} - B\mathbf{b}^{-1}\right) \tag{138}$$

The Cauchy stress also vanishes in the undeformed state. If desired, the Cayley–Hamilton theorem can be used to obtain alternative expressions to (137) and (138), in which the exponents of the three terms on the right-hand side are increased or decreased by a certain integer value. Note also that, as already indicated before, by setting $B = 0$ all these expressions collapse into those of the previous compressible model of Section 6.4 based on $\lambda_N$ only.

### 8.3. Reduction to small strain and initial elastic moduli

For the small-strain reduction, the microplane potential (133) may be alternatively expressed in terms of $\lambda_N$ and $\tan\gamma_N$, since intuition combined with the arguments in Section 8.1 suggest that this model should collapse into the "N–T" or $M^0$ small-strain microplane model with microplane stresses $\sigma_N = E_N\varepsilon_N$ and $\sigma_T = E_T\varepsilon_T$ (Bažant, 1984). To do that conversion, we replace $\bar{\lambda}_N$ by $\lambda_N\cos\gamma_N$ according to (44), which leads to the alternative form of the microplane potential

$$\Psi_\Omega[\lambda_N, \tan\gamma_N] = A\left(\frac{\lambda_N^2}{2} + \frac{\lambda_N^{-3}}{3} - \frac{5}{6}\right) + B\left(\frac{\lambda_N^{-2}}{2}\cos^{-2}\gamma_N + \frac{\lambda_N^3}{3}\cos^3\gamma_N - \frac{5}{6}\right) \tag{139}$$

and to the alternative microplane material stresses

$$\Sigma_N = \frac{\partial\Psi_\Omega}{\partial\lambda_N} = A(\lambda_N - \lambda_N^{-4}) + B(\lambda_N^2\cos^3\gamma_N - \lambda_N^{-3}\cos^{-2}\gamma_N) \tag{140}$$

$$\Sigma_T = \frac{\partial\Psi_\Omega}{\partial(\tan\gamma_N)} = B\tan\gamma_N(\lambda_N^{-2} - \lambda_N^3\cos^5\gamma_N) \tag{141}$$

Note that $\Psi_\Omega$, $\Sigma_N$ and $\Sigma_T$ all vanish in the undeformed configuration ($\lambda_N = 1$, $\gamma_N = 0$). Now, the normal small-strain variables may be approximated by

$$\varepsilon_N \approx \lambda_N - 1, \quad \sigma_N \approx \Sigma_N \tag{142}$$

and the initial modulus associated with $\varepsilon_N$ is given by

$$E_N = \left.\frac{\partial\Sigma_N[\lambda_N, \tan\gamma_N]}{\partial\lambda_N}\right|_{\lambda_N=1,\gamma_N=0} = A\left(1 + 4\lambda_N^{-5}\right) + B\left(2\lambda_N\cos^3\gamma_N + 3\lambda_N^{-4}\cos^{-2}\gamma_N\right)\Big|_{\lambda_N=1,\gamma_N=0} = 5(A + B) \tag{143}$$

which, to keep $E_N$ positive, requires that $A + B > 0$. In the same way, the small-strain tangential variables are identified as

$$\varepsilon_T \approx \tan\gamma_N, \quad \sigma_T \approx \Sigma_T \tag{144}$$

and the initial modulus associated with $\tan\gamma_N$ is given by

$$E_T = \left.\frac{\partial\Sigma_T[\lambda_N, \tan\gamma_N]}{\partial(\tan\gamma_N)}\right|_{\lambda_N=1,\gamma_N=0} = B\left(\lambda_N^{-2} - \lambda_N^3\cos^5\gamma_N(1 - 5\sin^2\gamma_N)\right)\Big|_{\lambda_N=1,\gamma_N=0} = 0 \tag{145}$$

Surprisingly, the initial shear tangential modulus turns out to be zero, which brings the model back to the "N-only" or M1 small-strain microplane formulation, similar to what happened with the $\lambda_N$ model in Section 6.3. This may be explained by the fact that, in spite of enriching considerably the model for large strain (from neo-Hookean to Mooney-Rivlin), the new microplane strain $\bar{\lambda}_N$ coincides in the infinitesimal range with $\lambda_N$, and therefore the only consequence that should be expected is an increase of $E_N$ as reflected by (143). Note that this result is consistent with the zero Cauchy tangential stresses (124b) obtained in the previous section.

Modulus $E_N$ can be introduced in the general formulae (24) and (25) derived in Section 3.2, with all the remaining microplane moduli equal to zero, to get the macroscopic elastic moduli

$$G = \frac{E_N}{5} = A + B, \quad K = \frac{E_N}{3} = \frac{5}{3}G, \quad E = \frac{E_N}{2} = \frac{5}{2}G, \quad \nu \equiv 0.25 \tag{146}$$

### 8.4. Some results under simple macroscopic loading scenarios

Here, the macroscopic behavior of the compressible Mooney-Rivlin microplane model with fixed Poisson ratio 0.25 is illustrated and compared to that of its neo-Hookean counterpart of Section 6.4. For this purpose, three pairs of values of constants $A$, $B$ have been selected, which always sum unity so that the shear modulus $G = A + B = 1$. These are $A = 1$, $B = 0$ (for which the neo-Hookean model of Section 6.4 is recovered), $A = B = 0.5$, which gives a balanced Mooney-Rivlin model, and $A = 0$, $B = 1$, which would correspond to a model dual to the first one with free energy dependent on $\operatorname{tr}\mathbf{C}^{-1}$ only.

Analytical solutions for the uniaxial, biaxial, volumetric and other simple loading cases may be obtained by first isolating the principal components of the Cauchy stress tensor (138),

$$J\sigma_i = A\lambda_i^2 - \frac{B}{\lambda_i^2} - \frac{A}{J} + BJ, \quad i = 1, 3 \tag{147}$$

In the uniaxial loading case, we have $\sigma_2 = \sigma_3 = 0$, $\lambda_2 = \lambda_3$ and $J = \lambda_1\lambda_2^2$. By enforcing $\sigma_2 = 0$, after simple manipulations one obtains the condition

$$(\lambda_1\lambda_2^4 - 1)(A + B\lambda_1) = 0 \tag{148}$$

which, for general values of $A$, $B$ can only be satisfied if $\lambda_2 = \lambda_1^{-1/4}$. Note that, for small deformations, this relation implies $\nu = -\partial\lambda_2/\partial\lambda_1|_{\lambda_1=1} = 0.25$, as expected. Substituting $\lambda_2 = \lambda_3 = \lambda_1^{-1/4}$ and $J = \lambda_1^{1/2}$ back into (147) written for $i = 1$, the uniaxial stress–strain relation is finally obtained in the form

$$\sigma_1 = A(\lambda_1^{3/2} - \lambda_1^{-1}) + B(1 - \lambda_1^{-5/2}) \tag{149}$$

The initial slope of the curve is

$$\left.\frac{\partial\sigma_1}{\partial\lambda_1}\right|_{\lambda_1=1} = A\left(\frac{3}{2}\lambda_1^{1/2} + \lambda_1^{-2}\right) + B\frac{5}{2}\lambda_1^{-7/2}\bigg|_{\lambda_1=1} = \frac{5}{2}(A + B) = \frac{5}{2}G \tag{150}$$

which, as expected, coincides with the elastic modulus $E$ given in Eq. (146).

In a similar way, the model response under biaxial loading (characterized by $\sigma_1 = \sigma_2$, $\sigma_3 = 0$, $\lambda_1 = \lambda_2$ and $J = \lambda_1^2\lambda_3$) leads to

$$\lambda_3 = \lambda_1^{-2/3}, \quad J = \lambda_1^{4/3} \tag{151}$$

with a resulting stress–strain relation

$$\sigma_1 = A(\lambda_1^{2/3} - \lambda_1^{-8/3}) + B(1 - \lambda_1^{-10/3}) \tag{152}$$

and initial slope

$$\left.\frac{\partial\sigma_1}{\partial\lambda_1}\right|_{\lambda_1=1} = A\left(\frac{2}{3}\lambda_1^{-1/3} + \frac{8}{3}\lambda_1^{-11/3}\right) + B\frac{10}{3}\lambda_1^{-13/3}\bigg|_{\lambda_1=1} = \frac{10}{3}(A + B) = \frac{10}{3}G \tag{153}$$

In the case of volumetric loading ($\sigma_1 = \sigma_2 = \sigma_3 = \sigma_v$, $\lambda_1 = \lambda_2 = \lambda_3 = J^{1/3}$), one directly obtains

$$\sigma_v = A(J^{-1/3} - J^{-2}) + B(1 - J^{-5/3}) \tag{154}$$

and initial slope

$$\left.\frac{\partial \sigma_v}{\partial (J^{1/3})}\right|_{J=1} = A(-J^{-2/3} + 6J^{-7/3}) + B\frac{5}{3}J^{-2}\bigg|_{J=1} = 5(A+B) = 5G \tag{155}$$

which, as expected, coincides with bulk modulus $3K$ given in Eq. (146).

The uniaxial and volumetric loading results for the three pairs of values of $A$, $B$ are represented in Fig. 5.

Biaxial tests give qualitatively similar curves at intermediate locations in between the previous two cases. Note that, in these tests, the three models give qualitatively similar results in compression. In tension, however, they are different. The third model ($A = 0, B = 1$) always tends to a horizontal asymptote at stress $\sigma_1 = B$, while the balanced Mooney-Rivlin and neo-Hookean models evolve progressively farther away from this behavior, with higher stresses in uniaxial tension, and lower stress after a maximum for the volumetric curve. This is due to the two negative exponents in terms with factor $A$ in the volumetric response equation (155), which do not appear in the uniaxial (149) or biaxial loading (152). This maximum is anyway obtained in Cauchy stresses, while the area surface on which they are applied is growing. More intuitive may be the nominal stress, which remains proportional to the load applied on a material specimen throughout a laboratory test, and therefore one would expect it to always grow with applied displacement for an *elastic* material. Because of coaxiality (always present in isotropic hyperelasticity), the nominal stress may be related to the Cauchy stress traction on the same plane by the ratio of the initial to the deformed area given by the factor $J/\lambda_i$ (Nanson formula) which, for volumetric deformation, is equal to $J^{2/3}$. In the expression of volumetric stresses (154), this product brings the exponent of the first term with factor $A$ to positive, thus finally yielding a monotonically increasing curve for the nominal stresses as expected.

In order to better highlight the improvement brought about by the compressible Mooney-Rivlin microplane model ($B \neq 0$) with respect to its neo-Hookean counterpart of Section 6.4 ($B = 0$), it is also of interest to consider the case of biaxial extension with no volume change ($J = 1 = \lambda_1\lambda_2\lambda_3$; $\lambda_3 = (\lambda_1\lambda_2)^{-1}$), which has been documented with tests on rubber materials such as those by Jones and Treloar (1975). In

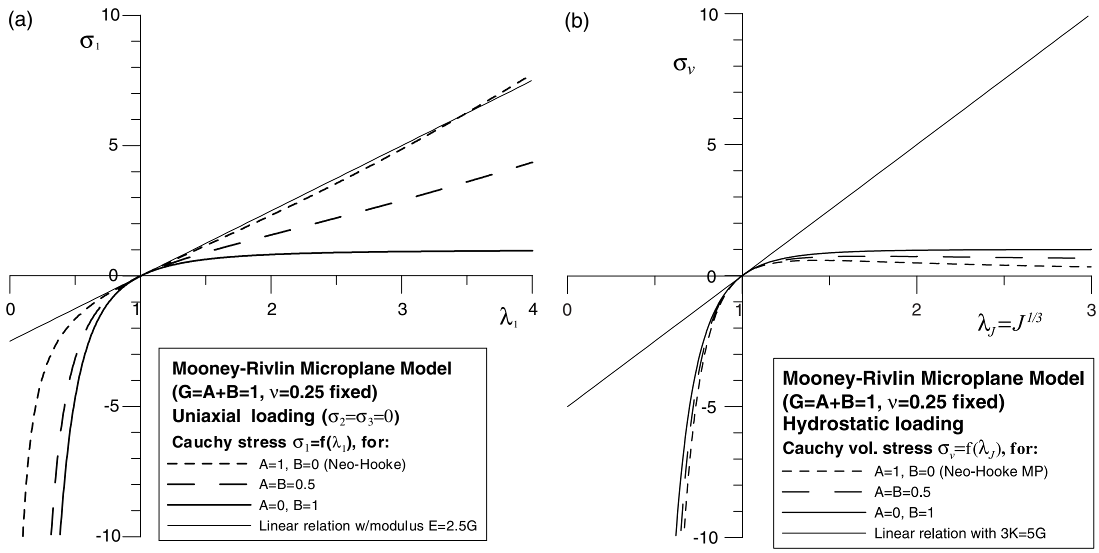


Fig. 5. Uniaxial (a) and volumetric (b) response curves for Mooney-Rivlin microplane model.

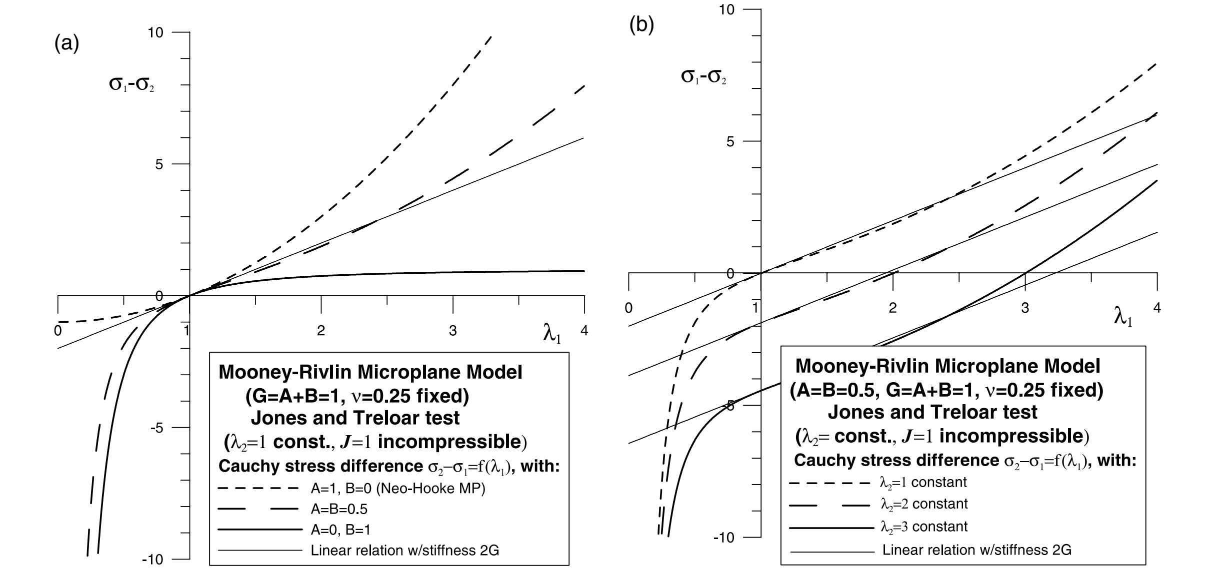


Fig. 6. Jones and Treloar test (biaxial extension, no volume change), for different values of $A$, $B$ at $\lambda_2 = 0$ (a), and different values of $\lambda_2$ for $A = B = 0.5$ (b).

those tests, $\lambda_2$ is held constant while $\lambda_1$ is increased, and what is represented is $\sigma_1 - \sigma_2$ against $\lambda_1$. In this case, from Eqs. (147) one obtains

$$\sigma_1 - \sigma_2 = A(\lambda_1^2 - 1) + B\left(1 - \frac{1}{\lambda_1^2}\right) - \sigma_2, \quad \sigma_2 = A(\lambda_2^2 - 1) + B\left(1 - \frac{1}{\lambda_2^2}\right) \tag{156}$$

For $\lambda_2 = 1$ we obtain $\sigma_2 = 0$ as expected. Three curves corresponding to the same three models as before, with $A = 1$, $B = 0$ (neo-Hookean), $A = B = 0.5$ (balanced Mooney-Rivlin) and $A = 0$, $B = 1$ are represented in Fig. 6a. In the figure it can be seen that in the compression regime the neo-Hookean model with $B = 0$ exhibits stresses that tend to a constant value equal to $-A$ for $\lambda_1 \to 0$. This is a physically questionable behavior contradicting the existing data, which clearly follow the tendency shown by the other two curves (Ogden, 1984). Note that, in this case, no compensation may be expected from consideration of nominal stresses, since the specimen areas decrease to zero causing actually vanishing nominal stresses for volume tending to zero.

The diagram on the right-hand side of Fig. 6 corresponds to three curves $\sigma_1 - \sigma_2$ vs. $\lambda_1$, obtained with fixed $\lambda_2$ values of 1, 2, and 3, for the balanced model with $A = B = 0.5$. As seen in the figure, this implies simply a translation of the curve downwards, by the amount of $\sigma_2$, which from expression above turns out to be by 0, 15/8, and 40/9, respectively. This property follows from the sum-type separation of the terms involving the various principal stretches in the energy function. This so-called Valanis–Landel hypothesis (Valanis and Landel, 1967) is known to hold very approximately for rubber-type materials up to stretches of about 10 (Ogden, 1984).

### 8.5. Model based on $\lambda_N$, $\bar{\lambda}_N$ with $-1 \leqslant \nu \leqslant 0.25$. Peculiarities and discussion

From the general formulation of Section 8.1, we know that microplane energy functions involving $\tan \gamma_N$ also generate shear stiffness and, therefore, in the general case, in the small-strain limit they should collapse into the traditional "N–T" or M1$^0$ microplane models in which Poisson ratio can take values between −1 and 0.25 (Bažant and Gambarova, 1984; Carol and Bažant, 1997). Since $\bar{\lambda}_N = \lambda_N \tan\gamma_N$, it would seem possible to find an energy function of $\lambda_N$ and $\bar{\lambda}_N$ that, unlike the one of the previous section, should also lead to a non-zero shear stiffness term and therefore non-fixed Poisson ratio.

To investigate this possibility, we consider the microplane energy function

$$\Psi_\Omega[\lambda_N, \bar{\lambda}_N] = \frac{A}{2}\lambda_N^2 + \frac{A-B}{3}\lambda_N^{-3} + \frac{B}{2}\bar{\lambda}_N^{-2} - \frac{5A-B}{6} \tag{157}$$

in which the term in $\bar{\lambda}_N^3$ from the previous formulation has been omitted, and the coefficients readjusted so that the microplane energy function vanishes for the undeformed state. By substituting (157) into integral formula (1) and applying the expressions derived in Appendix C for the integrals over the hemisphere, one obtains the macroscopic free energy

$$\rho_0\Psi = \frac{A}{2}\mathrm{tr}\,\mathbf{C} + \frac{B}{2}\mathrm{tr}\,\mathbf{C}^{-1} + \frac{A-B}{J} - \frac{5A-B}{2} \tag{158}$$

which again corresponds to a compressible formulation of the Mooney-Rivlin type and also vanishes in the undeformed state ($\mathbf{C} = \mathbf{I}$, $J = 1$).

The macroscopic sPK stress tensor may be simply obtained by partial derivative of this macroscopic potential with respect to the GL strain $\mathbf{E}$, leading to

$$\boldsymbol{\Sigma} = A\mathbf{I} - \frac{A-B}{J}\mathbf{C}^{-1} - B\mathbf{C}^{-2} \tag{159}$$

which vanishes in the undeformed state. Macroscopic Cauchy stresses may be evaluated by push-forward of the previous expression and scaling by $1/J$, which leads to

$$\boldsymbol{\sigma} = \frac{1}{J}\left(A\mathbf{b} - \frac{A-B}{J}\mathbf{I} - B\mathbf{b}^{-1}\right) \tag{160}$$

Note that by setting $B = 0$, all these expressions collapse into those of the previous simpler model of Section 6.4 based on $\lambda_N$ only.

For the small-strain reduction, it is better first to rewrite the microplane energy function in terms of $\lambda_N$ and $\tan\gamma_N$. After replacing $\bar{\lambda}_N$ by $\lambda_N \cos\gamma_N$ according to (44), we get

$$\Psi_\Omega[\lambda_N, \tan\gamma_N] = \frac{A}{2}\lambda_N^2 + \frac{A-B}{3}\lambda_N^{-3} + \frac{B}{2}\lambda_N^{-2}(1+\tan^2\gamma_N) - \frac{5A-B}{6} \tag{161}$$

which leads to the microplane stresses

$$\Sigma_N = A\lambda_N - B\lambda_N^{-3}(1+\tan^2\gamma_N) - (A-B)\lambda_N^{-4}, \quad \Sigma_T = B\lambda_N^{-2}\tan\gamma_N \tag{162}$$

From the first of these equations, the normal small-strain variables and modulus may be identified as

$$\varepsilon_N \approx \lambda_N - 1, \quad \sigma_N \approx \Sigma_N, \quad E_N = \left.\frac{\partial\Sigma_N}{\partial\lambda_N}\right|_{\lambda_N=1,\gamma_N=0} = 5A - B \tag{163}$$

To keep $E_N$ positive, the parameters are restricted by $B < 5A$. In the same way, the small-strain tangential variables and parameters may be identified as

$$\varepsilon_T \approx \tan\gamma_N, \quad \sigma_T \approx \Sigma_T, \quad E_T = \left.\frac{\partial\Sigma_T}{\partial\tan\gamma_N}\right|_{\lambda_N=1,\gamma_N=0} = B \tag{164}$$

To avoid negative $E_T$, parameter $B$ must be non-negative. The derived values of $E_N$ and $E_T$ can be introduced in the general formula (24) derived in Section 3.2, with all the remaining microplane moduli equal to zero, to get the macroscopic elastic moduli

$$3K = E_N = 5A - B, \quad 2G = \frac{2}{5}E_N + \frac{3}{5}E_T = 2A + \frac{B}{5} \tag{165}$$

which can be further substituted in the expressions for the elastic modulus and Poisson's ratio

$$E = \frac{9KG}{3K+G} = E_N\frac{2E_N + 3E_T}{4E_N + E_T} = (5A - B)\frac{10A + B}{20A - 3B}, \quad v = \frac{3K - 2G}{6K + 2G} = \frac{E_N - E_T}{4E_N + E_T} = \frac{5A - 2B}{20A - 3B} \tag{166}$$

Positiveness of $E$ is guaranteed if $E_N, E_T > 0$ (that is, if $B > 0$ and $A > B/5$). On the other hand, the Poisson's ratio can take any value between $-1$ (when $E_T \gg E_N$, or $A \to B/5$) and 0.25 (when $E_T \ll E_N$, or $B \ll A$). The impossibility to obtain Poisson's ratios larger than 0.25 is the same limitation as that found for the small-strain "N–T" or $M1^0$ microplane model (Bažant, 1984). Previous equations can be recombined to express constants $A$, $B$ in terms of the desired macroscopic elastic parameters $G$, $v$, taking into account the above mentioned limitations:

$$A = \frac{2}{5}\frac{(2 - 3v)}{(1 - 2)v}G, \quad B = \frac{1 - 4v}{1 - 2v}2G \tag{167}$$

Up to this point, the initial purpose of obtaining a Mooney-Rivlin microplane formulation based on $\lambda_N$ and $\bar{\lambda}_N$ which would collapse into N–T small strain microplane models with non-fixed Poisson ratio seems accomplished. However, the model obtained exhibits some peculiarities. First, we examine the microplane stresses $\Sigma_N$ and $\bar{\Sigma}_N$, obtained according to (123):

$$\Sigma_N = A(\lambda_N - \lambda_N^{-4}) + B\lambda_N^{-4}, \quad \bar{\Sigma}_N = -B\bar{\lambda}_N^{-3} \tag{168}$$

If introduced in (124), these expressions lead again to the right sPK stress tensor (159). However, one immediately notices that none of these microplane stresses vanish for the undeformed state (as they did for the previous model), but they take the opposite values $B$ and $-B$, respectively. If these values, together with initial values of the stretches equal to 1 are replaced into the first two terms of the Lagrangian integral expression of Cauchy stresses (129), we can group terms into common factor to $(\mathbf{n}\otimes\mathbf{n} - \bar{\mathbf{n}}\otimes\bar{\mathbf{n}})$, which reproduces the structure of a shear term even if $\Sigma_T$ itself is not considered. This can help understand how the model with only normal stretches can reproduce shear stiffness. However, it can also be seen that this is achieved at the expense of a 'trick' involving somewhat unphysical behavior on the microplanes.

Same as for the first model based on $\lambda_N$ only (Section 5), when integrated over the hemisphere, the non-vanishing microplane stresses compensate each other with the result that macroscopic sPK and Cauchy stress tensors, (159) and (160), indeed turn out to be zero.

Note that an entirely different version of "irregular" Mooney-Rivlin microplane model with initial shear stiffness could have been obtained if, from the "regular" energy function from Section 8.2, the term in $\lambda_N^3$ would have been omitted instead of the term $\bar{\lambda}_N^{-3}$ (the formulation with *both* terms omitted is a particular case already included above if $A = B$). But surely it would suffer by similar anomalies as observed in the present formulation.

The framework developed in Section 8.1, involving microplane strains $\lambda_N$, $\bar{\lambda}_N$ and $\tan\gamma_N$, certainly does not preclude the possibility of a model which collapses into small-strain "N–T" (or $M1^0$) microplane model. However, the results in previous paragraphs seem to suggest that to achieve that result one would need to add to the energy function some terms that involve $\tan\gamma_N$ directly, i.e. independently and separately from terms with $\lambda_N$ and $\bar{\lambda}_N$. Such models would exhibit additional challenges since the closed-form solutions currently available for the integrals over the hemisphere of microplane kinematic variables all involve simple power functions of $\lambda_N$ or $\bar{\lambda}_N$ *exclusively* (see Appendix C). Strictly speaking, this should not prevent to formulate such models, since the availability of closed-form equivalence to a macroscopic model is only a convenient feature for verification, not a requirement. But certainly it may make the development more cumbersome and trial-and-error based. In any case, any models collapsing into the small strain "N–T" microplane model will still exhibit a Poisson ratio restricted to the range $-1 \leqslant \nu \leqslant 0.25$. The only way to overcome this limitation necessarily involves considering separate volumetric and distortional microplane strains (rather than simply "normal" ones), as it was done in Section 7 for the neo-Hookean formulation, and is done in the following Section 9.

## 9. Formulation based on $J$, $\lambda_D$, $\bar{\lambda}_D$ and $\tan\gamma_N$: compressible Mooney-Rivlin material with unconstrained $\nu$

In analogy to the extension of Section 6 developed in Section 7, we will now introduce the volumetric strain $J$ (or, equivalently, $\lambda_J = J^{1/3}$), to account for pressure sensitivity of the bulk of the material independently of the stretch of the embedded fibers and "platelets". A straightforward extension would lead to a formulation with $\lambda_J$, $\lambda_N$, $\bar{\lambda}_N$ and, following the same arguments as in the previous section, also with $\tan\gamma_N$. However, it is more convenient to exploit the relations $\lambda_N = \lambda_D\lambda_J$ and $\bar{\lambda}_N = \bar{\lambda}_D\lambda_J$ and work with the deviatoric stretches instead of the normal ones. This type of extension leads to cleaner uncoupled expressions, which for small strain collapse into the existing "V–D–T" or M2 microplane model (Bažant and Prat, 1988a; Carol and Bažant, 1997).

### 9.1. General stress evaluation formulae

The microplane free energy takes the general form

$$\Psi_\Omega = \Psi_\Omega[\lambda_J, \lambda_D, \bar{\lambda}_D, \tan\gamma_N, \xi] \tag{169}$$

Introducing this energy function into Eq. (30), we obtain

$$\mathbf{\Sigma} = \frac{3}{2\pi}\int_\Omega \left( \Sigma_J \frac{\partial\lambda_J}{\partial\mathbf{E}} + \Sigma_D \frac{\partial\lambda_D}{\partial\mathbf{E}} + \bar{\Sigma}_D \frac{\partial\bar{\lambda}_D}{\partial\mathbf{E}} + \Sigma_T \frac{\partial \tan\gamma_N}{\partial\mathbf{E}} \right) d\Omega \tag{170}$$

where the individual microplane stress components

$$\Sigma_J = \frac{\partial\Psi_\Omega}{\partial\lambda_J}, \quad \Sigma_D = \frac{\partial\Psi_\Omega}{\partial\lambda_D}, \quad \bar{\Sigma}_D = \frac{\partial\Psi_\Omega}{\partial\bar{\lambda}_D}, \quad \Sigma_T = \frac{\partial\Psi_\Omega}{\partial(\tan\gamma_N)} \tag{171}$$

are the thermodynamic forces conjugate to the strain components. Substituting into (170) partial derivatives of $\lambda_J$, $\lambda_D$ and $\bar{\lambda}_D$ with respect to $\mathbf{E}$ from Appendix A, we get the final Lagrangian-type formula for the sPK stress tensor,

$$\begin{aligned}\mathbf{\Sigma} = \frac{3}{2\pi}\int_\Omega \Bigg( & \frac{\lambda_J\Sigma_J}{3}\mathbf{C}^{-1} + \Sigma_D\left(\frac{1}{\lambda_J^2\lambda_D}\mathbf{N}\otimes\mathbf{N} - \frac{\lambda_D}{3}\mathbf{C}^{-1}\right) + \bar{\Sigma}_D\left(\lambda_J^2\bar{\lambda}_D^3\mathbf{C}^{-1}\cdot\mathbf{N}\otimes\mathbf{N}\cdot\mathbf{C}^{-1} - \frac{\bar{\lambda}_D}{3}\mathbf{C}^{-1}\right) \\ & + \frac{\Sigma_T}{\tan\gamma_N}\left(\frac{1}{\bar{\lambda}_N^2}\mathbf{N}\otimes\mathbf{N} - \lambda_N^2\mathbf{C}^{-1}\cdot\mathbf{N}\otimes\mathbf{N}\cdot\mathbf{C}^{-1}\right)\Bigg) d\Omega\end{aligned} \tag{172}$$

Contravariant push-forward of this equation and further simplifications, including the assumption that $\Sigma_J$ is the same for all microplanes (i.e. it only depends on $\lambda_J$ and *not* on $\lambda_D$, $\bar{\lambda}_D$ or $\tan\gamma_N$), lead to the Cauchy stress tensor

$$\boldsymbol{\sigma} = \frac{\Sigma_J}{\lambda_J^2}\mathbf{I} + \frac{3}{2\pi J}\int_\Omega \left(\Sigma_D\lambda_D\left(\mathbf{n}\otimes\mathbf{n} - \frac{\mathbf{I}}{3}\right) + \bar{\Sigma}_D\bar{\lambda}_D\left(\bar{\mathbf{n}}\otimes\bar{\mathbf{n}} - \frac{\mathbf{I}}{3}\right) + \frac{\Sigma_T}{\cos\gamma_N\sin\gamma_N}\left(\mathbf{n}\otimes\mathbf{n} - \bar{\mathbf{n}}\otimes\bar{\mathbf{n}}\right)\right)\mathrm{d}\Omega \tag{173}$$

Introduction of the unit vectors $\bar{\bar{\mathbf{n}}}$ and $\bar{\bar{\mathbf{t}}}$ given by (127) and illustrated in Fig. 3, followed by substitution of $\mathbf{n} - \bar{\mathbf{n}}$ according to (128), lead to

$$\boldsymbol{\sigma} = \frac{\Sigma_J}{\lambda_J^2}\mathbf{I} + \frac{3}{2\pi J}\int_\Omega \left(\Sigma_D\lambda_D\left(\mathbf{n}\otimes\mathbf{n} - \frac{\mathbf{I}}{3}\right) + \bar{\Sigma}_D\bar{\lambda}_D\left(\bar{\mathbf{n}}\otimes\bar{\mathbf{n}} - \frac{\mathbf{I}}{3}\right) + \frac{\Sigma_T}{\cos\gamma_N}\left(\bar{\bar{\mathbf{n}}}\otimes\bar{\bar{\mathbf{t}}} + \bar{\bar{\mathbf{t}}}\otimes\bar{\bar{\mathbf{n}}}\right)\right)\mathrm{d}\Omega \tag{174}$$

Split of the integral and substitution of $\mathrm{d}\Omega$ for each of them in terms of the three different deformed solid angles $\mathrm{d}\omega$, $\mathrm{d}\bar{\omega}$ and $\mathrm{d}\bar{\bar{\omega}}$ (Appendix B) leads to the modified formula

$$\begin{aligned}\boldsymbol{\sigma} = {} & \frac{\Sigma_J}{\lambda_J^2}\mathbf{I} + \frac{3}{2\pi}\int_\Omega \frac{\lambda_D^4\Sigma_D}{J}\left(\mathbf{n}\otimes\mathbf{n} - \frac{\mathbf{I}}{3}\right)\mathrm{d}\omega + \frac{3}{2\pi}\int_\Omega \frac{\bar{\Sigma}_D}{J\bar{\lambda}_D^2}\left(\bar{\mathbf{n}}\otimes\bar{\mathbf{n}} - \frac{\mathbf{I}}{3}\right)\mathrm{d}\bar{\omega} \\ & + \frac{3}{2\pi}\int_\Omega \frac{\Sigma_T 8\cos^3[\gamma_N/2]}{J\bar{\bar{J}}_N\cos\gamma_N}\left(\bar{\bar{\mathbf{n}}}\otimes\bar{\bar{\mathbf{t}}} + \bar{\bar{\mathbf{t}}}\otimes\bar{\bar{\mathbf{n}}}\right)\mathrm{d}\bar{\bar{\omega}}\end{aligned} \tag{175}$$

The last step is the collection of terms for the same deformed solid angle, which for each integral will come from different initial orientations. The resulting, fully Eulerian formula is

$$\boldsymbol{\sigma} = \sigma_v\mathbf{I} + \frac{3}{2\pi}\int_\Omega \sigma_d\left(\mathbf{n}\otimes\mathbf{n} - \frac{\mathbf{I}}{3}\right)\mathrm{d}\omega + \frac{3}{2\pi}\int_\Omega \frac{\sigma_t}{2}(\mathbf{n}\otimes\mathbf{t} + \mathbf{t}\otimes\mathbf{n})\mathrm{d}\omega \tag{176}$$

where

$$\sigma_v = \frac{\Sigma_J}{\lambda_J^2}, \quad \sigma_d = \frac{\lambda_D^4\Sigma_D}{J} + \frac{\bar{\Sigma}_{\bar{D}}}{J\bar{\lambda}_{\bar{D}}^2}, \quad \sigma_t = \frac{16\Sigma_{\bar{\bar{T}}}\cos^3[\gamma_{\bar{\bar{N}}}/2]}{J\bar{\bar{J}}_N\cos\gamma_{\bar{\bar{N}}}} \tag{177}$$

## 9.2. Hyperelastic microplane model based on $\lambda_J$, $\lambda_D$ and $\bar{\lambda}_D$ and equivalent compressible Mooney-Rivlin model

Consider the following microplane free-energy function, with power terms in $\lambda_D$, $\bar{\lambda}_D$ analogous to the "N" terms in (133):

$$\Psi_\Omega[\lambda_J, \lambda_D, \bar{\lambda}_D] = A\left(\frac{\lambda_D^2}{2} + \frac{\lambda_D^{-3}}{3} - \frac{5}{6}\right) + B\left(\frac{\bar{\lambda}_D^{-2}}{2} + \frac{\bar{\lambda}_D^3}{3} - \frac{5}{6}\right) + \frac{1}{3}g[\lambda_J^3] \tag{178}$$

The volumetric function $g[J]$ is assumed to satisfy the same conditions as stated in (110). Note that the potential (178) vanishes in the undeformed state ($\lambda_D = \bar{\lambda}_D = J = 1$).

The material microplane stresses are obtained by taking partial derivatives of the potential:

$$\Sigma_J = \frac{\partial\Psi_\Omega}{\partial\lambda_J} = \lambda_J^2 g'[\lambda_J^3], \quad \Sigma_D = \frac{\partial\Psi_\Omega}{\partial\lambda_D} = A(\lambda_D - \lambda_D^{-4}), \quad \bar{\Sigma}_D = \frac{\partial\Psi_\Omega}{\partial\bar{\lambda}_D} = B(\bar{\lambda}_D^2 - \bar{\lambda}_D^{-3}) \tag{179}$$

These microplane stresses all vanish in the undeformed state ($\lambda_D = \bar{\lambda}_D = J = 1$). The evolution of $\Sigma_D$ with $\lambda_D$ is the same already shown in Fig. 2 for the previous model based on $\lambda_N$ only, and the evolution of $\bar{\Sigma}_N$ with $\bar{\lambda}_N$ exhibits a similar intuitive shape, as also represented in Fig. 4 (Section 8.2).

Cauchy microplane stresses are obtained applying (177), yielding

$$\sigma_v = g'[J], \quad \sigma_d = \frac{A}{J}(\lambda_D^5 - 1) + \frac{B}{J}(1 - \bar{\lambda}_{\bar{D}}^{-5}), \quad \sigma_t = 0 \tag{180}$$

where $\bar{\lambda}_{\bar{D}}$ is the distortional thickening of the plane with initial normal $\bar{\mathbf{N}}$, i.e. the plane whose normal after deformation became $\mathbf{n}$. Note that, similarly to the formulation of Section 8.2, tangential Cauchy stresses turn out to be zero in spite of the indirect involvement of the shear angle through $\bar{\lambda}_D = \lambda_D \cos\gamma_N$. This result is in accordance with the small strain reduction in the next section.

The macroscopic hyperelastic model equivalent to this microplane formulation is obtained by inserting the microplane potential (178) into integral formula (1) and applying the expressions derived in Appendix C for the integrals over the hemisphere of $\lambda_D^2$, $\lambda_D^{-3}$, $\bar{\lambda}_D^{-2}$ and $\bar{\lambda}_D^3$. This leads to the macroscopic free energy

$$\rho_0 \Psi = \frac{A}{2}(\operatorname{tr}\mathbf{C}_D - 3) + \frac{B}{2}(\operatorname{tr}\mathbf{C}_D^{-1} - 3) + g[J] \tag{181}$$

This potential corresponds to a compressible formulation of the Mooney-Rivlin type expressed in terms of the distortional left Cauchy–Green tensor $\mathbf{C}_D = J^{-2/3}\mathbf{C}$ (Ogden, 1984). Note also that this potential vanishes in the undeformed state ($\mathbf{C}_D = \mathbf{I}$, $J = 1$).

The macroscopic sPK stress tensor can be obtained either by partial derivative of the potential with respect to the GL strain tensor $\mathbf{E}$, or by integrating the material microplane stresses over the microplanes in the original configuration via Eq. (172). Either way, the result is

$$\mathbf{\Sigma} = Jg'[J]\mathbf{C}^{-1} + AJ^{-2/3}\left(\mathbf{I} - \frac{\operatorname{tr}\mathbf{C}}{3}\mathbf{C}^{-1}\right) - BJ^{2/3}\left(\mathbf{C}^{-2} - \frac{\operatorname{tr}\mathbf{C}^{-1}}{3}\mathbf{C}^{-1}\right) \tag{182}$$

which may also be expressed in terms of $\mathbf{C}_D$ as

$$\mathbf{\Sigma} = J^{1/3}g'[J]\mathbf{C}_D^{-1} + AJ^{-2/3}\left(\mathbf{I} - \frac{\operatorname{tr}\mathbf{C}_D}{3}\mathbf{C}_D^{-1}\right) - BJ^{-2/3}\left(\mathbf{C}_D^{-2} - \frac{\operatorname{tr}\mathbf{C}_D^{-1}}{3}\mathbf{C}_D^{-1}\right) \tag{183}$$

Contravariant push-forward divided by $J$ and substitution of $\operatorname{tr}\mathbf{C} = \operatorname{tr}\mathbf{b}$ leads to the macroscopic Cauchy stresses

$$\boldsymbol{\sigma} = g'[J]\mathbf{I} + AJ^{-5/3}\left(\mathbf{b} - \frac{\operatorname{tr}\mathbf{b}}{3}\mathbf{I}\right) - BJ^{-1/3}\left(\mathbf{b}^{-1} - \frac{\operatorname{tr}\mathbf{b}^{-1}}{3}\mathbf{I}\right) \tag{184}$$

which may also be expressed in terms of the distortional part of the left Cauchy–Green tensor $\mathbf{b}_D = J^{-2/3}\mathbf{b}$ and of its inverse:

$$\boldsymbol{\sigma} = g'[J]\mathbf{I} + \frac{A}{J}\left(\mathbf{b}_D - \frac{\operatorname{tr}\mathbf{b}_D}{3}\mathbf{I}\right) - \frac{B}{J}\left(\mathbf{b}_D^{-1} - \frac{\operatorname{tr}\mathbf{b}_D^{-1}}{3}\mathbf{I}\right) \tag{185}$$

In this expression it turns out that terms between parentheses correspond to the deviatoric part (in the traditional additive sense) of $\mathbf{b}_D$ and $\mathbf{b}_D^{-1}$. For this reason, the model exhibits a clear separation between the volumetric part of the Cauchy stresses, which depends on the scalar function $g[J]$, and its deviatoric part, which depends on the deviatoric parts of $\mathbf{b}_D$ and $\mathbf{b}_D^{-1}$. Due to this fact, the previous expression may also be written in the more compact form

$$\boldsymbol{\sigma} = g'[J]\mathbf{I} + \frac{A}{J}\mathbf{b}_D^{\text{dev}} - \frac{B}{J}(\mathbf{b}_D^{-1})^{\text{dev}} \tag{186}$$

All these expressions may be compared to those of the "$J$–D" model of Section 7, to which the model collapses if $B = 0$. Basically, in that model the expressions were linear in $\mathbf{C}_D$ and $\mathbf{b}_D$, while in this case new terms proportional to $\mathbf{C}_D^{-1}$ and $\mathbf{b}_D^{-1}$ are added. By application of the Cayley–Hamilton theorem, these expressions can be converted to other sequences of three consecutive powers of the same tensors.

### 9.3. Small-strain reduction and elastic constants

In the small-strain limit, the model collapses into the elastic part of the small-strain microplane model M2 with microplane stresses $\sigma_V = E_V\varepsilon_V$, $\sigma_D = E_D\varepsilon_D$ and $\sigma_T = E_T\varepsilon_T$ (Bažant and Prat, 1988a), also known as the "V–D–T" or M2 formulation. To achieve that result, first the microplane potential must be rephrased in terms of $\lambda_J$, $\lambda_D$ and $\tan\gamma_N$, by substituting $\bar{\lambda}_D = \lambda_D\cos\gamma_N$ into (178). In this way one obtains

$$\Psi_\Omega[\lambda_J, \lambda_D, \bar{\lambda}_D] = A\left(\frac{\lambda_D^2}{2} + \frac{\lambda_D^{-3}}{3} - \frac{5}{6}\right) + B\left(\frac{\lambda_D^{-2}}{2}\cos^{-2}\gamma_N + \frac{\lambda_D^3}{3}\cos^3\gamma_N - \frac{5}{6}\right) + \frac{1}{3}g[J] \tag{187}$$

Partial derivatives lead to the material microplane stresses

$$\Sigma_J = \lambda_J^2 g'[\lambda_J^3] \tag{188}$$

$$\Sigma_D = A(\lambda_D - \lambda_D^{-4}) + B(-\lambda_D^{-3}\cos^{-2}\gamma_N + \lambda_D^2\cos^3\gamma_N) \tag{189}$$

$$\Sigma_T = B\tan\gamma_N(\lambda_D^{-2} - \lambda_D^3\cos^5\gamma_N\tan\gamma_N) \tag{190}$$

Note that all these microplane stresses vanish in the undeformed state ($\lambda_J = \lambda_D = 1$, $\gamma_N = 0$).

From previous equations and some physical considerations, it follows that in the small strain regime

$$\varepsilon_V \approx \lambda_J - 1, \quad \sigma_V \approx \Sigma_J, \quad E_V = \left.\frac{\partial\Sigma_J}{\partial\lambda_J}\right|_{\lambda_J=1} = 3g''[1] \tag{191}$$

$$\varepsilon_D \approx \lambda_D - 1, \quad \sigma_D \approx \Sigma_D, \quad E_D = \left.\frac{\partial\Sigma_D}{\partial\lambda_D}\right|_{\lambda_D=1,\gamma_N=0} = 5(A+B) \tag{192}$$

$$\varepsilon_T \approx \tan\gamma_N, \quad \sigma_T \approx \Sigma_T, \quad E_T = \left.\frac{\partial\Sigma_T}{\partial(\tan\gamma_N)}\right|_{\lambda_D=1,\gamma_N=0} = 0 \tag{193}$$

where, same as in Section 8.3, we obtain $E_T = 0$. If the remaining microplane moduli have to be positive, this implies constraints $g''[1] > 0$ and $A + B > 0$. The expressions for $E_V$ and $E_D$ can be introduced into (24), with all the remaining microplane moduli (in this case $E_N$ and $E_T$) set to zero, to obtain the macroscopic linear elastic bulk and shear moduli

$$3K = E_V = 3g''[1], \quad G = \frac{E_D}{5} = A + B \tag{194}$$

The elastic modulus and Poisson ratio are given by the standard expressions

$$E = \frac{9KG}{3K+G} = 9g''[1]\frac{A+B}{3g''[1]+A+B}, \quad \nu = \frac{3K-2G}{6K+2G} = \frac{3g''[1]-2(A+B)}{6g''[1]+2(A+B)} \tag{195}$$

Note that all these expressions collapse into their counterparts in Section 7 if $B = 0$. Same as in that case, the elastic modulus $E$ is guaranteed to be positive if $A$, $B$ and $g''[1]$ satisfy restrictions stated above, and the Poisson coefficient can take any value between −1 (when $g''[1] \ll A + B$) and 0.5 (when $g''[1] \gg A + B$). There seem to be no restrictions to the sign of $A$ or $B$ as long as their sum remains positive as stated above. Previous equations may be inverted to obtain $A + B$ and $g''[1]$ from $E$ and $\nu$:

$$A + B = G = \frac{E}{2(1+\nu)}, \quad 3g''[1] = 3K = \frac{E}{1-2\nu} \tag{196}$$

### 9.4. Application results: uniaxial loading

The macroscopic behavior of the Mooney-Rivlin microplane formulation with unconstrained Poisson ratio just presented, is illustrated in this section with some results under uniaxial loading. But first, a particular form needs to be specified for the volumetric energy function $g[J]$. The expression adopted for this example is

$$g[J] = \frac{K}{2}\left(J + \frac{1}{J} - 2\right) \tag{197}$$

which satisfies the requirements $g[1] = 0$ and $g'[1] = 0$, and for which $K$ is identified as the bulk modulus because $g''[1] = K$. Note that this expression is similar to the $J$-dependent terms of the macroscopic free-energy function of the Mooney-Rivlin model with fixed Poisson ratio developed in Section 8.2, see Eq. (136). However, this does not mean that the response of both models under volumetric loading is the same, since in the previous case the other terms of the free energy including $\operatorname{tr}\mathbf{C}$ and $\operatorname{tr}\mathbf{C}^{-1}$ would also be involved in the volumetric response, while here the similar terms involve $\mathbf{C}_\mathrm{D}$ and will not participate.

With the function $g[J]$ above, volumetric Cauchy and Kirchhoff stresses turn out to be

$$\sigma_\mathrm{v} = g'[J] = \frac{K}{2}\left(1 - \frac{1}{J^2}\right), \quad \tau_\mathrm{v} = J\sigma_\mathrm{v} = Jg'[J] = \frac{K}{2}\left(J - \frac{1}{J}\right) \tag{198}$$

For large compression, both $\sigma_\mathrm{v}$ and $\tau_\mathrm{v}$ tend to $-\infty$; under large extension $\sigma_\mathrm{v}$ tends asymptotically to a limit value of $K/2$, while $\tau_\mathrm{v}$ tends to a linearly increasing response with slope $K/2$. All this is qualitatively similar to the example model of Section 8.2 with fixed Poisson ratio, for the case $A = 0$, $B = 1$ (Fig. 5b).

The previous choice of $g[J]$ was partially motivated by the feature that it can be inverted in closed form to obtain the useful expression

$$J = \frac{\tau_\mathrm{v}}{K} + \sqrt{\left(\frac{\tau_\mathrm{v}}{K}\right)^2 + 1} \tag{199}$$

Uniaxial loading is characterized by the total and deviatoric stress components

$$\sigma_2 = \sigma_3 = 0, \quad \sigma_\mathrm{v} = \frac{\sigma_1}{3}, \quad s_1 = \sigma_1 - \sigma_\mathrm{v} = \frac{2\sigma_1}{3}, \quad s_2 = s_3 = \sigma_2 - \sigma_\mathrm{v} = -\frac{\sigma_1}{3} \tag{200}$$

and by the strain components

$$\lambda_2 = \lambda_3, \quad J = \lambda_1\lambda_2^2, \quad \lambda_\mathrm{J} = J^{1/3} = \lambda_1^{1/3}\lambda_2^{2/3} \tag{201}$$

which lead to the distortional stretches $\mu_1$, $\mu_2$, $\mu_3$ (square roots of eigenvalues of $\mathbf{b}_\mathrm{D}$) given by

$$\mu_1 = \frac{\lambda_1}{\lambda_\mathrm{J}} = \lambda_1^{2/3}\lambda_2^{-2/3}, \quad \mu_2 = \mu_3 = \frac{\lambda_2}{\lambda_\mathrm{J}} = \lambda_1^{-1/3}\lambda_2^{1/3} \tag{202}$$

which are subjected to the constraint

$$\mu_2 = \mu_1^{-1/2} \tag{203}$$

From Eq. (185), one can write the deviatoric stress component of the Kirchhoff stress tensor, which on the right-hand side will only involve the deviatoric parts of $\mathbf{b}_\mathrm{D}$ and $\mathbf{b}_\mathrm{D}^{-1}$, and because of coaxiality will lead to

$$Js_1 = A\left(\mu_1^2 - \frac{\mu_1^2 + 2\mu_2^2}{3}\right) - B\left(\mu_1^{-2} - \frac{\mu_1^{-2} + 2\mu_2^{-2}}{3}\right) \tag{204}$$

By simplifying the right-hand side and substituting the constraint (203) and $s_1$ according to (200c), one gets the explicit expression of $\tau_1 = J\sigma_1$ in terms of $\mu_1$,

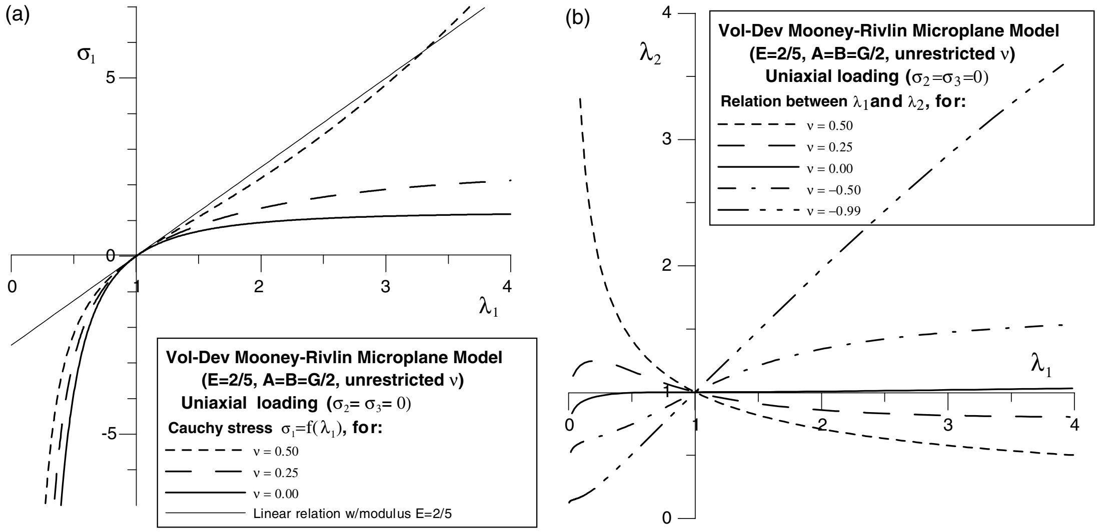


Fig. 7. Mooney-Rivlin microplane model with unrestricted $v$ under uniaxial loading: (a) uniaxial stress and (b) lateral stretch, in terms of axial stretch, for various values of $v$.

$$\tau_1 = A(\mu_1^2 - \mu_1^{-1}) - B(\mu_1^{-2} - \mu_1) \tag{205}$$

This equation may be used as a part of a closed-form parametric solution in terms of $\mu_1$. The solution is completed with the inverted volumetric equation (199) which, together with (200b), leads to

$$J = \frac{\tau_1}{3K} + \sqrt{\left(\frac{\tau_1}{3K}\right)^2 + 1} \tag{206}$$

Once $\tau_1$ and $J$ are known for a given value of $\mu_1$, it is immediate to calculate the corresponding values of

$$\sigma_1 = \frac{\tau_1}{J}, \quad \lambda_1 = \mu_1 J^{1/3}, \quad \lambda_2 = \mu_1^{-1/2} J^{1/3} \tag{207}$$

The pairs $\sigma_1$, $\lambda_1$ and $\lambda_2$, $\lambda_1$ obtained with previous relations for each value of $\mu_1$ have been plotted as curves in the left and right diagrams of Fig. 7. Different curves in each diagram correspond to the same initial elastic modulus $E = 5/2$ but different values of initial Poisson ratio $v$ ranging between $-1$ and 0.5. This leads to different values of constants $g''[1] = K$ and $A + B = G$ (196). For this example $A$, $B$ have been assumed to be equal, i.e., $A = B = G/2$.

The figure shows that, unlike in small-strain elasticity, in this case the choice of initial Poisson ratio not only changes the lateral strain development, but also may affect the uniaxial curve itself.

## 10. Concluding remarks

A consistent extension of microplane theory to large strain has been presented, based on the thermodynamic approach developed in recent years by the authors. It has been reassuring to verify, albeit in a simplified way, that the same ideas of microplane model had already been used in rubber elasticity many years ago. A variety of possible microplane strain variables have been described, and their meaning analyzed. The analysis has dealt with four microplane formulations which progressively incorporate additional microplane variables and become more complex; see Table 1. For each of them, specific hyperelastic

Table 1
Overview of general large-strain microplane formulations proposed in the paper

| Section | Formulation | Microplane strains | sPK stress | Cauchy (L) | Cauchy (E) |
|---|---|---|---|---|---|
| 5.1 | N | $\lambda_N$ | 47 | 49 | 52 |
| 7.1 | JD | $\lambda_J, \lambda_D$ | 103, 105 | 107 | 108 |
| 8.1 | NG | $\lambda_N, \bar{\lambda}_N, \gamma_N$ | 124 | 126, 129 | 130, 131 |
| 9.1 | JDG | $\lambda_J, \lambda_D, \bar{\lambda}_D, \gamma_N$ | 172 | 173, 174 | 175, 176 |

Table 2
Overview of specific hyperelastic microplane models developed in the paper

| Section | Underlying formulation | Microplane potential | Macroscopic potential | Compressible | Poisson's ratio | Related to |
|---|---|---|---|---|---|---|
| 5.3 | N | 64 | 74 | – | 0.5 | Gaussian network |
| 6.1 | N | 80 | N/A | – | 0.5 | Treloar model (1954) |
| 6.2 | N | 82 | N/A | – | 0.5 | Thomas model (1955) |
| 6.2 | N | 85 | 88 | – | 0.5 | Neo-Hookean material |
| 6.4 | N | 85 | 97 | + | 0.25 | Neo-Hookean material |
| 7.2 | JD | 111 | 114 | + | $[-1, 0.5]$ | Neo-Hookean material |
| 8.2 | NG | 133, 139 | 136 | + | 0.25 | Mooney-Rivlin material |
| 8.5 | NG | 157 | 158 | + | $[-1, 0.25]$ | Mooney-Rivlin material |
| 9.2 | JDG | 178, 187 | 181 | + | $[-1, 0.5]$ | Mooney-Rivlin (deviatoric) |

models, most of which correspond to specific forms of the well-known macroscopic neo-Hookean and Mooney-Rivlin materials, have been developed (Table 2). At the same time, for small strains the models collapse into the well-known small-strain microplane formulations of the N, V–D, N–T and V–D–T types, also called M1, $M1^0$ and M2.

Table 1 summarizes the general formulations, labeled as N, JD, NG and JDG, depending on the microplane strains listed in the third column of the table, which are used as arguments of the microplane free-energy potential. The numbers in the fourth to sixth column refer to the equations in the paper that give the stress evaluation formula for the sPK stress and for the Cauchy stress in either Lagrangian or Eulerian setting (integration over spatial directions performed either in the initial or in the deformed configuration).

Table 2 gives an overview of hyperelastic microplane models constructed as specific examples within the framework provided by the aforementioned general formulations. Each model is characterized by an expression for the microplane free-energy potential, given by the equation referred to in the third column of Table 2. For most of the models, this potential can be integrated in a closed form over the spatial directions, and the resulting expression for macroscopic free energy is referred to in the fourth column. Symbol "N/A" means that the closed-form expression is not available. The fifth column specifies whether the material is compressible, and the sixth column gives the value or range for Poisson's ratio of the model in the small-strain limit. The hyperelastic theory related to the respective microplane model is mentioned in the last column.

Overall, this paper intends to provide evidence that microplane formulations can also be used for rigorous constitutive modeling at large strain and that, as the first step, standard finite elasticity models can be recovered. But obviously, this exercise is only for the sake of verification. Once the framework has been established, a wide range of new possibilities emerge and should be explored in the future.

The real advantage of microplane-based constitutive modeling is that it provides the possibility of a large variety of formulations that do not have explicit equivalents in the traditional macroscopic (tensorial) context, and that may be much richer and more powerful. In small strain this has been shown extensively for materials undergoing damage such as concrete and, although not so extensively, also for metal plasticity. For instance, the simplest microplane model of the von Mises type, exhibiting perfectly plastic behavior on the microplanes, provides, by effect of the progressive yielding of the microplanes of different orientations, a spontaneous hardening response at the macroscopic level, with realistic unloading–reloading representation of the Bauschinger effect (Carol and Bažant, 1997). These and other features, which are very difficult or impossible to achieve with tensor-based models, may be naturally obtained with microplane models, as for instance the vertex or "loading to the side" non-linear response observed in concrete (Caner et al., 2002).

Applying these ideas within the new framework of large strain developed in this paper is a promising lead for a new generation of constitutive models with enhanced features and conceptual simplicity.

## Acknowledgements

Partial support from DGICYT (Madrid, Spain) under research grants MAT2000-1007 and BTE2001-2227 and from the Swiss Commission for Technology and Innovation under research grant CTI-5501.1 is gratefully acknowledged. The initial work at Northwestern University was supported by US-NSF grant CMS-9732791.

## Appendix A. Derivatives of microplane strain measures with respect to Green's Lagrangian strain tensor

In this appendix we differentiate the microplane strain measures $\lambda_J$, $\lambda_N$, $\bar{\lambda}_N$, $\lambda_D$, $\bar{\lambda}_D$ and $\tan\gamma_N$, defined in Section 4.2, with respect to the GL strain tensor $\mathbf{E}$. It is useful to recall the relation $\mathbf{C} = \mathbf{I} + 2\mathbf{E}$ and to realize that $\partial f/\partial\mathbf{E} = 2\partial f/\partial\mathbf{C}$ for any scalar or tensorial function $f$.

Differentiation of the Jacobian $J = \det\mathbf{F} = \sqrt{\det\mathbf{C}}$ is facilitated by the well-known formula $\partial(\det\mathbf{A})/\partial\mathbf{A} = (\det\mathbf{A})\mathbf{A}^{-t}$, valid for any regular matrix $\mathbf{A}$. Replacing $\mathbf{A}$ by the Green's Lagrangian strain $\mathbf{C}$, we get

$$\frac{\partial(\det\mathbf{C})}{\partial\mathbf{C}} = (\det\mathbf{C})\mathbf{C}^{-1} \tag{A.1}$$

because $\mathbf{C}$ is symmetric. Now we can express

$$\frac{\partial J}{\partial\mathbf{E}} = 2\frac{\partial J}{\partial\mathbf{C}} = 2\frac{\partial\sqrt{\det\mathbf{C}}}{\partial\mathbf{C}} = \frac{2}{2\sqrt{\det\mathbf{C}}}\frac{\partial(\det\mathbf{C})}{\partial\mathbf{C}} = \frac{1}{J}J^2\mathbf{C}^{-1} = J\mathbf{C}^{-1} \tag{A.2}$$

and

$$\frac{\partial\lambda_J}{\partial\mathbf{E}} = \frac{\partial(J^{1/3})}{\partial\mathbf{E}} = \frac{1}{3}J^{-2/3}\frac{\partial J}{\partial\mathbf{E}} = \frac{1}{3}J^{1/3}\mathbf{C}^{-1} = \frac{1}{3}\lambda_J\mathbf{C}^{-1} \tag{A.3}$$

Definition (33) of $\lambda_N$ can be rewritten as

$$\lambda_N = \sqrt{\mathbf{N}\cdot\mathbf{C}\cdot\mathbf{N}} = \sqrt{(\mathbf{N}\otimes\mathbf{N}):\mathbf{C}} = \sqrt{\mathcal{N}:\mathbf{C}} \tag{A.4}$$

and its differentiation leads to

$$\frac{\partial\lambda_N}{\partial\mathbf{E}} = 2\frac{\partial\sqrt{\mathcal{N}:\mathbf{C}}}{\partial\mathbf{C}} = \frac{2}{2\sqrt{\mathcal{N}:\mathbf{C}}}\frac{\partial(\mathcal{N}:\mathbf{C})}{\partial\mathbf{C}} = \frac{1}{\lambda_N}\mathcal{N} \tag{A.5}$$

To derive a similar formula for $\bar{\lambda}_N = (\mathcal{N}:\mathbf{C}^{-1})^{-1/2}$, we need to express $\partial\mathbf{C}^{-1}/\partial\mathbf{C}$. This is easier to do in the indicial notation. Let us start from the identity $\mathbf{C}^{-1}\cdot\mathbf{C} = \mathbf{I}$, rewritten as

$$(C^{-1})_{ik}C_{kl} = \delta_{il} \tag{A.6}$$

Differentiating with respect to $C_{mn}$ (assumed symmetric), we obtain

$$\frac{\partial (C^{-1})_{ik}}{\partial C_{mn}} C_{kl} + (C^{-1})_{ik}\frac{1}{2}(\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm}) = 0 \tag{A.7}$$

This identity must hold for an arbitrary combination of $i$, $l$, $m$, and $n$. Multiplying both sides by $(C^{-1})_{lj}$ leads to

$$\frac{\partial (C^{-1})_{ik}}{\partial C_{mn}} \delta_{kj} + \frac{1}{2}\left((C^{-1})_{im}(C^{-1})_{jn} + (C^{-1})_{in}(C^{-1})_{jm}\right) = 0 \tag{A.8}$$

from which

$$\frac{\partial (C^{-1})_{ij}}{\partial C_{mn}} = -\frac{1}{2}\left((C^{-1})_{im}(C^{-1})_{jn} + (C^{-1})_{in}(C^{-1})_{jm}\right) \tag{A.9}$$

Now we can evaluate

$$\frac{\partial \bar{\lambda}_{\mathrm{N}}}{\partial \mathbf{E}} = 2\frac{\partial\left((\mathcal{N} : \mathbf{C}^{-1})^{-1/2}\right)}{\partial \mathbf{C}} = 2\left(-\frac{1}{2}\right)(\mathcal{N} : \mathbf{C}^{-1})^{-3/2}\mathcal{N} : \frac{\partial \mathbf{C}^{-1}}{\partial \mathbf{C}} = -\bar{\lambda}_{\mathrm{N}}^{3}\mathcal{N} : \frac{\partial \mathbf{C}^{-1}}{\partial \mathbf{C}} \tag{A.10}$$

Switching to indicial notation we get

$$\frac{\partial \bar{\lambda}_{\mathrm{N}}}{\partial E_{mn}} = -\bar{\lambda}_{\mathrm{N}}^{3} N_i N_j \frac{\partial (C^{-1})_{ij}}{\partial C_{mn}} = \bar{\lambda}_{\mathrm{N}}^{3} N_i (C^{-1})_{im}(C^{-1})_{nj} N_j \tag{A.11}$$

and the result can be written as

$$\frac{\partial \bar{\lambda}_{\mathrm{N}}}{\partial \mathbf{E}} = \bar{\lambda}_{\mathrm{N}}^{3}\mathbf{N}\cdot\mathbf{C}^{-1}\otimes\mathbf{C}^{-1}\cdot\mathbf{N} = \bar{\lambda}_{\mathrm{N}}^{3}\mathbf{C}^{-1}\cdot\mathbf{N}\otimes\mathbf{N}\cdot\mathbf{C}^{-1} = \bar{\lambda}_{\mathrm{N}}^{3}\mathbf{C}^{-1}\cdot\mathcal{N}\cdot\mathbf{C}^{-1} \tag{A.12}$$

Note that we have exploited the symmetry of $\mathbf{C}$.

Once the basic relations (A.3), (A.5), and (A.12) have been established, it is easy to express

$$\frac{\partial \lambda_{\mathrm{D}}}{\partial \mathbf{E}} = \frac{\partial}{\partial \mathbf{E}}\left(\frac{\lambda_{\mathrm{N}}}{\lambda_{\mathrm{J}}}\right) = \frac{\lambda_{\mathrm{N}}^{-1}\mathcal{N}\lambda_{\mathrm{J}} - \lambda_{\mathrm{N}}\frac{1}{3}\lambda_{\mathrm{J}}\mathbf{C}^{-1}}{\lambda_{\mathrm{J}}^{2}} = \frac{1}{\lambda_{\mathrm{D}}\lambda_{\mathrm{J}}^{2}}\mathcal{N} - \frac{1}{3}\lambda_{\mathrm{D}}\mathbf{C}^{-1} \tag{A.13}$$

$$\frac{\partial \bar{\lambda}_{\mathrm{D}}}{\partial \mathbf{E}} = \frac{\partial}{\partial \mathbf{E}}\left(\frac{\bar{\lambda}_{\mathrm{N}}}{\lambda_{\mathrm{J}}}\right) = \frac{\bar{\lambda}_{\mathrm{N}}^{3}\mathbf{C}^{-1}\cdot\mathcal{N}\cdot\mathbf{C}^{-1}\lambda_{\mathrm{J}} - \bar{\lambda}_{\mathrm{N}}\frac{1}{3}\lambda_{\mathrm{J}}\mathbf{C}^{-1}}{\lambda_{\mathrm{J}}^{2}} = \bar{\lambda}_{\mathrm{D}}^{3}\lambda_{\mathrm{J}}^{2}\mathbf{C}^{-1}\cdot\mathcal{N}\cdot\mathbf{C}^{-1} - \frac{1}{3}\bar{\lambda}_{\mathrm{D}}\mathbf{C}^{-1} \tag{A.14}$$

and

$$\begin{aligned}\frac{\partial(\tan\gamma_{\mathrm{N}})}{\partial \mathbf{E}} &= \frac{\partial}{\partial \mathbf{E}}\left(\sqrt{\frac{\lambda_{\mathrm{N}}^{2}}{\bar{\lambda}_{\mathrm{N}}^{2}} - 1}\right) = \frac{1}{2}\left(\frac{\lambda_{\mathrm{N}}^{2}}{\bar{\lambda}_{\mathrm{N}}^{2}} - 1\right)^{-1/2}\left(2\lambda_{\mathrm{N}}\bar{\lambda}_{\mathrm{N}}^{-2}\frac{\partial \lambda_{\mathrm{N}}}{\partial \mathbf{E}} - 2\lambda_{\mathrm{N}}^{2}\bar{\lambda}_{\mathrm{N}}^{-3}\frac{\partial \bar{\lambda}_{\mathrm{N}}}{\partial \mathbf{E}}\right)\\ &= \frac{1}{\tan\gamma_{\mathrm{N}}}\left(\bar{\lambda}_{\mathrm{N}}^{-2}\mathcal{N} - \lambda_{\mathrm{N}}^{2}\mathbf{C}^{-1}\cdot\mathcal{N}\cdot\mathbf{C}^{-1}\right)\end{aligned} \tag{A.15}$$

Other useful expressions that are needed for the development of various hyperelastic models are

$$\frac{\partial(\mathrm{tr}\,\mathbf{C})}{\partial \mathbf{E}} = \frac{\partial(\mathbf{I} : \mathbf{C})}{\partial \mathbf{E}} = 2\mathbf{I} \tag{A.16}$$

and

$$\frac{\partial(\operatorname{tr}\mathbf{C}^{-1})}{\partial\mathbf{E}} = \frac{\partial(\mathbf{I}:\mathbf{C}^{-1})}{\partial\mathbf{E}} = 2\mathbf{I}:\frac{\partial(\mathbf{C}^{-1})}{\partial\mathbf{C}} = -2\mathbf{C}^{-2} \tag{A.17}$$

## Appendix B. Transformations of solid angle between initial and deformed configurations

In the initial configuration, each microplane is characterized by the unit normal $\mathbf{N}$. In the deformed configuration, we define the unit vector $\mathbf{n} = \mathbf{F}\cdot\mathbf{N}/|\mathbf{F}\cdot\mathbf{N}| = \lambda_N^{-1}\mathbf{F}\cdot\mathbf{N}$, which gives the direction of the fiber initially perpendicular to the microplane, and the unit vector $\bar{\mathbf{n}} = \mathbf{F}^{-t}\cdot\mathbf{N}/|\mathbf{F}^{-t}\cdot\mathbf{N}| = \bar{\lambda}_N\mathbf{F}^{-t}\cdot\mathbf{N}$, which is normal to the deformed microplane. Consider the set of all microplanes with initial normals in an infinitesimal solid angle $d\Omega$. After deformation, the vectors $\mathbf{n}$ associated with all these microplanes fill a solid angle $d\omega$, and vectors $\bar{\mathbf{n}}$ fill a solid angle $d\bar{\omega}$. Transformation rules for these solid angle differentials can be derived from the transformation rule for surfaces.

The intersection of the solid angle $d\Omega$ with the unit hemisphere is an infinitesimal surface characterized by the vector $d\mathbf{A} = \mathbf{N}\,dA$ where the unit vector $\mathbf{N}$ defines the normal to that surface and $dA = d\Omega$ is its area; see Fig. 8a. The deformation process transforms $d\mathbf{A}$ into a surface characterized by a vector $d\mathbf{a}$, as shown in Fig. 8b. The surface transformation is described by the Nanson formula,

$$d\mathbf{a} = J\mathbf{F}^{-t}\cdot d\mathbf{A} = J\mathbf{F}^{-t}\cdot\mathbf{N}\,dA = J\bar{\lambda}_N^{-1}\bar{\mathbf{n}}\,dA \tag{B.1}$$

To evaluate the corresponding solid angle $d\omega$, the deformed surface must be projected onto the plane perpendicular to the vector $\mathbf{F}\cdot\mathbf{N}$; this is achieved by contracting $d\mathbf{a}$ with $\mathbf{n}$. According to Fig. 8c, we can then write

$$|\mathbf{F}\cdot\mathbf{N}|^2\,d\omega = d\mathbf{a}\cdot\mathbf{n} = \frac{d\mathbf{a}\cdot\mathbf{F}\cdot\mathbf{N}}{|\mathbf{F}\cdot\mathbf{N}|} \tag{B.2}$$

from which

$$d\omega = \frac{d\mathbf{a}\cdot\mathbf{F}\cdot\mathbf{N}}{|\mathbf{F}\cdot\mathbf{N}|^3} = \frac{(J\mathbf{F}^{-t}\cdot\mathbf{N}\,dA)\cdot\mathbf{F}\cdot\mathbf{N}}{|\mathbf{F}\cdot\mathbf{N}|^3} = J\lambda_N^{-3}\,d\Omega \tag{B.3}$$

The expression for $d\bar{\omega}$ can be derived in an analogous way, with $\mathbf{F}$ replaced by $\bar{\mathbf{F}} = \mathbf{F}^{-t}$. The result is

$$d\bar{\omega} = J^{-1}\bar{\lambda}_N^3\,d\Omega \tag{B.4}$$

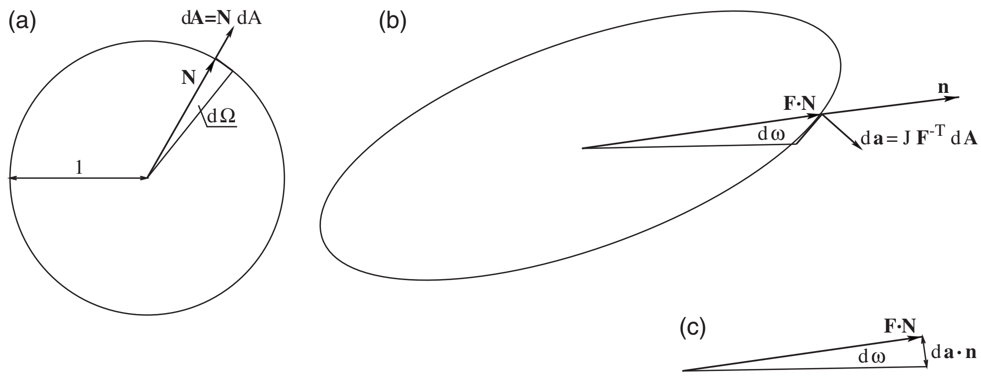


Fig. 8. Unit sphere in initial configuration and its ellipsoidal image in deformed configuration.

Both of previous expressions may be alternatively expressed in terms of the distortional stretches

$$d\omega = \lambda_D^{-3}\, d\Omega, \quad d\bar{\omega} = \bar{\lambda}_D^{3}\, d\Omega \tag{B.5}$$

The expression of $d\bar{\bar{\omega}}$ may be obtained by first identifying the fictitious deformation gradient tensor which transforms $\mathbf{N}$ into $\mathbf{n} + \bar{\mathbf{n}}$, which is

$$\mathbf{n} + \bar{\mathbf{n}} = \frac{1}{\lambda_N}\mathbf{F}\cdot\mathbf{N} + \bar{\lambda}_N\mathbf{F}^{-t}\cdot\mathbf{N} = \bar{\bar{\mathbf{F}}}_N\cdot\mathbf{N}, \quad \bar{\bar{\mathbf{F}}}_N = \frac{1}{\lambda_N}\mathbf{F} + \bar{\lambda}_N\mathbf{F}^{-t} \tag{B.6}$$

Note that $\bar{\bar{\mathbf{F}}}_N$ depends on the microplane direction $\mathbf{N}$, reason for which the symbol includes the subindex. From the fictitious deformation gradient one can also calculate a fictitious fiber stretch

$$\bar{\bar{\lambda}}_N = \sqrt{\mathbf{N}\cdot(\bar{\bar{\mathbf{F}}}^t\cdot\bar{\bar{\mathbf{F}}})\cdot\mathbf{N}} = \sqrt{\mathbf{N}\cdot(\mathbf{C}/\lambda_N^2 + \bar{\lambda}_N^2\mathbf{C}^{-1} + 2\cos\gamma_N\mathbf{I})\cdot\mathbf{N}} = \sqrt{2(1+\cos\gamma_N)}$$
$$= 2\cos[\gamma_N/2] \tag{B.7}$$

and the fictitious Jacobian

$$\bar{\bar{J}}_N = \det\bar{\bar{\mathbf{F}}}_N = \det\left[\frac{1}{\lambda_N}\mathbf{F} + \bar{\lambda}_N\mathbf{F}^{-t}\right] = \det\left[\frac{1}{\lambda_N}\mathbf{U} + \bar{\lambda}_N\mathbf{U}^{-1}\right] \tag{B.8}$$

in which the subindex $N$ has been maintained to indicate dependency on microplane direction. Finally, the desired relation is obtained by replacing $\mathbf{F}$, $\lambda_N$ and $J$ in (B.3) by their double bar counterparts, which leads to

$$d\bar{\bar{\omega}} = \frac{\bar{\bar{J}}_N}{8\cos^3[\gamma_N/2]}\, d\Omega \tag{B.9}$$

## Appendix C. Closed-form expressions for integrals over the unit hemisphere

Useful expressions can be derived from the well-known formula

$$\mathbf{I} = \frac{3}{2\pi}\int_\Omega \mathbf{N}\otimes\mathbf{N}\, d\Omega \tag{C.1}$$

Multiplying both sides from the left by $\mathbf{F}$ and from the right by $\mathbf{F}^t$, we obtain

$$\mathbf{F}\cdot\mathbf{I}\cdot\mathbf{F}^t = \mathbf{b} = \frac{3}{2\pi}\int_\Omega \mathbf{F}\cdot\mathbf{N}\otimes(\mathbf{F}\cdot\mathbf{N})\, d\Omega = \frac{3}{2\pi}\int_\Omega \lambda_N^2\mathbf{n}\otimes\mathbf{n}\, d\Omega = \frac{3}{2\pi J}\int_\Omega \lambda_N^5\mathbf{n}\otimes\mathbf{n}\, d\omega \tag{C.2}$$

where the relation between initial and deformed solid angle (B.3) has been used for the last equality. Tensor $\mathbf{b}$ has the same principal values $\lambda_I^2$, $I = 1, 2, 3$, as the right Cauchy–Green deformation tensor $\mathbf{C}$. Since $(\mathbf{n}\otimes\mathbf{n}) : \mathbf{I} = \mathbf{n}\cdot\mathbf{n} = 1$, the trace of (C.2) is

$$\operatorname{tr}\mathbf{b} = \operatorname{tr}\mathbf{C} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = I_1 = \frac{3}{2\pi}\int_\Omega \lambda_N^2\, d\Omega = \frac{3}{2\pi J}\int_\Omega \lambda_N^5\, d\omega \tag{C.3}$$

Therefore the average value of the squared microplane stretch over the unit hemisphere in initial configuration is equal to the average value of the squared principal stretches.

Repeating the procedure with $\mathbf{F}$ replaced by $\mathbf{F}^{-t}$, we obtain an integral expression for the Finger tensor

$$\mathbf{b}^{-1} = \mathbf{F}^{-t}\cdot\mathbf{F}^{-1} = \frac{3}{2\pi}\int_\Omega \mathbf{F}^{-t}\cdot\mathbf{N}\otimes(\mathbf{F}^{-t}\cdot\mathbf{N})\, d\Omega = \frac{3}{2\pi}\int_\Omega \bar{\lambda}_N^{-2}\bar{\mathbf{n}}\otimes\bar{\mathbf{n}}\, d\Omega = \frac{3J}{2\pi}\int_\Omega \bar{\lambda}_N^{-5}\bar{\mathbf{n}}\otimes\bar{\mathbf{n}}\, d\bar{\omega} \tag{C.4}$$

and its trace

$$\operatorname{tr}\mathbf{b}^{-1} = \operatorname{tr}\mathbf{C}^{-1} = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} = \frac{I_2}{J^2} = \frac{3}{2\pi}\int_\Omega \bar{\lambda}_N^{-2}\,\mathrm{d}\Omega = \frac{3J}{2\pi}\int_\Omega \bar{\lambda}_N^{-5}\,\mathrm{d}\bar{\omega} \tag{C.5}$$

Formulae analogous to (C.1) must hold in the deformed configuration, with $\mathrm{d}\Omega$ replaced by $\mathrm{d}\omega$ and $\mathbf{N}$ replaced by $\mathbf{n}$, or with $\mathrm{d}\Omega$ replaced by $\mathrm{d}\bar{\omega}$ and $\mathbf{N}$ replaced by $\bar{\mathbf{n}}$. The integration domain is still the unit hemisphere, $\Omega$. Using transformation rules (B.3) and (B.4) derived in Appendix B, we can construct integral expressions for the right Cauchy–Green tensor,

$$\mathbf{C} = \mathbf{F}^t\cdot\mathbf{I}\cdot\mathbf{F} = \mathbf{F}^t\cdot\left(\frac{3}{2\pi}\int_\Omega \bar{\mathbf{n}}\otimes\bar{\mathbf{n}}\,\mathrm{d}\bar{\omega}\right)\cdot\mathbf{F} = \frac{3}{2\pi J}\int_\Omega \bar{\lambda}_N^5\mathbf{N}\otimes\mathbf{N}\,\mathrm{d}\Omega = \frac{3}{2\pi}\int_\Omega \bar{\lambda}_N^2\mathbf{N}\otimes\mathbf{N}\,\mathrm{d}\bar{\omega} \tag{C.6}$$

and its inverse,

$$\mathbf{C}^{-1} = \mathbf{F}^{-1}\cdot\mathbf{I}\cdot\mathbf{F}^{-t} = \mathbf{F}^{-1}\cdot\left(\frac{3}{2\pi}\int_\Omega \mathbf{n}\otimes\mathbf{n}\,\mathrm{d}\omega\right)\cdot\mathbf{F}^{-t} = \frac{3J}{2\pi}\int_\Omega \lambda_N^{-5}\mathbf{N}\otimes\mathbf{N}\,\mathrm{d}\Omega$$
$$= \frac{3}{2\pi}\int_\Omega \lambda_N^{-2}\mathbf{N}\otimes\mathbf{N}\,\mathrm{d}\omega \tag{C.7}$$

Double contraction of (C.6) and (C.7) with the unit tensor $\mathbf{I}$ provides alternative expressions for the invariants

$$\operatorname{tr}\mathbf{C} = \operatorname{tr}\mathbf{b} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = I_1 = \frac{3}{2\pi J}\int_\Omega \bar{\lambda}_N^5\,\mathrm{d}\Omega = \frac{3}{2\pi}\int_\Omega \bar{\lambda}_N^2\,\mathrm{d}\bar{\omega} \tag{C.8}$$

and

$$\operatorname{tr}\mathbf{C}^{-1} = \operatorname{tr}\mathbf{b}^{-1} = \lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2} = \frac{I_2}{J^2} = \frac{3J}{2\pi}\int_\Omega \lambda_N^{-5}\,\mathrm{d}\Omega = \frac{3}{2\pi}\int_\Omega \lambda_N^{-2}\,\mathrm{d}\omega \tag{C.9}$$

All previous relations may be rephrased in terms of $\lambda_D$ and $\bar{\lambda}_D$ by only replacing $\lambda_N = J^{1/3}\lambda_D$ and $\bar{\lambda}_N = J^{1/3}\bar{\lambda}_D$, and extracting the $J$ terms out of the integral.

Equations of the solid angle transformations (B.3) and (B.4), together with the identities

$$2\pi = \int_\Omega \mathrm{d}\Omega = \int_\Omega \mathrm{d}\omega = \int_\Omega \mathrm{d}\bar{\omega} \tag{C.10}$$

lead to integral expressions of the Jacobian and its inverse

$$J = \frac{1}{2\pi}\int_\Omega \bar{\lambda}_N^3\,\mathrm{d}\Omega = \frac{1}{2\pi}\int_\Omega \lambda_N^3\,\mathrm{d}\omega, \quad \frac{1}{J} = \frac{1}{2\pi}\int_\Omega \lambda_N^{-3}\,\mathrm{d}\Omega = \frac{1}{2\pi}\int_\Omega \bar{\lambda}_N^{-3}\,\mathrm{d}\bar{\omega} \tag{C.11}$$

Since $J = \lambda_J^3$, relations (C.11) imply that the average value of $\bar{\lambda}_D^3$ as well as of $\lambda_D^{-3}$ over all microplanes is 1. Recall that $\lambda_D = \lambda_N/\lambda_J$ and $\bar{\lambda}_D = \bar{\lambda}_N/\lambda_J$ are the microplane stretch and microplane thickening corresponding to the distortional part of the deformation gradient. Conditions

$$\frac{1}{2\pi}\int_\Omega \bar{\lambda}_D^3\,\mathrm{d}\Omega = \frac{1}{2\pi}\int_\Omega \lambda_D^{-3}\,\mathrm{d}\Omega = \frac{1}{2\pi}\int_\Omega \lambda_D^3\,\mathrm{d}\omega = \frac{1}{2\pi}\int_\Omega \bar{\lambda}_D^{-3}\,\mathrm{d}\bar{\omega} = 1 \tag{C.12}$$

are large-strain generalizations of condition $\int_\Omega \varepsilon_D\,\mathrm{d}\Omega = 0$ valid for the small-strain microplane theory.

## Appendix D. Expansions used in the small-strain limit

Reduction of the general theory to the small-strain case is based on the assumption that the components of the displacement gradient are of the order of $\epsilon$ where $\epsilon \ll 1$ is a small parameter. It is useful to write the deformation gradient in the form

$$\mathbf{F} = \mathbf{I} + \boldsymbol{\varepsilon} + \boldsymbol{\omega} \tag{D.1}$$

where $\boldsymbol{\varepsilon}$ is a symmetric tensor describing the small strain and $\boldsymbol{\omega}$ is a skew-symmetric tensor describing the small rotation. The following approximations can then be derived:

$$\mathbf{C} = \mathbf{F}^{t} \cdot \mathbf{F} = \mathbf{I} + \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}^{t} + \boldsymbol{\omega} + \boldsymbol{\omega}^{t} + \mathrm{O}[\epsilon^2] = \mathbf{I} + 2\boldsymbol{\varepsilon} + \mathrm{O}[\epsilon^2] \tag{D.2}$$

$$\mathbf{F}^{-1} = (\mathbf{I} + \boldsymbol{\varepsilon} + \boldsymbol{\omega})^{-1} = \mathbf{I} - \boldsymbol{\varepsilon} - \boldsymbol{\omega} + \mathrm{O}[\epsilon^2] \tag{D.3}$$

$$\mathbf{C}^{-1} = \mathbf{F}^{-1} \cdot \mathbf{F}^{-t} = \mathbf{I} - 2\boldsymbol{\varepsilon} + \mathrm{O}[\epsilon^2] \tag{D.4}$$

$$\lambda_N = \sqrt{\mathbf{N} \cdot \mathbf{C} \cdot \mathbf{N}} = \sqrt{1 + 2\mathbf{N} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{N} + \mathrm{O}[\epsilon^2]} = 1 + \mathbf{N} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{N} + \mathrm{O}[\epsilon^2] = 1 + \varepsilon_N + \mathrm{O}[\epsilon^2] \tag{D.5}$$

$$\bar{\lambda}_N = (\mathbf{N} \cdot \mathbf{C}^{-1} \cdot \mathbf{N})^{-1/2} = (1 - 2\mathbf{N} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{N} + \mathrm{O}[\epsilon^2])^{-1/2} = 1 + \mathbf{N} \cdot \boldsymbol{\varepsilon} \cdot \mathbf{N} + \mathrm{O}[\epsilon^2] = 1 + \varepsilon_N + \mathrm{O}[\epsilon^2] \tag{D.6}$$

$$J^2 = \det \mathbf{C} = \lambda_1^2 \lambda_2^2 \lambda_3^2 = (1 + \varepsilon_1)^2 (1 + \varepsilon_2)^2 (1 + \varepsilon_3)^2 + \mathrm{O}[\epsilon^2] = 1 + 2(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + \mathrm{O}[\epsilon^2] \tag{D.7}$$

$$J = \sqrt{J^2} = \sqrt{1 + 2(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + \mathrm{O}[\epsilon^2]} = 1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \mathrm{O}[\epsilon^2] \tag{D.8}$$

$$\lambda_J = J^{1/3} = (1 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \mathrm{O}[\epsilon^2])^{1/3} = 1 + \tfrac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) + \mathrm{O}[\epsilon^2] \tag{D.9}$$

$$\mathbf{n} = \lambda_N^{-1} \mathbf{F} \cdot \mathbf{N} = (1 - \varepsilon_N + \mathrm{O}[\epsilon^2])(\mathbf{N} + \boldsymbol{\varepsilon} \cdot \mathbf{N} + \boldsymbol{\omega} \cdot \mathbf{N}) = \mathbf{N} + \boldsymbol{\varepsilon} \cdot \mathbf{N} + \boldsymbol{\omega} \cdot \mathbf{N} - \varepsilon_N \mathbf{N} + \mathrm{O}[\epsilon^2] \tag{D.10}$$

$$\bar{\mathbf{n}} = \bar{\lambda}_N \mathbf{F}^{-t} \cdot \mathbf{N} = (1 + \varepsilon_N)(\mathbf{N} - \boldsymbol{\varepsilon} \cdot \mathbf{N} + \boldsymbol{\omega} \cdot \mathbf{N}) + \mathrm{O}[\epsilon^2] = \mathbf{N} - \boldsymbol{\varepsilon} \cdot \mathbf{N} + \boldsymbol{\omega} \cdot \mathbf{N} + \varepsilon_N \mathbf{N} + \mathrm{O}[\epsilon^2] \tag{D.11}$$

$$\mathbf{n} \otimes \mathbf{n} = \mathcal{N} + \mathcal{N} \cdot \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon} \cdot \mathcal{N} - \mathcal{N} \cdot \boldsymbol{\omega} + \boldsymbol{\omega} \cdot \mathcal{N} - 2\varepsilon_N \mathcal{N} + \mathrm{O}[\epsilon^2] \tag{D.12}$$

$$\bar{\mathbf{n}} \otimes \bar{\mathbf{n}} = \mathcal{N} - \mathcal{N} \cdot \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon} \cdot \mathcal{N} - \mathcal{N} \cdot \boldsymbol{\omega} + \boldsymbol{\omega} \cdot \mathcal{N} + 2\varepsilon_N \mathcal{N} + \mathrm{O}[\epsilon^2] \tag{D.13}$$

## References

Batdorf, S.B., Budiansky, B., 1949. A mathematical theory of plasticity based on the concept of slip. Technical Note 1871, National Advisory Committee for Aeronautics.

Bažant, Z.P., 1984. Mathematical modeling of progressive cracking and fracture of rock. In: Dowding, C.H., Singh, M. (Eds.), Proceedings, 25th US National Symposium on Rock Mechanics, Northwestern University, Evanston, IL. Soc. of Mining Engrs.—Am. Institute of Mining, Met. & Petr. Engrs., New York.

Bažant, Z.P., Adley, M.D., Carol, I., Jirásek, M., Akers, S.A., Rohani, B., Cargile, J.D., Caner, F.C., 2000a. Large-strain generalization of microplane constitutive model for concrete and application. ASCE J. Engrg. Mech. 126 (9), 971–980.

Bažant, Z.P., Caner, F.C., Carol, I., Adley, M.D., Akers, S.A., 2000b. Microplane model M4 for concrete: I. Formulation with work-conjugate deviatoric stress. ASCE J. Engrg. Mech. 126 (9), 944–953.

Bažant, Z.P., Gambarova, P., 1984. Crack shear in concrete: crack band microplane model. ASCE J. Struct. Engrg. 110, 2015–2035.

Bažant, Z.P., Oh, B.H., 1985. Microplane model for progressive fracture of concrete and rock. ASCE J. Engrg. Mech. 111, 559–582.

Bažant, Z.P., Prat, P.C., 1988a. Microplane model for brittle-plastic material: I. Theory. ASCE J. Engrg. Mech. 114 (10), 1672–1688.

Bažant, Z.P., Prat, P.C., 1988b. Microplane model for brittle-plastic material: II. Verification. ASCE J. Engrg. Mech. 114 (10), 1689–1702.

Bažant, Z.P., Xiang, Y., Prat, P.C., 1996a. Microplane model for concrete: I. Stress–strain boundaries and finite strain. ASCE J. Engrg. Mech. 122 (3), 245–262.

Bažant, Z.P., Xiang, Y., Adley, M.D., Prat, P.C., Akers, S.A., 1996b. Microplane model for concrete: II. Data delocalization and verification. ASCE J. Engrg. Mech. 122 (3), 263–268.

Caner, F.C., Bažant, Z.P., 2000. Microplane model M4 for concrete: II. Algorithm and calibration. ASCE J. Engrg. Mech. 126, 954–961.

Caner, F.C., Bažant, Z.P., Červenka, J., 2002. Vertex effect in strain-softening concrete at rotating principal axes. ASCE J. Engrg. Mech. 128, 24–33.

Carol, I., Bažant, Z.P., 1997. Damage and plasticity in microplane theory. Int. J. Solids Struct. 34 (29), 3807–3835.

Carol, I., Bažant, Z.P., Prat, P.C., 1992. Microplane-type constitutive models for distributed damage and localized cracking in concrete structures. In: Bažant, Z.P. (Ed.), Fracture Mechanics of Concrete Structures (FraMCoS 1). Elsevier, Breckenridge, CO, USA, pp. 299–304.

Carol, I., Jirásek, M., Bažant, Z.P., 2001. A thermodynamically consistent approach to microplane theory. Part I: free energy and consistent microplane stresses. Int. J. Solids Struct. 38 (17), 2921–2931.

Carol, I., Jirásek, M., Bažant, Z.P., Steinmann, P., 1998. New thermodynamic approach to microplane model with application to finite deformations. Technical Report PI-145, Int. Center for Numerical Methods in Engrg. (CIMNE), ETSECCPB-UPC, E-08034 Barcelona, Spain.

Carol, I., Prat, P.C., 1995. A multicrack model based on the theory of multisurface plasticity and two fracture energies. In: Owen, D.R.J., Oñate, E., Hinton, E. (Eds.), Computational Plasticity (COMPLAS IV), vol. 2. Pineridge Press, Barcelona, pp. 1583–1594.

Coleman, B.D., Gurtin, M.E., 1967. Thermodynamics with internal state variables. J. Chem. Phys. 47, 597.

Flory, P.J., 1969. Statistical Mechanics of Chain Molecules. Interscience Publishers, New York.

Flory, T.J., 1961. Thermodynamic relations for high elastic materials. Trans. Faraday Soc. 57, 829–838.

James, H.M., Guth, E., 1943. Theory of the elastic properties of rubber. J. Chem. Phys. 11, 455–481.

Jones, D.F., Treloar, L.R.G., 1975. The properties of rubber in pure homogeneous strain. J. Phys. D, Appl. Phys. 8, 1285–1304.

Kuhl, E., Ramm, E., 2000. Microplane modelling of cohesive frictional materials. Euro. J. Mech./A: Solids 19, S121–S143.

Kuhl, E., Steinmann, P., Carol, I., 2001. A thermodynamically consistent approach to microplane theory. Part II: Dissipation and inelastic constitutive modelling. Int. J. Solids Struct. 38 (17), 2933–2952.

Kuhn, W., Grün, F., 1942. Beziehungen zwischen elastischen Konstanten und Dehnungsdoppel-brechung hochelastischer Stoffe. Kolloid-Zeitschrift 101, 248–271.

Lubarda, V.A., Krajcinovic, D., 1993. Damage tensors and the crack density distribution. Int. J. Solids Struct. 30, 2859–2877.

Mohr, O., 1900. Welche Umstande bendingen der Bruch und der Elastizitätsgrenze des Materials. Z. Vereins Deutscher Ingenieure 44, 1–12.

Ogden, R.W., 1984. Non-Linear Elastic Deformations. John Wiley & Sons (reprinted by Dover Publications).

Ogden, R.W., 1982. Elastic deformations of rubberlike solids. In: Hopkins, H.G., Sewell, M.J. (Eds.), Mechanics of Solids. Pergamon Press, Oxford, pp. 499–537.

Ožbolt, J., Li, Y., Kožar, I., 2001. Microplane model for concrete with relaxed kinematic constraint. Int. J. Solids Struct. 38, 2683–2711.

Taylor, G.I., 1938. Plastic strain in metals. J. Inst. Metals 62, 307–324.

Thomas, A.G., 1955. The departures from the statistical theory of rubber elasticity. Transactions of the Faraday Society 51, 569–582.

Treloar, L.R.G., 1954. The photoelastic properties of short-chain molecular networks. Trans. Faraday Soc. 50, 881–896.

Treloar, L.R.G., 1975. The Physics of Rubber Elasticity, third ed. Clarendon Press, Oxford.

Treloar, L.R.G., Riding, G., 1979. A non-Gaussian theory for rubber in biaxial strain. I. Mechanical properties. Proc. Roy. Soc. London A 369, 261–280.

Valanis, K.C., Landel, R.F., 1967. The strain-energy function of a hyperelastic material in terms of the extension ratios. J. Appl. Phys. 38, 2997–3002.