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Strength distribution of dental restorative ceramics: Finite weakest link model with zero threshold

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ABSTRACT

Ensuring a small enough failure probability is important for the design and selection of restorative dental ceramics. For this purpose, the two-parameter Weibull distribution, which is based on the weakest link model with infinitely many links, is usually adopted to model the strength distribution of dental ceramics. This distribution has been thoroughly validated for perfectly brittle materials. However, dental ceramics are generally quasibrittle because the inhomogeneity size is not negligible compared to the size of the ceramic part. For such materials, the experimental histograms of many quasibrittle materials have been shown to exhibit strong deviations from the two-parameter Weibull distribution. As a remedy, the three-parameter Weibull distribution, which has a nonzero threshold, has been proposed. However, the improvement of the fits of histograms of quasibrittle materials has been only partial. Instead of making the threshold non-zero, the correct remedy is to consider the weakest link model to have a finite number of links, each of them representing one finite-size representative volume element of material. This model has recently been justified on the basis of the probability of random jumps of atomic lattice cracks over the activation energy barriers on the free energy potential of the lattice. It is shown that, in similarity to other quasibrittle materials, this new model allows excellent fits of the experimental strength histograms of various types of dental ceramics.

1. Introduction

The flexural strength is one of the main limiting factors in the modern applications of dental restorative ceramics [24,16,14]. A fairly low failure probability P_f, such as 10^{-2} per lifetime, is usually required [1,15]. The strength corresponding to such a failure probability may be determined through extensive strength histogram testing. But the costs are high and the results, as it now appears, misleading. Therefore, it is necessary to determine the strength distribution by a theory that is more realistic and can be calibrated by tests easily.

The statistical distribution of flexural strength of dental restorative ceramics has been studied extensively. The experimental strength histograms have initially been fitted with the two-parameter Weibull distribution, which has a zero threshold [24,23,15,17,16]. It has been found, though, that the two-parameter Weibull distribution cannot fit the histograms closely.

As found upon closer scrutiny, the reason for poor fits is that the size of the inhomogeneities, and thus of the fracture process zone (FPZ) and of the representative volume of material (RVE), is not negligible compared to the size D of den-

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tal restorative ceramic parts. A new theory, in which the size of the FPZ (or the RVE) is taken into account, will be shown capable of providing excellent fits of the experimental strength histograms of alumina-, feldspar-, leucite-, and zirconia-based ceramics, commonly used in dentistry. It will also be demonstrated that the current practice of fitting the histograms by the two-parameter Weibull distribution can lead to serious overestimation of safety margins for fracture of the ceramic parts and underestimation of the Weibull modulus.

The statistical distribution of strength is well known for two simple limiting cases: (1) perfectly ductile (or plastic) structures, for which the cumulative distribution function (cdf) of strength distribution must be Gaussian (or normal), and (2) for perfectly brittle structures, for which the pdf must be Weibullian [10,2,3]. The statistics of plastic behavior can be described by the fiber-bundle model and is characterized by the absence of size effect, while the coefficient of variation (C.o.V.) of strength decreases as $D^{-1/2}$ (D = structure size). The statistics of brittle failure is described by a weakest link model with infinitely many links in the chain, and is known to lead to a power-law statistical size effect on material strength with a size-independent C.o.V., provided that the structure fails as soon as a macro-crack initiates from one RVE. Such failure behavior is typical of many applications, including the dental ones (though not of reinforced concrete).

Perfectly brittle structures consist of fine-grained brittle materials for which the RVE of the material is negligibly small compared to the structural dimension D. The width of the FPZ at the front of a crack is approximately equal to the RVE size, which typically equals 2–3 inhomogeneity sizes (grain sizes). Quasibrittle structures are those for which the RVE is not negligibly small compared to D, which is typical of dental applications. Therefore, the weakest link model must be considered to have a finite, rather than infinite, number of links in the chain. Consequently, the classical formulas (based on the gamma function) for the mean and standard deviation of strength and for the size effect do not apply. The finite number of links explains why the behavior of quasibrittle materials is at a transition between ductile and brittle behaviors. Theoretically, this has two implications: (1) The upper right part of strength histogram in Weibull scale deviates to the right from a straight line fit of the lower left part and (2) the size effect plot deviates from the power law of classical Weibull theory.

Both of these properties have recently been amply verified experimentally for various quasibrittle materials (e.g. concretes, mortars, fiber–polymer composites, industrial ceramics, sea ice, wood, rigid foams) [8,9,18,6,7]. The purpose of this paper is to demonstrate it for dental restorative ceramics.

2. Review of finite weakest link model

Strong deviations of the strength histograms of quasibrittle ceramics from Weibull distribution [18] have recently been documented in the testing of industrial ceramics. It has been thought that this problem could be overcome by using a three-parameter Weibull distribution, which has a finite threshold. However, this brings about only a partial improvement of histogram fits [8,9]. Histograms of a broad enough range (> 1000 tests) still reveal significant deviations at the high probability tail. Likewise, for some quasibrittle materials for which deviations from power-law size effect curves have been documented, the use of a finite threshold has been found to improve the fits of experimental size effect curves only partly, and to lead to excessive strength prediction for very large sizes.

Recent analysis [8,9] showed that the problem lies elsewhere—in the assumption of an infinite weakest link model, which underlies the Weibull distribution of strength. The number of links must be finite, rather than infinite, because the size of the RVE (or FPZ) is not negligible compared to the size of the ceramic part. This is the salient feature of quasibrittle materials, which endows them with some degree of ductility.

That the finiteness of the weakest link model is generally the correct remedy is verified by attaining, for many quasibrittle materials, close fits of both the histograms of [8,9,18] and the size effect curves. The use of a finite weakest link model might be regarded as inconvenient, since the mean and coefficient of variation can no longer be given by explicit formulas involving the gamma function. But there still exists a simple integral for the failure probability distribution whose numerical evaluation is easy [9].

The power-law nature of the strength distribution tail, and the vanishing of the strength threshold, are not merely empirical findings. Under certain plausible hypotheses, they have been shown to be a logical consequence of fracture mechanics of random jumps of the front of cracks propagating through the nano-structure, either a regular atomic lattice or a disordered nano-structure [5]. The jumps are controlled by activation energy barriers separating a series of metastable states on the surface of the free energy potential of the nano-structure. These barriers depend on the energy release increment from the nano-structure due to nano-crack advance by one atomic spacing in atomic lattice or one nano-inhomogeneity in a disordered nano-structure. By applying the Griffith theory to the propagation of the nano-crack, the amount of energy release increment can be expressed as a function of the remote stress applied on the nano-structure. Based on the fact that the activation energy barriers for forward and backward jumps can differ only little, the left tail of the pdf of strength has been shown to be a power law of exponent 2, having a threshold that is virtually zero (since the effect of crack front diffusion at low Péclet number is entirely negligible).

By approximating the multi-scale transition from nano-scale to the RVE scale with a hierarchical model consisting of series and parallel couplings (i.e., of chains and bundles) (Fig. 1), it has been shown that the strength distribution on the material scale of one RVE has a Gaussian distribution onto which a remote power-law tail with zero threshold is grafted from the left [8,9,11]. The exponent of power-law tail is found to increase in passing to higher scales until, on the RVE scale, it becomes equal to the Weibull modulus, m [9].

So, as this theory shows, the rate processes such as the phase changes, creep rate, dislocation mobility, etc., are not the only phenomena that translate simply and directly from the atomic scale to the material scale. The power-law tail of the cdf of strength is another such phenomenon.

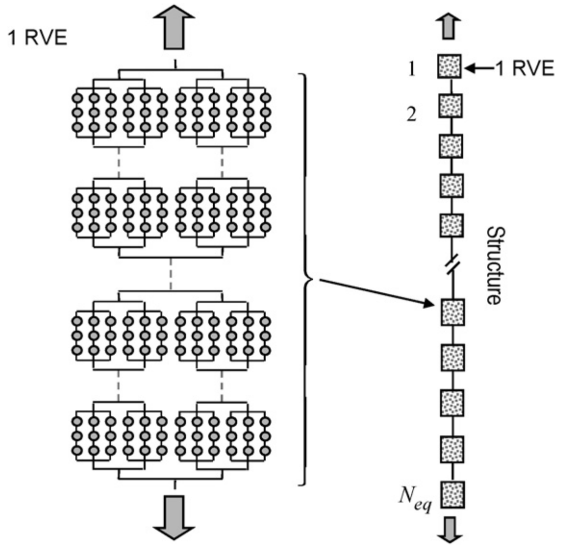

Fig. 1 – Models of series and parallel couplings: (a) hierarchy of sub-chains and sub-bundles and (b) a weakest link model—series coupling (or chain) of elements, each representing one RVE.

To describe the probability density function (pdf) of material strength on the RVE level, the Weibull tail (which is a power-law for small stress) may be considered to be grafted from the left onto a Gaussian pdf at the failure probability of about 0.01–1% [8,9]:

$$p_1(\sigma_N) = r_f \phi_W(\sigma_N) \quad \text{for} \quad \sigma_N < \sigma_{N,gr} \tag{1}$$

$$p_1(\sigma_N) = r_f \phi_G(\sigma_N) \quad \text{for} \quad \sigma_N \geq \sigma_{N,gr} \tag{2}$$

where $\phi_W(\sigma_N) = (m/s_1)(\sigma_N/s_1)^m e^{-(\sigma_N/s_1)^m}$; $\phi_G(\sigma_N) = e^{-(\sigma_N-\mu_G)^2/2\delta_G^2}/(\delta_G\sqrt{2\pi})$; σ_N = nominal strength of structure = P_{max}/bD or P_{max}/D^2 for two- for three-dimensional scaling (P_{max} = maximum load or its parameter, b = structure thickness); μ_G and δ_G are the mean and standard deviation of the Gaussian core; m and s_1 are the shape and scale parameters of the Weibull tail; r_f is the scaling parameter needed to normalize the grafted pdf; i.e. $\int_{-\infty}^{\infty} p_1(\sigma_N)d\sigma_N = 1$, and $\sigma_{N,gr}$ is the grafting stress corresponding to the grafting probability P_{gr}. In total there are 6 statistical parameters: m, s_0, μ_G, δ_G, P_{gr}, and r_f. Only 4 of them are free to be adjusted to statistical data because of the aforementioned normalization condition and a second condition that ensures the continuity of pdf at the grafting point, i.e. $\phi_W(\sigma_{N,gr}) = \phi_G(\sigma_{N,gr})$ [9].

Since, for the purpose of failure, one RVE must be defined as the smallest material volume whose failure causes the whole structure to fail, the failure probability of quasibrittle structures with positive geometry can be calculated by applying the joint probability theorem to the survival probability of the structure:

$$1 - P_f(\sigma_N) = \prod_{i=1}^{n} [1 - P_1(\langle \sigma_i(\mathbf{x}_i) \rangle)] \tag{3}$$

or

$$P_f(\sigma_N) = 1 - \prod_{i=1}^{n} [1 - P_1(\langle s(\mathbf{x}_i) \rangle \sigma_N)] \tag{4}$$

where σ_N = maximum stress in the structure, $\sigma_i(\mathbf{x}_i) = \sigma_N s(\mathbf{x}_i)$ = stress at the center $\mathbf{x}_i$ of the i-th RVE, s = dimensionless stress (if $\sigma_N = \sigma_{max}$, then the maximum of s is 1); $\langle x \rangle = \max(x, 0)$, and $P_1(\sigma) = \int_0^{\sigma} p_1(\sigma')\,d\sigma'$ = strength cdf of one RVE.

Here, it is useful to introduce the concept of equivalent number of RVEs, N_{eq}, for which a chain of N_{eq} elements subjected to a uniform stress σ_N gives the same cdf:

$$P_f(\sigma_N) = 1 - [1 - P_1(\sigma_N)]^{N_{eq}} \tag{5}$$

In general, N_{eq} is a function of the actual number of RVEs n, modified by the stress distribution $s(\mathbf{x})$ and by σ_N. It can further be shown that if all of the RVEs have Weibull distribution, then N_{eq} depends only on n and the stress field $s(\mathbf{x})$: $N_{eq} = \sum_{i=1}^{n} \langle s_i(\mathbf{x}_i) \rangle^m$. For large-size structures, the structure will fail at very small σ_N, and so only the tail of the cdf of each RVE (Weibull tail) is relevant: $P_1(\sigma_N) = (\sigma_N/s_0)^m$. Noting that $\lim_{N\to\infty}(1 + z/N)^N = e^z$, and setting $z = N_{eq}(\sigma_N/s_0)^m$, one can see that the strength cdf for large-size structures approaches the Weibull distribution, which corresponds to the case of perfectly brittle behavior, i.e.

$$P_f(\sigma_N) = 1 - \left[1 - \frac{N_{eq}(\sigma_N/s_0)^m}{N_{eq}}\right]^{N_{eq}} \underset{N_{eq}\to\infty}{\Rightarrow} 1 - e^{-N_{eq}(\sigma_N/s_0)^m} \tag{6}$$

3. Problems with non-zero threshold

For some quasibrittle materials, it has been demonstrated that the Weibull distribution with a finite threshold σ_0, i.e.,

$$P_f = 1 - e^{-[(\sigma-\sigma_0)/s_0]^m} \tag{7}$$

fits the strength histograms better than the Weibull distribution with a zero threshold [13,22,12]. However, these histograms were limited to several hundred specimens and, according to the present theory, it is likely that the upper part of histogram would not be fitted well for much broader histograms with many more tests. This is clear from the plot of Weibull's data [26] in Fig. 10 of [9]. Anyway, the excellent fit of the strength histograms obtained by the finite weakest link model with a zero threshold demonstrates that the nonzero threshold is unnecessary (besides, if one has doubts, it is the less conservative choice).

Within the framework of the present theory, the strength distributions of a structure on the macro-scale, and of an atomic lattice at the nano-scale, are linked through parallel and series couplings. It has been shown that a power-law tail with a zero threshold is a logical consequence of the activation energy control of crack length jumps in the atomic lattice, an that it is preserved by all parallel and series couplings. Therefore, the theory requires the strength cdf at the macro-continuum scale to also have a power-law tail with zero threshold.

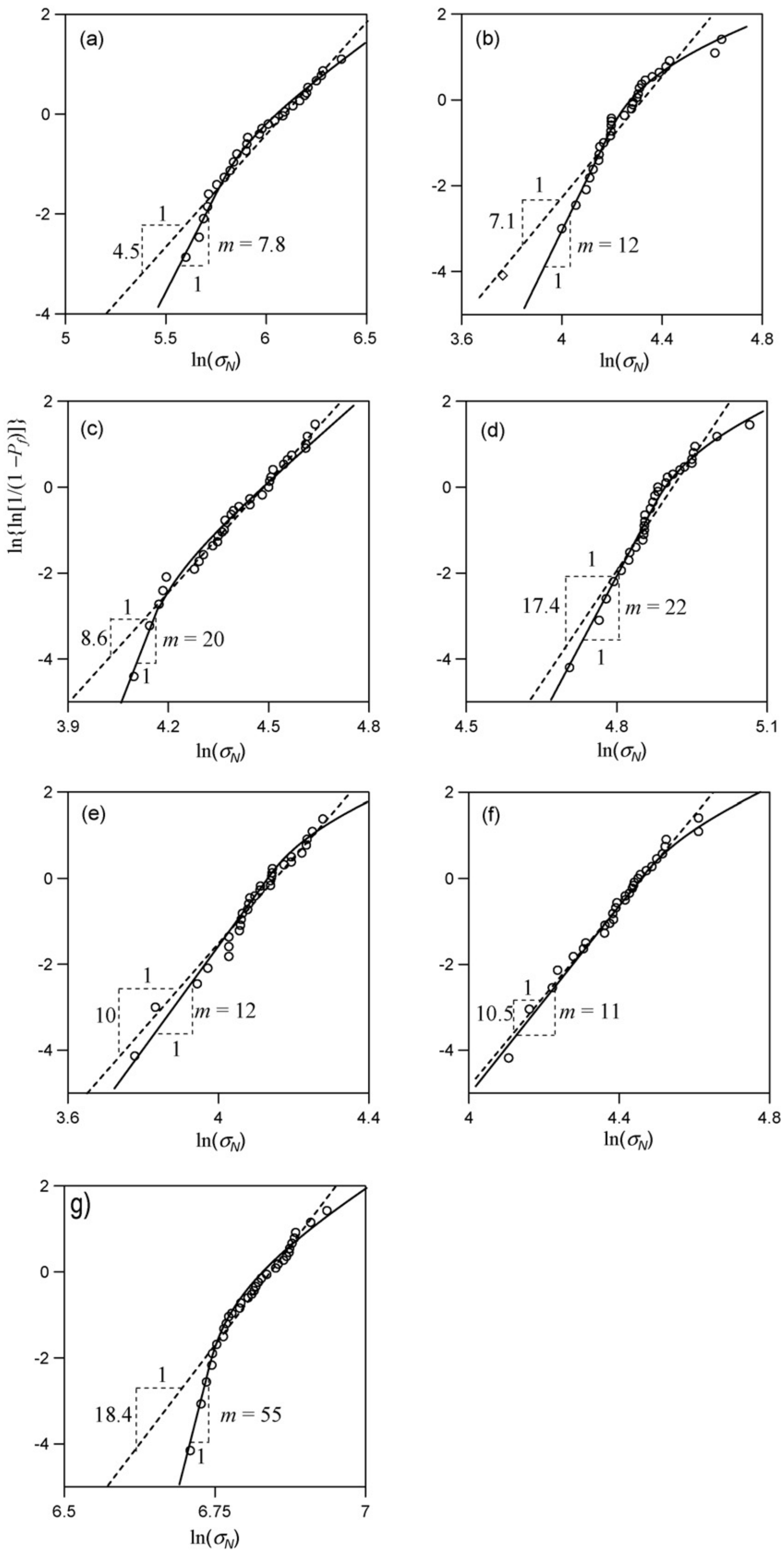

Fig. 2 – Optimum fits of experimental strength histograms of dental ceramics by the present theory and by the two-parameter Weibull distribution: (a) Alumina glass Composite, (b) Dicor, (c)IPS Empress, (d) Vitadur Alpha Core, (e) Vitadur Alpha Dentin, (f) Vitadur VMK 68 and (g) Zironia-TZP.

Table 1 – Test specimens

Product	Code	Type	Specimen size
Alumina glass Composite	AG	Lanthannum-glass-infiltrated alumina glass-ceramic	3 mm × 4 mm × 45 mm
Dicor	D	Tetrasilicic fluoromica glass-ceramic	1.5 mm × 3 mm × 30 mm
IPS Empress	IE	Leucite-reinforced porcelain	1.5 mm × 3 mm × 30 mm
Vitadur Alpha Core	VAC	Alumina-reinforced feldspathic porcelain	1.5 mm × 3 mm × 30 mm
Vitadur Alpha Dentin	VAD	Feldspathic Porcelain	1.5 mm × 3 mm × 30 mm
Vita VMK 68	VMK	Feldspathic Porcelain	1.5 mm × 3 mm × 30 mm
Zironia-TZP	Z	Partially stabilized zirconia ceramic	1.5 mm × 3 mm × 30 mm

As another argument, it may be remarked that a nonzero threshold would be in conflict with the nonlocal Weibull theory for quasibrittle materials [10,6,2,3], which underlies the well-established Type I size effect law for mean strength [7,4]). To see what size effect would result if a non-zero threshold were assumed, consider that each single RVE has a power-law tail with non-zero threshold, i.e. $P_1 = [\langle \sigma_N - \sigma_0 \rangle / s_0]^m$. Then the mean strength of large-size structures can be calculated as:

$$\bar{\sigma}_N = \int_0^\infty (1 - P_N)\, d\sigma_N = \int_0^\infty (1 - P_1)^{N_{eq}}\, d\sigma_N \tag{8}$$

$$\bar{\sigma}_N = \sigma_0 + N_{eq}^{1/m} s_0 \Gamma\left(1 + \frac{1}{m}\right) \tag{9}$$

Therefore, for increasing structure size, the mean strength will approach a constant value σ_0 and the size effect on strength will asymptotically vanish. However, well-established models such as the nonlocal model and the crack band model, predict a power-law size effect for large structure sizes [6,2,3], and these predictions are well supported by many experimental observations for quasibrittle structures [7]. In fact, the power-law for the size effect on mean strength is inextricably linked to a zero threshold in the Weibull distribution.

The problems with assuming a nonzero threshold are also apparent upon examining the existing test data for quasibrittle materials in general, not limited to dental ceramics. They are blatant when trying to fit the data on the size effect on mean strength of geometrically similar specimens. Fitting these data with the Weibull size effect gives different values of Weibull modulus m than fitting the strength histograms for one fixed size. Some researchers have even been misled to suggest that m depends on the structure size as well as geometry, although this is in principle impossible as m is a material property.

A related problem with nonzero threshold is that fitting the strength histograms for different sizes of geometrically similar specimens gives different m values. To avoid misleading erroneous results, it is, therefore, useful to test not only specimens of one size but also specimens of different sizes. The specimens must be geometrically similar and the size ratio must not be too small (compared to the coefficient of variation of strength).

4. Optimum fitting of experimental data for dental ceramics

In recent literature, one can find strength data for seven dental restorative ceramics that are commonly used in crown, veneer and inlay construction [24,15]. They represent almost the full range of commercially available dental restorative ceramics, ranging from low strength ceramics such as feldspathic ceramics, to high-tech ceramics such as Zirconia ceramics [16,14]. For each type of ceramic, four-point loaded flexural strength tests using prismatic beam specimens were reported. Table 1 shows the product name, the ceramics type, and the specimen size for each material [24,15].

Optimum least-square fits of data have been obtained by the algorithm presented in Appendix A. Fig. 2 presents the fits of the strength histograms, obtained by both the present model and the two-parameter Weibull model. The plots are made in the Weibull scale, in which the Weibull distribution is a straight line.

A salient feature of these histograms is a kink separating each histogram into two segments. Similar kinks have also been found for other quasibrittle materials such as cement mortar and fibrous composites [26,25,21,19,20,13]. Obviously, the two-parameter Weibull distribution cannot fit both segments simultaneously.

By contrast, the current model allows an excellent fit of these histograms over the entire range, with both segments and the kink location matched well. The histogram can be subdivided into three ranges of failure probability. For the low probability range, the stress is so low that the strength of all the RVEs lies in the Weibullian tail, causing that the cdf of the strength of the whole structure follows the Weibull distribution.

For the intermediate failure probability range, each RVE has a different type of cdf because of the non-uniformity of the stress field. Some of the RVEs are subjected to high stress which corresponds to the Gaussian part of the strength cdf, while the remaining RVEs still experience a low stress corresponding to the Weibull tail of the strength cdf.

For the high failure probability range, the stress is high enough for most RVEs to follow the Gaussian distribution, though with different stress value. Note that the transitions between these three portions are smooth. The transition between the first and second ranges creates in the histogram a kink. The kink point may be identified with the grafting probability, and its location is a measure of brittleness or quasibrittleness of the structure. It is impossible to identify the exact kink location (or the grafting probability) by mere visual examination of the histogram. But if the lower part of the histogram is fitted by Weibull cdf and the upper part by the Eqs. (1–3) for the weakest link model, the grafting point is unique (see Appendix A).

Many researchers attempted to determine the Weibull modulus by fitting the strength histograms that were not quite

Table 2 – Prediction of design strength

Material Code	σ_{design} (MPa) at $P_f = 0.01$			σ_{design} (MPa) at $P_f = 0.05$		
	Weibull	Current Model	Error (%)	Weibull	Current Model	Error (%)
AG	161.13	215.05	25.07	228.04	264.35	13.75
D	39.30	47.96	18.05	49.50	55	10.00
IE	52.10	59.07	11.80	62.90	64.21	2.05
VAC	104.38	108.46	3.76	114.58	116.62	1.75
VAD	40.20	42.49	5.38	47.30	48.60	2.67
VMK	55.92	56.84	1.62	65.27	65.94	1.01
Z	729.80	810.92	10.00	797.20	835.17	4.55

straight [15,24]. However, fitting a two-parameter Weibull cdf to the entire histogram will always underestimate the Weibull modulus. Since the kink (or grafting) point separates the distribution into two segments, only the slope of the lower segment, which is steeper, represents the true Weibull modulus. The error in m obtained by fitting the histograms with a two-parameter Weibull model is nearly undetectable if only specimens of one size are used, and if the structure size (compared to the inhomogeneity size or the RVE size) is so large that the kink (or the grafting point) lies at the upper margin of the histogram. This happens to be the case for some of the histogram tests analyzed here, i.e. Vita VMK 68, Vitadur Alpha Dentin, and Vitadur Alpha Core. Nevertheless, for some dental ceramics studied here (Zironia-TZP, IPS Empress and Alumina glass Composite), the kink lies in the middle range of the histogram, and the Weibull modulus identified by data fitting with the two-parameter Weibull model can be as low as one third of the correct value (Fig. 2).

To be sufficiently reliable, the restorative dental ceramics should be designed for failure probability $P_f \leq 10^{-2}$ [1,15]. If 100 tests are carried out, only one test is likely to lie below 10^{-2}. To demonstrate $P_f \leq 10^{-2}$ experimentally, one would need to obtain almost 100 realizations in the tail beyond 10^{-2}, which would require increasing the number of test about 100 times. Therefore, one would need $> 10^4$ tests to verify such a probability purely empirically, i.e., by histogram testing at real size. Therefore, to reduce the cost, it is preferable to deduce P_f from a well founded statistical model that can be calibrated by much fewer tests.

Many researchers predicted flexural strength for 1 and 5% failure probabilities based on two-parameter Weibull model calibrated by the strength histograms [15,24]. Table 2 shows the comparisons between the prediction of flexural strength by the two-parameter Weibull cdf and the present model. It is clear that, at the failure probabilities of 1 and 5% the two-parameter Weibull cdf underestimate the predicted strength. At $P_f = 5\%$, the two-parameter Weibull cdf underestimates the design strength by nearly 3% for most of the materials, and for some even by 14%. At $P_f = 1\%$, the underestimation is around 6% for most of the materials, and for some even 25%. If lower P_f were specified, the underestimation of design strength would be even greater.

Therefore, while the two-parameter Weibull cdf does give a safe design strength, it can lead to significant over-design. On the other hand, the three-parameter Weibull cdf, which has a finite threshold, can lead to a severe over-estimation of design strength.

5. Closing remarks

In closing, the present theory, which is based on atomistic fracture mechanics, gives more realistic predictions of the strength distribution of restorative dental ceramics. The two-parameter Weibull distribution used so far cannot fit the entire strength histograms and often leads to erroneous predictions of the design strength.

Finally, it should be emphasized that that the best way to calibrate the strength distribution is to test histograms not only for one size but also for some significantly different size.

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Appendix A. Optimum least-square fitting algorithm used

(1) First estimate the RVE size and calculate the stress field. According to the nonlocal model, the RVE size is roughly the double or triple of the maximum size of material inhomogeneities [8,9]. The structure may be discretized by elements having approximately the RVE size, and then the elastic stress field $s(\mathbf{x})$ can be generally obtained by finite element method. For simple structures such as the four-point flexure test considered here, the stress field can be simply calculated according to the engineering theory of beam bending.

(2) By fitting the lower portion of the histogram, which appears to be a straight line in the Weibull scale, one can obtain the Weibull modulus m and the scale parameter S of the Weibull portion of the strength cdf. If the stress field $s(\mathbf{x})$ is known, the N_{eq} value for the Weibull portion is calculated as $N_{eq} = \sum_{i=1}^{n} \langle s_i(\mathbf{x}_i) \rangle^m$. The scale parameter of the Weibull tail of the strength cdf of one RVE is obtained as $s_0 = SN_{eq}^{1/m}$. Thus the Weibull modulus m and the scale parameter s_0 for the Weibull tail are fixed.

(3) To define the entire strength cdf for one RVE, two more statistical parameters must be determined. Convenient parameters are the grafting probability P_{gr} and the coefficient of variation ω_0 of the entire strength distribution of one RVE. An empirical equation was calibrated to relate the normalized standard deviation of the Gaussian core to

P_{gr} and ω_0 [9]:

$$\delta_{Gn} = \left(\frac{\delta_G}{s_0}\right) = \exp\left\{-3.254 + 11.566\omega_0 - \left[\frac{1000P_{gr}}{108.8P_{gr} + 0.1334}\right]\omega_0^2\right\} \quad (10)$$

Then the normalized grafting stress can be calculated as

$$\varsigma_{gr} = \left(\frac{\sigma_{gr}}{s_0}\right) = [-\ln(1 - P_{gr})]^{1/m} \quad (11)$$

and the normalized mean of the Gaussian core can be obtained as

$$\mu_{Gn} = \left(\frac{\mu}{s_0}\right) = \varsigma_{gr} + \delta_{Gn}\{-2\ln[\sqrt{2\pi}m\delta_{Gn}\varsigma_{gr}^{m-1}e^{-\varsigma_{gr}^m}]\}^{1/2} \quad (12)$$

Finally, the normalizing parameter r_f can be calculated as

$$r_f = (1 - P_{gr})\left[1 - \Phi\left(\frac{\varsigma_{gr} - \mu_{Gn}}{\delta_{Gn}}\right)\right]^{-1} \quad (13)$$

where $\Phi(x) = (1/\sqrt{2\pi})\int_{-\infty}^{x} e^{-x^2/2}\,dx$ = error function representing the standard Gaussian cdf. Knowing all these parameters, one can obtain the entire strength cdf of one RVE from Eqs. 1 and 2 .

(4) P_{gr} and ω_0 are the free parameters to adjust so as to obtain the optimum fit of the given histogram by the method of least squares. For a given set of P_{gr} and ω_0, the failure probability of the structure P_f can be calculated from Eq. (4). Let N = number of data points in the histogram and $y_i(\sigma_i)(i = 1, \cdots, N)$ = failure probability corresponding to stress σ_i. For P_{gr}, the reasonable parameter range is $(10^{-5} < P_{gr} < 10^{-2})$, and for ω_0 it is $(0.05 < \omega_0 < 1.0)$. Since these ranges are relatively narrow, the simplest programming is to choose sets of discrete values such as $P_{gr} = 10^{-5.00}, 10^{-4.75}, 10^{-4.50}, 10^{-4.25}, \ldots 10^{-2.25}, 10^{-2.00}$ and $\omega_0 = 0.05, 0.10, 0.15, 0.20, \ldots 1.00$. There are only 520 possible combinations of these values, for each of which the computer calculates the objective function

$$F = \sum_{i=1}^{N} [\ \ln(\ \ln\left\{\frac{1}{1 - P_f(\sigma_i)}\right\}) - \ln(\ \ln\left\{\frac{1}{1 - y_i(\sigma_i)}\right\})\ ]^2 \quad (14)$$

and then chooses the combination giving the minimum F. Alternatively, the Levenberg–Marquardt optimization algorithm can be used to locate the minimum of F precisely.

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