

# FRACTURE AND SIZE EFFECT in Concrete and Other Quasibrittle Materials

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# Contents

Preface	v
Vector and Tensor Notation	vii
<b>1 Why Fracture Mechanics?</b>	<b>1</b>
1.1 Historical Perspective	1
1.1.1 Classical Linear Theory	1
1.1.2 Classical Nonlinear Theories	3
1.1.3 Continuum-Based Theories	4
1.1.4 Trends in Fracture of Quasibrittle Materials	5
1.2 Reasons for Fracture Mechanics Approach	5
1.2.1 Energy Required for Crack Formation	5
1.2.2 Objectivity of Analysis	5
1.2.3 Lack of Yield Plateau	7
1.2.4 Energy Absorption Capability and Ductility	7
1.2.5 Size Effect	7
1.3 Sources of Size Effect on Structural Strength	9
1.4 Quantification of Fracture Mechanics Size Effect	11
1.4.1 Nominal Stress and Nominal Strength	11
1.4.2 Size Effect Equations	13
1.4.3 Simple Explanation of Fracture Mechanics Size Effect	13
1.5 Experimental Evidence for Size Effect	16
1.5.1 Structures with Notches or Cracks	18
1.5.2 Structures Without Notches or Cracks	19
<b>2 Essentials of LEFM</b>	<b>23</b>
2.1 Energy Release Rate and Fracture Energy	23
2.1.1 The General Energy Balance	24
2.1.2 Elastic Potentials and Energy Release Rate	25
2.1.3 The Linear Elastic Case and the Compliance Variation	28
2.1.4 Graphical Representation of Fracture Processes	30
2.1.5 Rice's $J$ -Integral	31
2.1.6 Fracture Criterion and Fracture Energy	34
2.2 LEFM and Stress Intensity Factor	37
2.2.1 The Center Cracked Infinite Panel and the Near-Tip Fields	37
2.2.2 The General Near-Tip Fields and Stress Intensity Factors	39
2.2.3 Relationship Between $K_I$ and $\mathcal{G}$	40
2.2.4 Local Fracture Criterion for Mode I: $K_{Ic}$	41
2.3 Size Effect in Plasticity and in LEFM	42
2.3.1 Size Effect for Failures Characterized by Plasticity, Strength, or Allowable Stress	43
2.3.2 General Forms of the Expressions for $K_I$ and $\mathcal{G}$	44
2.3.3 Size Effect in LEFM	45
2.3.4 Structures Failing at Very Small Cracks Whose Size is a Material Property	46

<b>3 Determination of LEFM Parameters</b>	<b>49</b>
3.1 Setting up Solutions from Closed-Form Expressions	49
3.1.1 Closed-Form Solutions from Handbooks	49
3.1.2 Superposition Methods	51
3.2 Approximate Energy-Based Methods	55
3.2.1 Examples Approximately Solvable by Bending Theory	55
3.2.2 Approximation by Stress Relief Zone	56
3.2.3 Herrmann's Approximate Method to Obtain $\mathcal{G}$ by Beam Theory	58
3.2.4 Subsurface Cracking in Compression by Buckling	59
3.3 Numerical and Experimental Procedures to Obtain $K_I$ and $\mathcal{G}$	60
3.3.1 Numerical Procedures	60
3.3.2 Experimental Procedures	63
3.4 Experimental determination of $K_{Ic}$ and $G_f$	64
3.5 Calculation of Displacements from $K_I$ -Expressions	67
3.5.1 Calculation of the Displacement	67
3.5.2 Compliances, Energy Release Rate, and Stress Intensity Factor for a System of Loads	68
3.5.3 Calculation of the Crack Mouth Opening Displacement	69
3.5.4 Calculation of the Volume of the Crack	71
3.5.5 Calculation of the Crack Opening Profile	72
3.5.6 Bueckner's Expression for the Weight Function	73
<b>4 Advanced Aspects of LEFM</b>	<b>75</b>
4.1 Complex Variable Formulation of Plane Elasticity Problems	75
4.1.1 Navier's Equations for the Plane Elastic Problem	75
4.1.2 Complex Functions	76
4.1.3 Complex Form of Hooke's and Navier's Equations	77
4.1.4 Integration of Navier's Equation: Complex Potentials	77
4.2 Plane Crack Problems and Westergaard's Stress Function	80
4.2.1 Westergaard Stress Function	80
4.2.2 Westergaard's Solution of Center-Cracked Infinite Panel	80
4.2.3 Near-Tip Expansion for the Center-Cracked Panel	82
4.3 The General Near-Tip Fields	83
4.3.1 In-Plane Near-Tip Asymptotic Series Expansion	83
4.3.2 The Stress Intensity Factors	85
4.3.3 Closer View of the Near-Tip Asymptotic Expansion for Mode I	86
4.3.4 The Antiplane Shear Mode	87
4.3.5 Antiplane Near-Tip Asymptotic Series Expansion	88
4.3.6 Summary: The General Singular Near-Tip Fields	89
4.4 Path-Independent Contour Integrals	90
4.4.1 Path Independence of the $J$ -Integral	90
4.4.2 Further Contour Integral Expressions for $\mathcal{G}$ in LEFM	91
4.4.3 Further Proof of the Irwin Relationship	92
4.4.4 Other Path-Independent Integrals	93
4.4.5 Exercises	94
4.5 Mixed Mode Fracture Criteria	94
4.5.1 Maximum Energy Release Rate Criterion	95
4.5.2 Maximum Principal Stress Criterion	96
Appendix: Strain Energy Density Criterion	98

<b>5 Equivalent Elastic Cracks and R-curves</b>	<b>101</b>
5.1 Variability of Apparent Fracture Toughness for Concrete	101
5.2 Types of Fracture Behavior and Nonlinear Zone	103
5.2.1 Brittle, Ductile, and Quasibrittle Behavior	104
5.2.2 Irwin's Estimate of the Size of the Inelastic Zone	105
5.2.3 Estimate of the Fracture Zone Size for quasibrittle Materials	106
5.3 The Equivalent Elastic Crack Concept	108
5.3.1 Estimate of the Equivalent LEFM Crack Extension	109
5.3.2 Deviation from LEFM	109
5.3.3 Intrinsic Size	110
5.3.4 How Large the Size Must Be for LEFM to Apply?	111
5.4 Fracture Toughness Determinations Based on Equivalent Crack Concepts	112
5.4.1 Compliance Calibration of Equivalent Crack Length	112
5.4.2 Modified Compliance Calibration Method	113
5.4.3 Nallathambi-Karihaloo Method	114
5.5 Two-Parameter Model of Jenq and Shah	116
5.5.1 The Basic Equations of Jenq-Shah Model	117
5.5.2 Experimental Determination of Jenq-Shah Parameters	119
5.6 R-Curves	121
5.6.1 Definition of an R- $\Delta a$ Curve	121
5.6.2 Description of the Fracture Process	123
5.6.3 The Peak Load Condition	124
5.6.4 Positive and Negative Geometries	126
5.6.5 R-Curve Determination from Tests	126
5.6.6 R-CTOD Curves	128
5.7 Stability Analysis in the R-Curve Approach	130
5.7.1 Stability under Load-Control Conditions	130
5.7.2 Stability under Displacement-Control Conditions	131
5.7.3 Stability under Mixed-Control Conditions	131
<b>6 Determination of Fracture Properties From Size Effect</b>	<b>135</b>
6.1 Size Effect in Equivalent Elastic Crack Approximations	135
6.1.1 Size Effect in the Large Size Range	135
6.1.2 Size Effect in the Jenq-Shah Model	136
6.2 Size Effect Law in Relation to Fracture Characteristics	138
6.2.1 Defining Objective Fracture Properties	138
6.2.2 Determination of Fracture Parameters from Size Effect	138
6.2.3 Determination of Fracture Parameters from Size and Shape Effects and Zero Brittleness Method	139
6.2.4 Intrinsic Representation of the Size Effect Law	139
6.3 Size Effect Method: Detailed Experimental Procedures	140
6.3.1 Outline of the Method	140
6.3.2 Regression Relations	140
6.3.3 RILEM Recommendation Using the Size Effect Method: Experimental Procedure	143
6.3.4 RILEM Recommendation Using the Size Effect Method: Calculation Procedure	144
6.3.5 Performance of the Size Effect Method	147
6.3.6 Improved Regression Relations	147
6.4 Determination of R-Curve from Size Effect	150
6.4.1 Determination of R-Curve from Size Effect	150
6.4.2 Determination of R-Curve from Bažant's Size Effect Law	152
6.4.3 Determination of the Structural Response from the R-Curve	154

<b>7 Cohesive Crack Models</b>	<b>157</b>
7.1 Basic Concepts in Cohesive Crack Model	157
7.1.1 Hillerborg's Approach: The Cohesive Crack as a Constitutive Relation	158
7.1.2 Other Approaches to Cohesive Cracks	160
7.1.3 Softening Curve, Fracture Energy, and Other Properties	162
7.1.4 Extensions of the Cohesive Crack Model	164
7.1.5 Cohesive Cracks with Tip Singularity	165
7.1.6 Cohesive Cracks with Bulk Energy Dissipation	165
7.2 Cohesive Crack Models Applied to Concrete	167
7.2.1 Softening Curves for Concrete	167
7.2.2 Experimental Aspects	170
7.2.3 Computational Procedures for Cohesive Crack Analysis	172
7.2.4 Size Effect Predictions	175
7.2.5 Cohesive Crack Models in Relation to Effective Elastic Crack Models	177
7.2.6 Correlation of Cohesive Crack with Bažant's and Jenq and Shah's Models	178
7.3 Experimental Determination of Cohesive Crack Properties	180
7.3.1 Determination of the Tensile Strength	181
7.3.2 Determination of the Initial Part of the Softening Curve	182
7.3.3 Determination of Fracture Energy $G_F$	184
7.3.4 Determination of a Bilinear Softening Curve	188
7.4 Pseudo-Boundary-Integral Methods for Mode I Crack Growth	190
7.4.1 The Underlying Problem	190
7.4.2 Petersson's Influence Method	191
7.4.3 Improved Solution Algorithm of Planas and Elices	192
7.4.4 Smeared-Tip Method	193
7.4.5 Scaling of the Influence Matrices	195
7.4.6 Inclusion of Shrinkage or Thermal Stresses	196
7.4.7 Inclusion of a Crack-Tip Singularity	197
7.4.8 Computation of Other Variables	198
7.4.9 Limitations of the Pseudo-Boundary Integral (PBI) Methods	199
7.5 Boundary-Integral Methods for Mode I Crack Growth	199
7.5.1 A Basic Boundary Integral Formulation	199
7.5.2 Size-Dependence of the Equations	202
7.5.3 The Dugdale and Rectangular Softening Cases	203
7.5.4 Eigenvalue Analysis of the Size Effect	204
7.5.5 Eigenvalue Analysis of Stability Limit and Ductility of Structure	206
7.5.6 Smeared-Tip Superposition Method	207
7.5.7 Asymptotic Analysis	209
<b>8 Crack Band Models and Smeared Cracking</b>	<b>213</b>
8.1 Strain Localization in the Series Coupling Model	213
8.1.1 Series Coupling of Two Equal Strain Softening Elements: Imperfection Approach	214
8.1.2 Series Coupling of Two Equal Strain Softening Elements: Thermodynamic Approach	215
8.1.3 Mean Stress and Mean Strain	215
8.1.4 Series Coupling of $N$ Equal Strain Softening Elements	216
8.2 Localization of Strain in a Softening Bar	217
8.2.1 Localization and Mesh Objectivity	217
8.2.2 Localization in an Elastic-Softening Bar	218
8.2.3 Summary: Necessity of Localization Limiters	219
8.3 Basic Concepts in Crack Band Models	220
8.3.1 Elastic-Softening Crack Band Models	220
8.3.2 Band Models with Bulk Dissipation	222
8.3.3 Unloading and Reloading	223
8.3.4 Fracture Energy for Crack Bands With Prepeak Energy Dissipation	224

8.3.5	Simple Numerical Issues	225
8.3.6	Crack Band Width	226
8.4	Uniaxial Softening Models	228
8.4.1	Elastic-Softening Model with Stiffness Degradation	228
8.4.2	Elastic-Softening Model with Strength Degradation	229
8.4.3	Elastic-Softening Model with Stiffness and Strength Degradation	229
8.4.4	A Simple Continuum Damage Model	230
8.4.5	Introducing Inelasticity Prior to the Peak	231
8.4.6	Crack Closure in Reverse Loading and Compression	231
8.4.7	Introducing Other Inelastic Effects	232
8.5	Simple Triaxial Strain-Softening Models for Smeared Cracking	234
8.5.1	Cracking of Single Fixed Orientation: Basic Concepts	234
8.5.2	Secant Approach to Cracking of Fixed Orientation	235
8.5.3	Scalar Damage Model for Cracking of Fixed Orientation	237
8.5.4	Incremental Approach to Cracking of Fixed Orientation	238
8.5.5	Multi-Directional Fixed Cracking	239
8.5.6	Rotating Crack Model	240
8.5.7	Generalized Constitutive Equations with Softening	242
8.5.8	Mazars' Scalar Damage Model	243
8.5.9	Rankine Plastic Model with Softening	243
8.5.10	A Simple Model with Stiffness and Strength Degradation	244
8.6	Crack Band Models and Smeared Cracking	246
8.6.1	Stress-Strain Relations for Elements of Arbitrary Size	246
8.6.2	Skew Meshes: Effective Width	248
8.6.3	Stress Lock-In	250
8.6.4	Use of Elements of Large Size	251
8.6.5	Energy Criterion for Crack Bands with Sudden Cracking	252
8.7	Comparison of Crack Band and Cohesive Crack Approaches	255
8.7.1	Localized fracture: Moot Point Computationally	255
8.7.2	Nonlocalized Fracture: Third Parameter	255
8.7.3	Relation to Micromechanics of Fracture	257
8.7.4	Fracture of Arbitrary Direction	258
9	Advanced Size Effect Analysis	261
9.1	Size Effect Law Refinements	261
9.1.1	The Generalized Energy Balance Equation	261
9.1.2	Asymptotic Analysis for Large Sizes	263
9.1.3	Matching to the Effective Crack Model	263
9.1.4	Asymptotic Formula for Small Sizes and Its Asymptotic Matching with Large Sizes	264
9.1.5	Asymptotic Aspects of Bažant's Extended Size Effect Law	265
9.1.6	Size Effect for Failures at Crack Initiation from Smooth Surface	266
9.1.7	Universal Size Effect Law for Cracked and Uncracked Structures	268
9.1.8	Asymptotic Scaling Law for Many Loads	269
9.1.9	Asymptotic Scaling Law for a Crack with Residual Bridging Stress	270
9.2	Size Effect in Notched Structures Based on Cohesive Crack Models	271
9.2.1	The General Size Effect Equation	271
9.2.2	Asymptotic Analysis for Large Sizes	273
9.2.3	Asymptotic Analysis for Small Sizes	274
9.2.4	Interpolation Formula	275
9.2.5	Application to Notched Beams with Linear Softening	277
9.2.6	Application to Notched Beams with Bilinear Softening	277
9.2.7	Experimental Evidence	279
9.3	Size Effect on the Modulus of Rupture of Concrete	280
9.3.1	Notation and Definition of the Rupture Modulus	280
9.3.2	Modulus of Rupture Predicted by Cohesive Cracks	281

9.3.3	Further Analysis of the Influence of the Initial Softening	284
9.3.4	Modulus of Rupture According to Bažant and Li's Model, Bažant's Universal Size Effect Law, and Zero-Brittleness Method	284
9.3.5	Modulus of Rupture Predicted by Jenq-Shah Model	287
9.3.6	Carpinteri's Multifractal Scaling Law	288
9.3.7	Comparison With Experiments and Final Remarks	289
9.4	Compression Splitting Tests of Tensile Strength	291
9.4.1	Cracking Process in Stable Splitting Tests	292
9.4.2	Modified Bažant's Size Effect Law	294
9.4.3	Size Effect Predicted by Jenq-Shah Model	295
9.4.4	Size Effect Predicted by Cohesive Crack Models	296
9.5	Compression Failure Due to Propagation of Splitting Crack Band	297
9.5.1	Concepts and Mechanisms of Compression Fracture	297
9.5.2	Energy Analysis of Compression Failure of Column	300
9.5.3	Asymptotic Effect for Large Size	305
9.5.4	Size Effect Law for Axial Compression of Stocky Column	305
9.5.5	Effect of Buckling Due to Slenderness	307
9.5.6	Comparison with Experimental Data	308
9.5.7	The Question of Variation of Microcrack Spacing with Size $D$	310
9.5.8	Special Case of Compression with Transverse Tension	310
9.5.9	Distinction Between Axial Splitting and Failure Appearing as Shear	311
9.6	Scaling of Fracture of Sea Ice	312
9.6.1	Derivation of Size Effect for Thermal Bending Fracture of Ice Plate	314
9.6.2	General Proof of 3/8-Power Scaling Law	316
10	Brittleness and Size Effect in Structural Design	319
10.1	General Aspects of Size Effect and Brittleness in Concrete Structures	319
10.1.1	Conditions for Extending Bažant's Size Effect Law to Structures	320
10.1.2	Brittleness Number	321
10.1.3	Brittleness of High Strength Concrete	323
10.1.4	Size Effect Correction to Ultimate Load Formulas in Codes	323
10.1.5	Size Effect Correction to Strength-Based Formulas	324
10.1.6	Effect of Reinforcement	325
10.2	Diagonal Shear Failure of Beams	326
10.2.1	Introduction	326
10.2.2	Bažant-Kim-Sun Formulas	327
10.2.3	Gustafsson-Hillerborg Analysis	330
10.2.4	LEFM Analyses of Jenq and Shah and of Karihaloo	331
10.2.5	Finite Element Solutions with Nonlocal Microplane Model	334
10.2.6	Influence of Prestressing on Diagonal Shear Strength	334
10.3	Fracturing Truss Model for Shear Failure of Beams	335
10.3.1	Basic Hypotheses of Fracturing Truss Model	336
10.3.2	Analysis Based on Stress Relief Zone and Strain Energy for Longitudinally Reinforced Concrete Beams Without Stirrups	337
10.3.3	Analysis Based on Stress Relief Zone and Strain Energy for Longitudinally Reinforced Concrete Beams With Stirrups	341
10.3.4	Analysis Based on Stress Redistribution and Complementary Energy	344
10.3.5	Size Effect on Nominal Stress at Cracking Load	346
10.3.6	Conclusions	349
10.4	Reinforced Beams in Flexure and Minimum Reinforcement	349
10.4.1	Lightly Reinforced Beams: Overview	349
10.4.2	Models Based on LEFM	350
10.4.3	Simplified Cohesive Crack Models	356
10.4.4	Models Based on Cohesive Cracks	357
10.4.5	Formulas for Minimum Reinforcement Based on Fracture Mechanics	363

10.5	Other Structures	365
10.5.1	Torsional Failure of Beams	365
10.5.2	Punching Shear Failure of Slabs	366
10.5.3	Anchor Pullout	367
10.5.4	Bond and Slip of Reinforcing Bars	368
10.5.5	Beam and Ring Failures of Pipes	371
10.5.6	Concrete Dams	372
10.5.7	Footings	375
10.5.8	Crack Spacing and Width, with Application to Highway Pavements	376
10.5.9	Keyed Joints	377
10.5.10	Fracture in Joints	377
10.5.11	Break-Out of Boreholes	379
10.5.12	Hillerborg's Model for Compressive Failure in Concrete Beams	380
<b>11</b>	<b>Effect of Time, Environment, and Fatigue</b>	<b>383</b>
11.1	Phenomenology of Time-Dependent Fracture	384
11.1.1	Types of Time-Dependent Fracture	384
11.1.2	Influence of Loading Rate on Peak Load and on Size Effect	385
11.1.3	Load Relaxation	386
11.1.4	Creep Fracture Tests	388
11.1.5	Sudden Change of Loading Rate	388
11.1.6	Dynamic Fracture	389
11.2	Activation Energy Theory and Rate Processes	390
11.2.1	Elementary Rate Constants	391
11.2.2	Physical Rate Constants	391
11.2.3	Fracture as a Rate Process	394
11.2.4	General Aspects of Isothermal Crack Growth Analysis	395
11.2.5	Load-Controlled Processes for Power-Law Rate Equation	396
11.2.6	Displacement-Controlled Processes for Power-Law Rate Equation	397
11.3	Some Applications of the Rate Process Theory to Concrete Fracture	398
11.3.1	Effect of Temperature on Fracture Energy of Concrete	398
11.3.2	Effect of Humidity on the Fracture Energy of Concrete	399
11.3.3	Time-Dependent Generalization of R-Curve Model	401
11.3.4	Application of the Time-Dependent $R$ -Curve Model to Limestone	403
11.4	Linear Viscoelastic Fracture Mechanics	404
11.4.1	Uniaxial Linear Viscoelasticity	404
11.4.2	Compliance Functions for Concrete	407
11.4.3	General Linear Viscoelastic Constitutive Equations	408
11.4.4	The Correspondence Principle (Elastic-Viscoelastic Analogy)	408
11.4.5	Near-Tip Stress and Displacement Fields for a Crack in a Viscoelastic Structure	409
11.4.6	Crack Growth Resistance in a Viscoelastic Medium	412
11.4.7	Steady Growth of a Cohesive Crack with Rectangular Softening in an Infinite Viscoelastic Plate	413
11.4.8	Analysis of Crack Growth in a Viscoelastic Plate	416
11.4.9	Crack Growth Analysis at Controlled Displacement	417
11.5	Rate-Dependent R-Curve Model with Creep	418
11.5.1	Basic Equations	418
11.5.2	Approximate Solution for Small Crack Extensions	419
11.5.3	Comparison with Tests	419
11.5.4	Rate-Dependence of Process Zone Length	420
11.5.5	Sudden Change of Loading Rate and Load Relaxation	420
11.5.6	Summary	422
11.6	Time Dependent Cohesive Crack and Crack Band Models	422
11.6.1	Time-Independent Softening in a Viscoelastic Body	423
11.6.2	Time-Dependent Softening in an Elastic Body	424

11.6.3	Time-Dependent Cohesive Crack Model	425
11.6.4	Analysis of Viscoelastic Structure with Rate-Dependent Cohesive Crack by Finite Elements	426
11.6.5	Analysis of Viscoelastic Structure with Rate-Dependent Cohesive Crack by Compliance Functions	428
11.7	Introduction to Fatigue Fracture and Its Size Dependence	429
11.7.1	Fatigue Crack Growth in Metals	430
11.7.2	Fatigue Crack Growth in Brittle Materials	431
11.7.3	Size Effect in Fatigue Crack Growth in Concrete	432
11.7.4	Fatigue Description by History-Dependent Cohesive Models	434
<b>12</b>	<b>Statistical Theory of Size Effect and Fracture Process</b>	<b>437</b>
12.1	Review of Classical Weibull Theory	439
12.1.1	The Weakest-Link Discrete Model	439
12.1.2	The Weakest-Link Model for Continuous Structures under Uniaxial Stress	440
12.1.3	The Weibull Statistical Probability Distribution	441
12.1.4	Structures with Nonhomogeneous Uniaxial Stress	443
12.1.5	Generalization to Triaxial Stress States	445
12.1.6	Independent Failure Mechanisms: Additivity of the Concentration Function	446
12.1.7	Effective Uniaxial Stress	447
12.1.8	Summary: Nonhomogeneous States of Stress	447
12.2	Statistical Size Effect due to Random Strength	449
12.2.1	General Strength Probability Distribution and Equivalent Uniaxial Volume	449
12.2.2	Statistical Size Effect Laws	451
12.2.3	Divergence of Weibull Failure Probability for Sharply Cracked Bodies	452
12.2.4	The Effect of Surface Flaws	454
12.3	Basic Criticisms of Classical Weibull-Type Approach	456
12.3.1	Stress Redistribution	456
12.3.2	Equivalence to Uniaxially Stressed Bar	457
12.3.3	Differences between Two- and Three-Dimensional Geometric Similarities	458
12.3.4	Energy Release Due to Large Stable Crack Growth	459
12.3.5	Spatial Correlation	460
12.3.6	Summary of the Limitations	460
12.4	Handling of Stress Singularity in Weibull-Type Approach	460
12.4.1	A Simplified Approach to Crack Tip Statistics	461
12.4.2	Generalization of the Thickness Dependence of the Crack Tip Statistics	462
12.4.3	Asymptotic Size Effect	463
12.4.4	Extending the Range: Bulk Plus Core Statistics	463
12.4.5	More Fundamental Approach Based on Nonlocal Concept	464
12.5	Approximate Equations for Statistical Size Effect	465
12.5.1	Bažant-Xi Empirical Interpolation Between Asymptotic Size Effects	465
12.5.2	Determination of Material Parameters	465
12.5.3	The Question of Weibull Modulus $m$ for the Fracture-Process Zone	466
12.5.4	Comparison with Test Results	466
12.5.5	Planas' Empirical Interpolation Between Asymptotic Size Effects	467
12.5.6	Limitations of Generalized Weibull Theory	470
12.6	Another View: Crack Growth in an Elastic Random Medium	470
12.6.1	The Strongest Random Barrier Model	471
12.6.2	The Statistical R-Curve	472
12.6.3	Finite Bodies	472
12.6.4	Fréchet's Failure Probability Distribution	474
12.6.5	Random $R$ -curve	476
12.6.6	Limitations of the Random Barrier Model	479
12.7	Fractal Approach to Fracture and Size Effect	479
12.7.1	Basic Concepts on Fractals	480

12.7.2	Invasive Fractal and Multifractal Size Effect for $G_F$	482
12.7.3	Lacunar Fractal and Multifractal Size Effect for $\sigma_{Nu}$	482
12.7.4	Fracture Analysis of Fractal Crack Propagation	483
12.7.5	Bažant's Analysis of Fractal Crack Initiation	485
12.7.6	Is Fractality the Explanation of Size Effect?	486
<b>13</b>	<b>Nonlocal Continuum Modeling of Damage Localization</b>	<b>489</b>
13.1	Basic Concepts in Nonlocal Approaches	490
13.1.1	The Early Approaches	490
13.1.2	Models with Nonlocal Strain	491
13.1.3	Gradient Models	492
13.1.4	A Simple Family of Nonlocal Models	493
13.1.5	A Second-Order Differential Model	495
13.1.6	An Integral-Type Model of the First Kind	496
13.1.7	An Integral-Type Model of the Second Kind	497
13.1.8	Nonlocal Damage Model	498
13.2	Triaxial Nonlocal Models and Applications	501
13.2.1	Triaxial Nonlocal Smeared Cracking Models	502
13.2.2	Triaxial Nonlocal Models with Yield Limit Degradation	502
13.2.3	Nonlocal Microplane Model	506
13.2.4	Determination of Characteristic Length	506
13.3	Nonlocal Model Based on Micromechanics of Crack Interactions	507
13.3.1	Nonlocality Caused by Interaction of Growing Microcracks	507
13.3.2	Field Equation for Nonlocal Continuum	510
13.3.3	Some Alternative Forms and Properties of the Nonlocal Model	511
13.3.4	Admissibility of Uniform Inelastic Stress Fields	513
13.3.5	Gauss-Seidel Iteration Applied to Nonlocal Averaging	514
13.3.6	Statistical Determination of Crack Influence Function	515
13.3.7	Crack Influence Function in Two Dimensions	517
13.3.8	Crack Influence Function in Three Dimensions	520
13.3.9	Cracks Near Boundary	522
13.3.10	Long-Range Decay and Integrability	523
13.3.11	General Formulation: Tensorial Crack Influence Function	523
13.3.12	Constitutive Relation and Gradient Approximation	524
13.3.13	Localization of Oriented Cracking into a Band	525
13.3.14	Summary	525
<b>14</b>	<b>Material Models for Damage and Failure</b>	<b>527</b>
14.1	Microplane Model	528
14.1.1	Macro-Micro Relations	529
14.1.2	Volumetric-Deviatoric Split of the Microstrain and Microstress Vectors	532
14.1.3	Elastic Response	533
14.1.4	Nonlinear Microplane Behavior and the Concept of Stress-Strain Boundaries	535
14.1.5	Numerical Aspects	537
14.1.6	Constitutive Characterization of Material on Microplane Level	538
14.1.7	Microplane Model for Finite Strain	540
14.1.8	Summary of Main Points	542
14.2	Calibration by Test Data, Verification and Properties of Microplane Model	543
14.2.1	Procedure for Delocalization of Test Data and Material Identification	543
14.2.2	Calibration of Microplane Model and Comparison with Test Data	545
14.2.3	Vertex Effects	545
14.2.4	Other Aspects	547
14.3	Nonlocal Adaptation of Microplane Model or Other Constitutive Models	548
14.4	Particle and Lattice Models	550
14.4.1	Truss, Frame, and Lattice Models	552

14.4.2	Directional Bias	554
14.4.3	Examples of Results of Particle and Lattice Models	555
14.4.4	Summary and Limitations	559
14.5	Tangential Stiffness Tensor Via Solution of a Body with Many Growing Cracks	560
<b>References</b>		<b>565</b>
<b>Reference Citation Index</b>		<b>599</b>
<b>Index</b>		<b>607</b>

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# Preface

Our book is intended to serve as both a textbook for graduate level courses in engineering and a reference volume for engineers and scientists. We assume that the reader has the background of the B.S. level mechanics courses in the departments of civil, mechanical, or aerospace engineering. Aside from synthesizing the main results already available in the literature, our book also contains some new research results not yet published and many original derivations.

The subject of our book is important to structural, geotechnical, mechanical, aerospace, nuclear, and petroleum engineering, as well as materials science and geophysics. In our exposition of this subject, we try to proceed from simple to complex, from special to general. We try to be as concise as possible and use the lowest level of mathematics necessary to treat the subject clearly and accurately. We include the derivations or proofs of all the important results, as well as their physical justifications. We also include a large number of fully worked out examples and an abundance of exercise problems, the harder ones with hints. Our hope is that the reader will gain from the book true understanding rather than mere knowledge of the facts.

A special feature of our book is the theory of scaling of the failure loads of structures, and particularly the size effect on the strength of structures. We present a systematic exposition of this currently hot subject, which has gained prominence in current research. It has been only two decades that the classical model of size effect, based on Weibull-type statistical theory of random material strength, was found to be inadequate in the case of quasibrittle materials. Since then, a large body of results has been accumulated and is scattered throughout many periodicals and proceedings. We attempt to bring it together in a single volume. In treating the size effect, we try to be comprehensive, dealing even with aspects such as statistical and fractal, which are not normally addressed in the books on fracture mechanics.

Another special feature of our book is the emphasis on quasibrittle materials. These include concrete, which is our primary concern, as well as rocks, toughened ceramics, composites of various types, ice, and other materials. Owing to our concern with the size effect and with quasibrittle fracture, much of the treatment of fracture mechanics in our book is different from the classical treatises, which were concerned primarily with metals.

In its scope, our book is considerably larger than the subject matter of a single semester-length course. A graduate level course on fracture of concrete, with proper treatment of the size effect and coverage relevant also to other quasibrittle materials, may have the following contents: Chapter 1, highlights of Chapters 2, 3, and 4, then a thorough presentation of the main parts of Chapters 5, 6, 7, and 8, parts of Chapters 9 and 12, and closing with mere comments on Chapters 10, 11, and 13. A quarter-length course obviously requires a more reduced coverage.

The book can also serve as a text for a basic course on fracture mechanics. In that case, the course consists of a thorough coverage of Section 1.1 and Chapters 2, 3, 4, 5, and 7.

Furthermore, the book can be used as a text for a course on the scaling of fracture (i.e., the size effect), as a follow-up to the aforementioned basic course on fracture mechanics (or to courses on fracture mechanics based on other books). In that case, the coverage of this second course may be as follows: the rest of Chapters 1 and 5, a thorough exposition of Chapter 6, the rest of Chapters 7 and 8, much of Chapter 9, followed by highlights only of Chapter 10, bits of Chapter 11, and a thorough coverage of Chapter 12.

Chapters 13 and 14, the detailed coverage of which is not included in the foregoing course outlines, represent extensions important for computational modeling of fracture and size effect in structures. They alone can represent a short course, or they can be appended to the course on fracture of concrete or the course on scaling of fracture, although at the expense of the depth of coverage of the preceding chapters.

We were stimulated to write this book by our teaching of various courses on fracture mechanics, damage, localization, material instabilities, and scaling.<sup>1</sup> Our collaboration on this book began already in 1990, but had to proceed with many interruptions, due to extensive other commitments and duties. Most of the book was written between 1992 and 1995.

Our book draws heavily from research projects at Northwestern University funded by the Office of Naval Research, National Science Foundation, Air Force Office of Scientific Research, Waterways Experiment Station of the U.S. Army Corps of Engineers, Argonne National Laboratory, Department of Energy, and Sandia National Laboratories, as well as from research projects at the Universidad Politécnica de Madrid, funded by Dirección General de Investigación Científica y Técnica (Spain) and Comisión Interministerial de Ciencia y Tecnología (Spain). We are grateful to these agencies for their support.

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Z.P.B. and J.P.  
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<sup>1</sup>In the case of the first author: The course on Fracture of Concrete, introduced at Northwestern University in 1988, and intensive short courses on these subjects taught at Politecnico di Milano (1981, 1993, 1997), Swiss Federal Institute of Technology, Lausanne (1987, 1989, 1994), Ecole Normale Supérieure de Cachan, France (1992), and Lulea University, Sweden (1994). In the case of the second author: The undergraduate courses on Fracture Mechanics and Continuum Mechanics and the doctoral-level courses of Physics of Continuum Media and Advanced Fracture Mechanics at the Universidad Politécnica de Madrid, and intensive short courses on Fracture Mechanics taught at Universidad Politécnica and at Universidad Carlos III in Madrid (1994, 1995), and at Universidad de la Plata, Argentina (1995).