Bažant, Z.P. (1993). Discussion of "Use of engineering strain and Trefftz theory in buckling of columns" by C.M. Wang and W.A.M. Alwis, ASCE *J. of Engineering Mechanics*, 119(12), 2536--2537.

Use of Engineering Strain and Trefftz Theory in Buckling of Columns[a]

Discussion by Zdeněk P. Bažant,[3] Fellow, ASCE

This paper is interesting but misleading. A number of such papers questioning the differences in critical load predictions apparently associated with different possible definitions of finite strain and different incremental equilibrium relations at finite strain have appeared in the past [see the review in Bažant (1971) and Bažant and Cedolin (1991)]. The question raised by the authors is of the same nature as in another recent discussion (Bažant 1992), and therefore no more than a brief explanation is needed.

The authors' interpretation of the differences between critical loads for the engineering strain definition and the Green strain definition is erroneous. The authors implicitly assume that the tensile (incremental) Young's elastic modulus is the same in both cases, but it cannot be the same. There is an infinite number of various possible second-order approximations to the incremental finite strain tensor, which have the form:

$$\varepsilon_{ij}^{(m)} = \varepsilon_{ij} - \alpha e_{ki} e_{kj} \quad (19a)$$

$$\alpha = 1 - \frac{m}{2} \quad (19b)$$

in which ε_{ij} = incremental Green's (Langranian) finite strain tensor in Cartesian coordinates $x_i (i = 1, 2, 3)$; $\varepsilon_{ij}^{(m)}$ = other possible incremental finite strain tensors characterized by any real number m; and ε_{ij} = incremental linearized (small) strain tensor. When incremental stability formulations corresponding to different $\varepsilon_{ij}^{(m)}$ are used, the same material must be represented by different fourth-order tensors of incremental (tangential) moduli $C_{ijkl}^{(m)}$ (Bažant and Cedolin 1991) which have been known since 1971 (Bažant 1971) to be related as follows:

$$C_{ijkl}^{(m)} = C_{ijkl} + \frac{2-m}{4}(S_{ik}\delta_{jl} + S_{jk}\delta_{il} + S_{il}\delta_{jk} + S_{jl}\delta_{ik}) \quad (20)$$

where C_{ijkl} = the incremental moduli corresponding to the choice of Green's finite strain tensor, for which $m = 2$; S_{ij} = initial (Cauchy) stress tensor; and δ_{ij} = Kronecker delta. For the special case of a column under initial stress, S_{ij} reduces to initial uniaxial stress $S_{11} = \sigma_0 = -P/A$, where P = axial force (positive for compression) and A = cross-section area.

The case $m = 2$ corresponds to Green's finite strain tensor, while the case $m = 1$ corresponds to Biot's strain tensor, called by the authors "engineering strain." For these cases (and when the Poisson effect is neglected, $C_{1133}^{(m)} = 0$), (20) reduces to

$$E^{(1)} = E + G_0 \quad (21)$$

where E and $E^{(1)}$ are the tangential moduli at initial stress G_0 for the Green's strain and for Biot's strain. If (21) is used, then the author's expression [(19)] for the critical stress corresponding to the Green's strain tensor can be transformed to their expression [(16)] for the critical stress corresponding to the "engineering strain" (i.e., Biot strain).

Thus there is no contradiction, no difference between the results based on the engineering strain definition and on Green's strain definition. In particular, there is no difference between the critical loads. The difference between the curves in the author's Fig. 2 is merely a manifestation of considering different materials, characterized by different incremental constitutive laws (with different values of modulus E depending on G_0), and it is incorrect to interpret it as a discrepancy between different theories.

[a]October, 1992, Vol. 118, No. 10, by C. M. Wang and W. A. M. Alwis (Paper 2715).

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Appendix. References

Bažant, Z. P. (1971). "Correlation study of formulations of incremental deformations and stability of continuous bodies." *J. Appl. Mech. Trans. ASME*, 38, 919–28.

Bažant, Z. P. (1992). Discussion of "Stability of built-up columns," by A. Gjelsvik. *J. Engrg. Mech.*, ASCE, 118(6), 1279–1281.

Bažant, Z. P., and Cedolin, L. (1991). *Stability of structures: Elastic, inelastic, fracture and damage theories*. Oxford University Press, New York, N.Y.

Closure by C. M. Wang[4] and W. A. M. Alwis[5]

The writers would like to thank the discusser for his clarification on the apparent difference between the critical load solutions resulting from the use of two strain functions. The difference is attributed to the use of a common Young's modulus E in place of the different elastic moduli (representing the same material) for the engineering strain and Green strain formulations. Accordingly, the critical load solutions may be implicitly expressed as either

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$$P_{cr} = \frac{\pi^2 E^e I}{l^2} \text{ (engineering strain formulation)} \quad (22)$$

or

$$P_{cr} = \frac{\pi^2 \left(E^g - \dfrac{P_{cr}}{A}\right) I}{l^2} \text{ (Green strain formulation)} \quad (23)$$

where E^e, E^g = incremental elastic moduli at load P_{cr} corresponding to engineering strain and Green strain, respectively; and l = the length of the column under the axial load P_{cr}. As pointed out by the discusser, the relationship between the two moduli is $E^e = E^g - P_{cr}/A$ and hence the critical loads given by (22) and (23) are identical.

Although a unique solution is ensured through the application of appropriate moduli for different incremental strain expressions, the differences in solutions discussed by the writers are of importance from an engineering viewpoint. In the discusser's description of material behavior, the elastic moduli are functions of stress. However, in the engineering practice, elastic moduli are treated and extensively used as constants. Characterisation of a material for design purposes becomes unduly complicated for most practical applications if a single set of constant moduli is not adopted. A compromise would be to approximate one among the alternative moduli to a constant and to derive all necessary formulae using the appropriate strain expression, so as to avoid any confusion. Such a strong measure or any other corrective action need not be taken unless the error, due to adopting a solution derived from an incompatible strain expression and common constant elastic moduli, is justifiably large. Thus, it is important to investigate the differences in solutions due to adopting various strain definitions together with common constant elastic moduli. Naturally, such investigations would become simpler if the discusser's description of incremental moduli is exploited.