Bažant, Z.P. (1971). "Micropolar medium as a model for buckling of grid frameworks." *Developments in Mechanics*, Vol. 6 (Proc. of the 12th Midwestern Mechanics conference, U. of Notre Dame), 587–593.

Developments in Mechanics, Vol. 6. Proceedings of the 12th Midwestern Mechanics Conference: (1971) 40

Micropolar Medium as a Model for Buckling of Grid Frameworks

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ABSTRACT

Attention is focused on large rectangular frameworks of constant mesh size and constant properties of members in each direction. The framework is considered to be under initial axial loads. A continuous approximation for the expression of potential energy is formulated and, postulating an equivalent micropolar continuum under initial stress, differential equations of equilibrium in terms of displacements and rotations are derived. Expressions for stresses, couple stresses and constitutive relations are also presented.

INTRODUCTION

Although the methods for the analysis of buckling of frames are theoretically well-known, buckling of truly large grid frameworks, such as high-rise buildings, is practically intractable with the classical methods since the size of problem overtaxes the capacity of computers presently available. In practice, the assessment of stability is restricted, as a rule, to local behavior of columns within the frame, and very slender buildings are either avoided, in order to insure that investigation of the overall loss of stability is not important, or very rigid bracings (or stiffening walls) are provided, which secures stability even without the framework. Nevertheless, even in such structures the response to horizontal forces is affected by the initial loads in the columns. In the future development toward higher, lighter and slenderer structures, it can be expected that even the overall stability modes with axial extensions of columns will become an important consideration in design.

Experience with the exact solutions of the overall behavior of large frames indicates that the displacements and rotations of joints usually vary relatively smoothly from floor to floor and bay to bay. This suggests that a certain continuum approximation can be used as one method of overcoming the difficulties.

The development of various theories of structured continua has been accomplished only recently [1, 2, 3]. Their common characteristic feature is the existence of couple stresses and asymmetric shear stresses. As will be shown later, an appropriate approximation to a grid framework is Eringen's micropolar medium [3], characterized by the dependence of the elastic potential on the gradient of microrotation and the difference between micro- and macrorotation, in addition to the dependence on the symmetric part of the displacement gradient as in classical elasticity.

The possibility of applying these theories as approximations to frameworks and lattices has been mentioned in many papers. Some very general discussions were made, e.g., by Woźniak [4]. First specific treatment was presented by Banks and Sokolowski [5]. In their paper, however, the special case of a Cosserat continuum, in which the micro- and macrorotations are equal [1], was assumed. This model is, however, inadequate because the microrotation, which corresponds to the rotation of joints in a framework, and the macrorotation, which characterizes the rotation of a line connecting two adjacent joints, are in general unequal. A further significant contribution was made by Askar and Cakmak [6] who considered a rectangular gridwork with diagonals and correctly arrived at a micropolar medium. However, their model is also not fully consistent because certain important terms in the expression of elastic potential, namely those which contain second derivatives of microrotation but can be transformed on integration by parts to terms with first derivatives only, have been neglected. Buckling and deformations of frameworks under initial stress probably have not yet been treated in this light. The intent of the present paper is to formulate a consistent continuum analogy for such problems.

POTENTIAL ENERGIES OF GRIDWORK AND CONTINUUM

Consider a member of a planar framework (Figure 1) which is initially straight and in equilibrium under a large axial force, P^0. Assume that small end moments M_a, M_b, shear force T and axial force P is superposed at the ends of member,

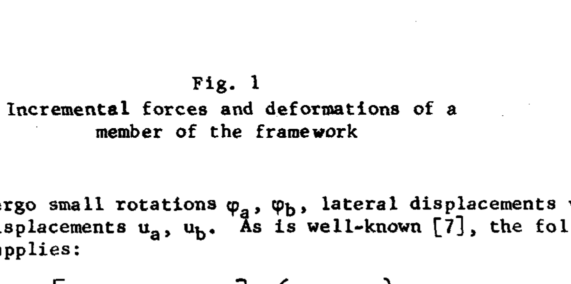

Fig. 1
Incremental forces and deformations of a member of the framework

which thus undergo small rotations φ_a, φ_b, lateral displacements v_a, v_b and longitudinal displacements u_a, u_b. As is well-known [7], the following relationship then applies:

$$\begin{Bmatrix} M_a \\ M_b \\ P \end{Bmatrix} = \begin{bmatrix} ks, & ksc, & 0 \\ ksc, & ks, & 0 \\ 0, & 0, & E' \end{bmatrix} \begin{Bmatrix} \psi - \varphi_a \\ \psi - \varphi_b \\ u_a - u_b \end{Bmatrix} \quad (1)$$

where $\psi = (v_a - v_b)/L$; L = initial length of member; $\psi - \varphi_a$ and $\psi - \varphi_b$ = rotations relative to ab; k = EI/L; E' = EA/L; I and A = inertia moment and area of the cross-section; E = Young's modulus. Coefficients s and c are functions of P^0, called stability functions. The expressions and tables for these functions are available in the literature [7]. For a zero axial force, s = 4, c = 1/2.

The expression for the incremental strain energy U_1 of a single member is

$$U_1 = \frac{1}{2}\left[M_a(\varphi_a - \psi) + M_b(\varphi_b - \psi) + P(u_b - u_a)\right] - P^0(L\psi^2/2) \tag{2}$$

plus a linear term $P^0(u_b - u_a)$ which need not be considered because it governs only the initial equilibrium. The value $(L\psi^2/2)$ represents, with an error $O(\psi^4)$, the axial extension of the member due to small incremental lateral displacements v_a, v_b. If the expressions for M_a and M_b, and P according to Equation (1) are substituted, Equation (2) may be brought, after rearrangements, to the form:

$$U_1 = \frac{1}{2}E'(u_b - u_a)^2 + \frac{1}{2}ks(\varphi_b - \varphi_a)^2 + ks'(\psi - \varphi_a)(\psi - \varphi_b) - P^0\frac{1}{2}L\psi^2 \tag{3}$$

where

$$s' = s(1 + c) \tag{3a}$$

Consider now a plane rectangular grid framework with members parallel to Cartesian axes x and y (Figure 2). Assume that the properties in each direction are uniform, including the value of the axial forces. Quantities related to the directions x and y will be distinguished by subscripts x and y. Subscripts x or y preceded by a comma will denote partial derivatives, e.g., $v_{,x} = \partial v/\partial x$,

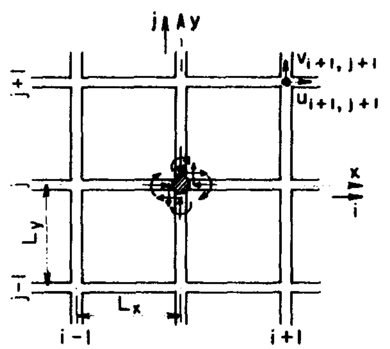

Fig. 2

Notation for the members of framework and forces acting on a joint

$\varphi_{,xx} = \partial^2\varphi/\partial x^2$. The individual joints will be referred to by subscripts i and j expressing the number of the vertical or the horizontal row of members (Figure 2). The displacements of joint (i,j) in the x- and y- directions will be denoted as $u_{i,j}$, $v_{i,j}$, and its rotation as $\varphi_{i,j}$. Here a comma between the subscripts does not refer to a derivative.

The transition from a discrete to a continuous system may be achieved by defining (sufficiently smooth) continuous functions u, v, φ and f_x, f_y, m of the variables x, y, such that their values in points (x_i, y_j) are sufficiently close to the values $u_{i,j}$, $v_{i,j}$, $\varphi_{i,j}$, and $\bar{X}_{i,j}$, $\bar{Y}_{i,j}$, $\bar{M}_{i,j}$, respectively. The latter three values represent prescribed incremental loads and moments applied in the joint and f_x, f_y, m are the equivalent incremental distributed loads and moments per unit area of the gridwork.

The smoothing operation, by which the continuous approximation of gridwork may be obtained, consists in introducing the continuous functions u, v, φ into the expression for potential energy and neglecting higher order derivatives in the Taylor series expansions of u, v, φ. This is, of course, justified only if the change of u, v, φ from joint to joint is sufficiently small.

The incremental strain energy U_x contained in a pair of horizontal members between the joints (i - j, j) and (i + j, j) is a sum of two expressions of form (2). Expanding the values of u, v, and φ in joints (i - 1, j) and (i + 1, j) in Taylor series about the point (i, j) yields the following continuum approximation:

$$U_x = L_x^2 E_x' u_{,x}^2 + L_x^2 k_x s_x \varphi_{,x}^2 + L_x^2 k_x s_x' \varphi\varphi_{,xx} + 2k_x s_x' (v_{,x} - \varphi)^2 - P_x^0 L_x v_{,x}^2 \qquad (4)$$

In this expression, the terms with higher than first derivatives of u, v, and φ have in general been dropped. An exception must be made, however, with the term $\varphi\varphi_{,xx}$ because integration by parts in the expression for energy of the whole structure converts this term into a term with first order derivatives. (This point has been overlooked in Reference [6].) It deserves mention that without the term $\varphi\varphi_{,xx}$ an agreement with the continuum approximation derived from the equilibrium equations of a joint could not be reached. The legitimacy of dropping the terms with other combinations of higher derivatives, with regard to integration by parts, can be easily verified.

The incremental strain energy U_y stored in a pair of vertical members meeting in the joint (i, j) can be expressed in a similar manner. The strain energy corresponding to the area $L_x L_y$ of the frame is $(U_x + U_y)/2$.

The incremental potential energy of the whole structure, $\mathcal{U}$, approximately equals

$$\int_{(x)}\int_{(y)} (U_x + U_y - f_x u - f_y v - m\varphi) \frac{dx\, dy}{2L_x L_y} \qquad (5)$$

minus the work of the loads applied at the boundary of frame. Integrating the terms involving the products $\varphi\varphi_{,xx}$ and $\varphi\varphi_{,yy}$ by parts (or applying the Green's theorem), the integral (5) takes on the form:

$$\int_{(x)}\int_{(y)} U\, dx\, dy - \int_{(x)}\int_{(y)} (f_x u + f_y v + m\varphi) \frac{dx\, dy}{2L_x L_y} \qquad (6)$$

where

$$U = \Big[L_x^2 E_x' u_{,x}^2 + L_y^2 E_y' v_{,y}^2 - L_x^2 k_x s_x c_x \varphi_{,x}^2 - L_y^2 k_y s_y c_y \varphi_{,y}^2 + $$
$$+ 2k_x s_x' (v_{,x} - \varphi)^2 + 2k_y s_y' (u_{,y} + \varphi)^2 - P_x^0 L_x v_{,x}^2 - P_y^0 L_y u_{,y}^2 \Big] / (2L_x L_y) \qquad (7)$$

plus a certain contour integral of terms involving products $\varphi\varphi_{,x}$ and $\varphi\varphi_{,y}$. Expression U can be regarded as the specific incremental elastic potential of the continuum approximating the framework.

Inspecting Equation (7) to determine the mutually independent variables of which U is a function, the special case of our continuum for $P_x^0 = P_y^0 = 0$ is found to represent the micropolar medium as defined by Eringen [3]. This also shows that the classical Cosserat's medium [1] is insufficient, while theories more general than micropolar medium are unnecessarily complex [2, 3].

DIFFERENTIAL EQUILIBRIUM EQUATIONS

The first variation of the incremental potential $\mathcal{U}$ may be written in the form:

$$\delta \mathcal{U} = \int_{(x)}\int_{(y)} \Big[L_x^2 E'_x u_{,x}\delta u_{,x} + L_y^2 E'_y v_{,y}\delta v_{,y} - L_x^2 k_x s_x c_x \varphi_{,x}\delta\varphi_{,x}$$

$$- L_y^2 k_y s_y c_y \varphi_{,y}\delta\varphi_{,y} + 2k_x s'_x(\varphi - v_{,x})(\delta\varphi - \delta v_{,x}) + 2k_y s'_y(\varphi + u_{,y})(\delta\varphi + \delta u_{,y})$$

$$- P_x^0 L_x v_{,x}\delta v_{,x} - P_y^0 L_y u_{,y}\delta u_{,y} - f_x\delta u - f_y \delta v - m\delta\varphi\Big]\frac{dx\ dy}{L_x L_y} \qquad (8)$$

plus a certain contour integral expressing the work of prescribed boundary loads. If in Equation (8) the terms containing derivatives of the variations are integrated by parts (or if Green's theorem is applied), the condition that $\delta\mathcal{U} = 0$ for any δu, δv and $\delta\varphi$ results in the following differential equations:

$$L_x^2 E'_x u_{,xx} + k_y s''_y u_{,yy} + 2k_y s'_y \varphi_{,y} + f_x L_x L_y = 0 \qquad (9a)$$

$$L_y^2 E'_y v_{,yy} + k_x s''_x v_{,xx} - 2k_x s'_x \varphi_{,x} + f_y L_x L_y = 0 \qquad (9b)$$

$$2k_x s'_x(\varphi - v_{,x}) + 2k_y s'_y(\varphi + u_{,y}) + L_x^2 k_x s_x c_x \varphi_{,xx} + L_y^2 k_y s_y c_y \varphi_{,yy} - mL_x L_y = 0 \qquad (9c)$$

where s''_x and s''_y are defined as follows (omitting subscript x or y):

$$s'' = 2s' - \pi^2 P^0/P_E\ ,\ P_E = EI\,\pi^2/L^2 \qquad (9d)$$

Equations (9a)-(9c) represent the differential equations of equilibrium in terms of displacements and rotations for the continuous medium approximating the framework.

CONSTITUTIVE RELATIONS FOR MICROPOLAR CONTINUUM

The components of stress, σ_{xx}, σ_{xy}, σ_{yx}, σ_{yy}, and couple stress, m_{xz}, m_{yz} for a micropolar medium in plane stress may be defined with the help of the specific potential energy as is indicated in the following relations,

$$\left.\begin{aligned}
\sigma^0_{xx} + \sigma_{xx} &= \partial U/\partial u_{,x} = \sigma^0_{xx} + E'_x u_{,x} L_x/L_y \\
\sigma^0_{yy} + \sigma_{yy} &= \partial U/\partial v_{,y} = \sigma^0_{yy} + E'_y v_{,y} L_y/L_x \\
\sigma_{xy} &= \partial U/\partial(v_{,x} - \varphi) = (k_x s''_x v_{,x} - 2k_x s'_x \varphi)/(L_x L_y) \\
\sigma_{yx} &= \partial U/\partial(u_{,y} + \varphi) = (k_y s''_y u_{,y} + 2k_y s'_y \varphi)/(L_x L_y) \\
m_{xz} &= \partial U/\partial\varphi_{,x} = -\,k_x s_x c_x \varphi_{,x} L_x/L_y \\
m_{yz} &= \partial U/\partial\varphi_{,y} = -\,k_y s_y c_y \varphi_{,y} L_y/L_x
\end{aligned}\right\} \qquad (10)$$

in which also the expressions obtained after substitution of Equation (7) are introduced. The values σ^0_{xx}, σ^0_{yy} represent initial stresses in the micropolar medium, $\sigma^0_{xx} = -P^0_x/L_y$, $\sigma^0_{yy} = -P^0_y/L_x$. (The reason for their appearance in the stress definitions is that the work of the initial stress on the incremental displacement u is $\sigma^0_{xx}u_{,x}$.) Equations (10) have the significance of stress-strain relationships of the micropolar medium.

It is of interest to investigate the relationship of the above stresses to the internal forces in members of the frame. To this end, let us consider their

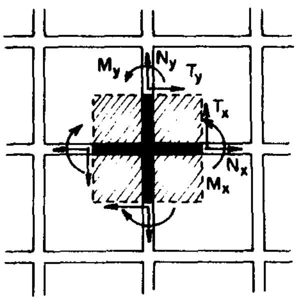

Fig. 3

Internal forces at the midspans and their analogy with the stresses and couple stresses acting on an element of a micropolar continuum

values at the midspan (Figure 3). According to the equilibrium conditions of the member shown in Figure 1,

$$T = (M_a + M_b)/L - P^0\psi \tag{10a}$$

Then, using Equation (1) and considering the equilibrium of the half-length of the member in Figure 1, the internal forces in the midspan can be obtained as follows:

$$\left.\begin{aligned} &N = E'(u_b - u_a) = -P, \quad T = k\left[s''(v_b - v_a)/L - s'(\varphi_a + \varphi_b)\right]/L \;, \\ &M = (M_b - M_a)/2 = \frac{1}{2}\,ks(1 - c)(\varphi_b - \varphi_a) \end{aligned}\right\} \tag{11}$$

where N is the axial force, T is the shear force and M is the bending moment at the midspan which is taken, by definition, about the point located on the straight line connecting the ends of member in the deformed position. Notice that these values characterize the end moments as well;

$$M_a = -M - (T + P^0\psi)L/2 \;, \quad M_b = M - (T + P^0\psi)L/2 \tag{11a}$$

Expanding u_a, u_b, v_a, v_b and φ_a, φ_b in Taylor series and dropping all terms containing higher than first order derivatives, the following expressions are obtained:

$$\left.\begin{aligned} &N_x = L_x E'_x u_{,x}, && N_y = L_y E'_y v_{,y}\;, \\ &T_x = k_x s''_x v_{,x}/L_x - 2k_x s'_x \varphi/L_x, && T_y = k_y s''_y u_{,y}/L_y + 2k_y s'_y \varphi/L_y, \\ &M_x = \frac{1}{2}\,L_x k_x s_x (1 - c_x)\,\varphi_{,x}, && M_y = \frac{1}{2}\,L_y k_y s_y (1 - c_y)\,\varphi_{,y} \end{aligned}\right\} \tag{12}$$

These expressions may be regarded as the continuous counterparts of the internal forces (11) at the midspan. Values of all functions in these expressions ought to be evaluated for the midspan.

Comparing expressions (12) and (10), it follows that

$$\left.\begin{aligned} N_x &= L_y\sigma_{xx}\,, & N_y &= L_x\sigma_{yy}\,, \\ T_x &= L_y\sigma_{yy}\,, & T_y &= L_x\sigma_{yx}\,, \\ N_x &= -\,L_y m_{xz}(1 - c_x)/(2c_x)\,, & M_y &= -\,L_x m_{yz}(1 - c_y)/(2c_y) \end{aligned}\right\} \quad (13)$$

For a medium without initial stress, the latter of these relationships reduces to

$$M_x = -\frac{1}{2}L_y m_{xz}\,, \qquad M_y = -\frac{1}{2}L_x m_{yz} \qquad (14)$$

It is interesting to note that, in Equation (14), M_x is not equal to the resultant of the couple stresses m_{xz} over length element L_y in the micropolar medium but rather equals minus one-half of it. (The formulation in Reference [5] implies incorrectly that $M_x = L_y m_{xz}$.) With varying initial stress, the ratio m_{xz}/M_x changes. The reason for the lack of any simple, intuitive correspondence between m_{xz} and M_x lies obviously in the fact that M_x varies along the member. By contrast, T and P are constant within each member and N_x or T_x do represent the resultants of stresses σ_{xx} or σ_{yy} over the length element L_y.

Expressions for stresses, Equations (10), and their relations to internal forces in framework, Equation (13), allow to formulate the boundary conditions of micropolar bodies approximating grid grameworks. The boundary conditions can, of course, be also deduced from the first variation of the full expression for potential energy.

CONCLUDING REMARKS

The equations presented above fully define the analogy between a grid framework and a micropolar medium under initial stress.

The equations of equilibrium could have been, alternatively, also derived by determining the continuum approximation to the equations of equilibrium of a joint in the framework (Figure 2). It has been verified that such a procedure does indeed yield the same results. For the correct expression of couple stresses, however, the potential energy approach is inevitable.

Application to practical problems is left to a subsequent paper.

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