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CONTINUOUS APPROXIMATION OF LARGE REGULAR FRAMEWORKS AND THE PROBLEM OF A SUBSTITUTE FRAME

BY
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Synopsis: Large regular frames under initial axial forces may be approximated by a continuum which is identical with Eringen's micropolar continuum. The finite difference method is then applied to the continuum problem. For a grid step equal to the distance between joints this solution is exact and coincides with the displacement method, while for a larger step, which allows considerable reduction in the number of unknowns, an approximate solution is obtained. Furthermore, it is found that a substitute (or equivalent) frame, which has been sought by structural analysts, in general does not exist. But the solution presented serves, in fact, the same purpose. Finally, numerical examples are given to verify the method.

Keywords: approximation; axial loads; buckling; failure; finite difference theory; frames; high rise buildings; joints (junctions); lateral pressure; moments; multistory buildings; reinforced concrete; stiffness; structural analysis; wind pressure.

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When classical methods are applied to the spatial analysis of the taller building frames, the capacity of computers presently available is overtaxed. If the problems of dynamics, stability, and nonlinear behavior are considered, this difficulty is encountered even in the case of large planar frameworks. One method of reducing the number of unknowns in such analyses may be based on the fact that in large regular frames with smoothly variable properties the displacements also vary smoothly from joint to joint so that a continuum approximation may be adopted. A consistent form of such an approximation was proposed in Ref. 1 and developed in detail in Ref. 2, including the case of frames under initial stress, as in buckling problems. It appears that the only correct continuum approximation to a frame is Eringen's micropolar continuum [3], which is characterized by the existence of couple stresses, asymmetric shear stresses $\tau_{xy} \neq \tau_{yx}$, and the independence of microrotation from the displacement field.

In the present paper, after a brief review of the basic theory, the approximate solution of the continuum problem by the finite difference method will be discussed and some peculiarities of the solution with regard to boundary conditions will be examined. Furthermore, certain implications for the method of a substitute (or equivalent) frame will be studied. Namely, such a frame, which has been widely utilized by structural analysts, in general does not exist, as has been mentioned in Ref. 2. This fact will be demonstrated in detail and a consistent method of analysis will be shown. Finally, some applications to overall buckling of large frames will be described.

REVIEW OF BASIC THEORY

Consider a framework whose members are parallel (in an unstressed state) to cartesian axes x and y, and are in equilibrium under initial axial loads P^0. Assume that the frame is then subjected to a small incremental deformation accompanied by increments of axial force, shear force, and bending

moments in the frame members, denoted by P, V, M_a, and M_b, respectively. Further assuming that there are no incremental loads between the extremities of the members, the following linearized relationship holds between the incremental end reactions and displacements (Fig. 1) [4]:

$$\begin{Bmatrix} M_a \\ M_b \\ V \\ P \end{Bmatrix} = \begin{bmatrix} ks & ksc & -ks'/L & 0 \\ ksc & ks & -ks'/L & 0 \\ -ks'/L & -ks'/L & ks''/L^2 & 0 \\ 0 & 0 & 0 & E' \end{bmatrix} \begin{Bmatrix} \phi_a \\ \phi_b \\ v_b - v_a \\ u_a - u_b \end{Bmatrix} \qquad (1)$$

L = the length of the member in its unstressed state; $\psi = (v_b - v_a)/L$ = the counterclockwise rotation of the line $\overline{ab}$ joining the member ends; I and A = the cross-sectional moment of inertia and area which are constant within a member; E = Young's modulus of the material; k = EI/L; E' = EA/L; s and c are the well-known stability functions [4] of the initial axial load P^0; $s' = s(1 + c)$ and $s'' = 2s' - P^0L/k$; for $P^0 = 0$, $s = 4$, $c = \frac{1}{2}$.

Let each joint of the framework be denoted by two numbers (i,j), which refer to the numbers of the line of vertical members and the line of horizontal members, respectively. In general, the stiffness of framework members varies from floor to floor or bay to bay either because of a change in cross-section or a change in initial axial force. So, let k_x, s_x, etc. represent the average value at joint (i,j) of the corresponding quantity between the two adjacent members parallel to the x-direction, and let k_y, s_y, etc. be similarly used for the members parallel to the y-direction. Also, let $\Delta_x(\cdot)$, $\Delta_y(\cdot)$, similarly represent the differences of the corresponding quantity between two adjacent members.

The conditions of equilibrium of joint (i,j) may then be expressed using Eq. (1) to determine the end reactions for each of the four members meeting in the joint. After algebraic rearrangements the conditions of equilibrium can be brought into the following form:

$$E_x'(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + \frac{k_y s_y'}{L_y}(\phi_{i,j+1} - \phi_{i,j-1})$$
$$+ \frac{k_y s_y''}{L_y^2}(u_{i,j+1} - 2u_{i,j} + u_{i,j-1}) + \tfrac{1}{2}\Delta_x\left(E_x'\right)(u_{i+1,j} - u_{i-1,j})$$
$$+ \tfrac{1}{2}\Delta_y\left(\frac{k_y s_y'}{L_y}\right)(\phi_{i,j+1} - 2\phi_{i,j} + \phi_{i,j-1} + 4\phi_{i,j})$$
$$+ \tfrac{1}{2}\Delta_y\left(\frac{k_y s_y''}{L_y^2}\right)(u_{i,j+1} - u_{i,j-1}) + f_{x_{i,j}} = 0 \qquad (2a)$$

These values fully characterize the end moments M_a, M_b because

$$M_a = -M - (V + P^o\psi)L/2, \qquad M_b = M - (V + P^o\psi)L/2. \tag{6}$$

The continuous approximations of Eqs. (5) in the x- and y-directions are straight forward and given by

$$\begin{aligned} N_x &= L_x E'_x u_{,x} & N_y &= L_y E'_y v_{,y} \\ T_x &= (k_x s''_x v_{,x} - 2k_x s'_x \phi)/L_x, & T_y &= (k_y s''_y u_{,y} + 2k_y s'_y \phi)/L_y \\ M_x &= \frac{1}{2} L_x k_x s_x (1 - c_x)\phi_{,x}, & M_y &= \frac{1}{2} L_y k_y s_y (1 - c_y)\phi_{,y} \end{aligned} \tag{7}$$

where N_x, N_y, T_x, T_y, M_x, M_y are continuous functions whose values at a point coinciding with the midspan approximate the internal forces (5), provided all derivatives, such as $u_{,x}$, are evaluated for that point. The shear force T_y was taken as $-V_y$ so that its positive direction would correspond to the usual continuum convention.

To complete the formulation of the continuum problem it is necessary to discuss the appropriate boundary conditions. At each boundary joint of the frame, one quantity of each of the pairs (u, f_x), (v, f_y), (ϕ, m) must be given. In the case of a continuum approximation, Eqs. (7) have an error of higher than second order only if the functions u, v, ϕ, and their derivatives are evaluated at the midspans of the members. Therefore, when the applied loads f_x, f_y, or m at the boundary joint are prescribed, the simplest formulation of the boundary conditions is achieved by imagining the gridwork to be extended one grid step beyond the actual boundary. The values of the displacements of the hypothetical nodes outside the physical boundary are then used in such a way that the internal forces at the midspan of the imagined members crossing the boundary transmit the prescribed forces into the actual boundary joints (Fig. 3). Thus, from Fig. 3 and Eq. (6), the conditions, for example, on the left vertical and top horizontal boundaries of a frame are:

$$\begin{aligned} N_x &= -P_x^{\,B} & N_y &= -P_y^{\,B} \\ T_x &= V_x^{\,B} & T_y &= -V_y^{\,B} \\ M_x &= M_x^{\,B} + (V_x^{\,B} + P^o_x\psi_x)\frac{L_x}{2} & M_y &= -M_y^{\,B} - (V_y^{\,B} + P^o_y\psi_y)\frac{L_y}{2} \end{aligned} \tag{8}$$

where P^B, V^B, M^B denote the prescribed incremental normal load, tangential load, and moment, respectively, at the actual boundary joint of the frame. These conditions are imposed all along the line connecting the midspans of the imagined members. The continuum boundary can thus be imagined as

located beyond the boundary joints of the frame by $L_x/2$ and $L_y/2$, respectively.

APPROXIMATE SOLUTION BY THE FINITE DIFFERENCE METHOD

One method of solving practical continuum problems represented by Eqs. (6) is the finite difference method. If the finite difference approximations for the indicated derivatives are written using a grid with h_x and h_y equal to the horizontal and vertical distances between grid lines, the following finite difference form of Eqs. (3) results:

$$\begin{aligned} & L_x^2 E'_x \frac{u_{r+1,s} - 2u_{r,s} + u_{r-1,s}}{h_x^2} + k_y s'_y \frac{\phi_{r,s+1} - \phi_{r,s-1}}{h_y} \\ & + k_y s''_y \frac{u_{r,s+1} - 2u_{r,s} + u_{r,s-1}}{h_y^2} + L_x^2 \frac{\Delta_x(E'_x)}{h_x} \frac{u_{r+1,s} - u_{r-1,s}}{2h_x} \\ & + 2L_y \frac{\Delta_y(k_y s'_y/L_y)}{h_y} \phi_{r,s} + \frac{L_y^3}{2} \frac{\Delta_y(k_y s'_y/L_y)}{h_y} \frac{\phi_{r,s+1} - 2\phi_{r,s} + \phi_{r,s-1}}{h_y^2} \\ & + L_y^2 \frac{\Delta_y(k_y s''_y/L_y^2)}{h_y} \frac{u_{r,s+1} - u_{r,s-1}}{2h_y} + f_{x_{r,s}} = 0 \end{aligned} \tag{9a}$$

$$\begin{aligned} & L_y^2 E'_y \frac{v_{r,s+1} - 2v_{r,s} + v_{r,s-1}}{h_y^2} - k_x s'_x \frac{\phi_{r+1,s} - \phi_{r-1,s}}{h_x} \\ & + k_x s''_x \frac{v_{r+1,s} - 2v_{r,s} + v_{r-1,s}}{h_x^2} + L_y^2 \frac{\Delta_y(E'_y)}{h_y} \frac{v_{r,s+1} - v_{r,s-1}}{2h_y} \\ & - 2L_x \frac{\Delta_x(k_x s'_x/L_x)}{h_x} \phi_{r,s} - \frac{L_x^3}{2} \frac{\Delta_x(k_x s'_x/L_x)}{h_x} \frac{\phi_{r+1,s} - 2\phi_{r,s} + \phi_{r-1,s}}{h_x^2} \\ & + L_x^2 \frac{\Delta_x(k_x s''_x/L_x^2)}{h_x} \frac{v_{r+1,s} - v_{r-1,s}}{2h_x} + f_{y_{r,s}} = 0 \end{aligned} \tag{9b}$$

Subsequently, the solution of the boundary disturbance due to the mismatch in the actual and modified boundary conditions is superimposed. In a general case, the solution for the boundary disturbance can be obtained by writing the finite difference equilibrium equations for only a small region near the surface and solving them as a system of algebraic equations (the interior of the frame being considered as rigid).

THE PROBLEM OF A SUBSTITUTE FRAME

By choosing the steps h_x, h_y of the grid used with Eqs. (9) to be much larger than the distances between the joints of the given frame (e.g., $h_y = 5L_y$), the number of unknowns in the analysis of a given problem is considerably reduced. Structural analysts have been seeking this objective by formulating a "substitute" or "equivalent" frame [5],[6],[7] which would approximate the given frame and have greater distances between joints. The substitute frame was usually determined from the requirement of equal deflections in a certain typical problem. However, a rational definition of a substitute frame would require that its continuum approximation be the same as for the actual frame. For frames whose members in a longitudinal line have equal properties, the condition that the continuum approximation for the given frame be the same as or proportional to the continuum approximation of a substitute frame follows by comparison of Eqs. (2) and (9). This leads to the following relations, where the substitute frame properties are labeled with bars, and β is an arbitrary parameter:

$$\left.\begin{aligned} E'_x L_x^2 &= \beta \bar{E}'_x \bar{L}_x^2 & \qquad E'_y L_y^2 &= \beta \bar{E}'_y \bar{L}_y^2 \\ k_x s'_x &= \beta \bar{k}_x \bar{s}'_x & \qquad k_y s'_y &= \beta \bar{k}_y \bar{s}'_y \\ k_x s''_x &= \beta \bar{k}_x \bar{s}''_x & \qquad k_y s''_y &= \beta \bar{k}_y \bar{s}''_y \\ k_x s_x c_x L_x^2 &= \beta \bar{k}_x \bar{s}_x \bar{c}_x \bar{L}_x^2 & \qquad k_y s_y c_y L_y^2 &= \beta \bar{k}_y \bar{s}_y \bar{c}_y \bar{L}_y^2 \end{aligned}\right\} \quad (12)$$

This is a system of eight equations which only five unknowns, namely β, $\bar{k}_x$, $\bar{E}'_x$, $\bar{k}_y$, $\bar{E}'_y$, must satisfy. Therefore, since there are too many conditions on these five unknowns, a substitute frame in general does not exist. (If one would admit $\bar{P}_x^0$ and $\bar{P}_y^0$ to differ from values P_x^0 and P_y^0 for the actual frame, seven unknowns would be available, which is still insufficient to satisfy relations (12).)

However, the finite difference equations in fact serve the same purpose. They have the same form as the equations of a displacement method, and thus existing programs for displacement analysis of frames may be easily generalized to include the finite difference solution in terms of grid steps h_x, h_y.

When one substitutes $h_x = L_x$, $h_y = L_y$, the exact solution is obtained, while for $h_x > L_x$, $h_y > L_y$, the number of unknowns is reduced and an approximate solution is obtained.

For setting up the finite difference equations (9a)-(9c) it may be convenient to imagine that the node (r,s) is connected to each of the four adjacent nodes by a fictitious elastic element whose end forces F_x, F_y and moments M are related to end rotations and displacements as follows:

$$\begin{Bmatrix} F_{x_r} \\ F_{y_r} \\ M_r \\ F_{x_{r+1}} \\ F_{y_{r+1}} \\ M_{r+1} \end{Bmatrix} = \begin{bmatrix} -\frac{L^2E'}{h^2} & 0 & 0 & \frac{L^2E'}{h^2} & 0 & 0 \\ & -\frac{ks''}{h^2} & \frac{ks'}{h} & 0 & \frac{ks''}{h^2} & -\frac{ks'}{h} \\ & & \left(-ks'-\frac{L^2ksc}{h^2}\right) & 0 & -\frac{ks'}{h} & \frac{L^2ksc}{h^2} \\ & \text{symmetric} & & -\frac{L^2E'}{h^2} & 0 & 0 \\ & & & & -\frac{ks''}{h^2} & -\frac{ks'}{h} \\ & & & & & \left(-ks'-\frac{L^2ksc}{h^2}\right) \end{bmatrix} \begin{Bmatrix} u_r \\ v_r \\ \varphi_r \\ u_{r+1} \\ v_{r+1} \\ \varphi_{r+1} \end{Bmatrix} \quad (12a)$$

where subscripts x are omitted in the stiffness matrix. Superposition of the forces transmitted into the joint (r,s) from the four adjacent fictitious elements yields then Eqs. (9a)-(9c), provided the step size and member properties are constant. Thus, the assembly of the finite difference equations (9a)-(9c) can be programmed in the same manner as is used for setting up the equilibrium equations in stiffness method. However, for nodes in which the step size or member properties change the use of Eq. (12a) would bring in an error of lower order. The present computing experience does not allow stating how significant this error would be. To avoid it, a more complicated matrix of Eq. (12a) whose elements also depend on the size of the adjacent grid step and the properties of the adjacent member would have to be introduced. (It should be also noted that it is impossible to find any real beam which would give the above stiffness matrix, although the zeros are in the same places as for a beam.)

Because five of the eight conditions (12) can be satisfied, a satisfactory substitute frame can be found in those special cases in which the terms in Eqs.(2a-c) that give rise to three of the conditions (12) have negligible values. Such a situation occurs for free standing rectangular frames under uniform or almost uniform lateral load; the rotations in each floor are almost the same and the vertical displacements v are almost linearly distributed, so that the terms $\varphi_{,x}$, $\varphi_{,xx}$ and $v_{,xx}$ in Eqs. (3b) are negligible. In a wide frame, v is not linearly distributed but is of negligible value. Moreover, in all of these frames horizontal displacements of the joints of each floor are almost the same, so that the terms $u_{,x}$ and $u_{,xx}$ are negligible. In these cases, as is well known, a satisfactory substitute frame can be found [8]; it consists of a single-bay frame of the actual bay width, with no floors deleted, the bending stiffnesses of beams and columns being lumped together and the column cross-sections modified so as to give the moment of inertia of the building cross-section the correct value [8]. Nevertheless, even in these cases the displacements and internal forces on or near the boundaries are not obtained accurately and further intuitive calculations are needed to improve their estimate. Furthermore, in a wide frame, the correct distribution of column forces (which is not linear) cannot be found by the substitute frame. The same is true for frames of variable width. The spreading of vertical or horizontal forces or moments into the frame also cannot be determined by the substitute frame.

NUMERICAL EXAMPLES

To verify the applicability of the proposed finite difference method, an analysis of a large rectangular plane frame with constant member properties and constant initial axial loads in columns was programmed for computer and the efficiency of various grid spacings, shown in Fig. 5, was investigated. To accurately represent the boundary conditions, the grid configurations were chosen so that the stress boundary of the continuum coincided with the middle of the grid step. The frame was assumed to have 52 floors, 12 column lines, L_x = 18 ft., I_x = 516 in.^{4}, A_x = 12 in.^{2}, and L_y = 12 ft., I_y = 7000 in.^{4}, A_y = 130 in.^{2}.

The solution was carried out for a constant horizontal distributed load on the left boundary, $P_x = 30\ L_xL_y$ lbs. per joint, considering various values of initial axial force in columns. The results are shown in Figs. 6 and 7. The exact solution, shown by solid lines, involves 1872 equations with a band width of 77, while the approximate solutions from grids 2 and 3 for example, shown by dashed and dotted lines, involve 1248 unknowns with band width 53, and 648 unknowns with band width 53, respectively. It is seen that the approximate solutions give quite accurate results for the lower values of vertical initial load, provided that the first two (or three) rows of nodal points near the vertical boundaries are chosen to coincide with the column lines.

It was mentioned previously that the boundary disturbance in a half-space filled by a frame with equal and constant properties in the x- and y-direction could be removed by imposing the condition of equal rotations in the two rows of joints j = 0, j = 1. This condition also lessens the boundary disturbance at the base of a finite frame. To demonstrate this, an analysis of the same frame was also carried out using the condition that the first two rows of grid joints at the base have equal rotations. Fig. 7 shows a comparison of the exact solution (solid line) with two approximate solutions from the same coarse grid. In one approximate solution, the boundary disturbance is present (dash-dot line) while in the other it is removed (dashed line). As is seen, the approximate solution with boundary disturbance removed does indeed give results away from the boundary that are very close to the exact solution.

It should also be mentioned that although the approximate finite-difference solution using a coarse grid becomes less accurate as the initial vertical load is increased, it can still be used for determining the load which causes long-wave (overall) buckling of the frame, as is evident from Fig. 8.

For the purpose of comparison, the approximate solution by finite difference grid shown in Fig. 9b was compared with the solutions by various substitute frames shown in Fig. 9c,d,e. The results are given in Figs. 10, 11,12. It is seen from Figs. 10 and 11 that for horizontal deflections and joint rotations the finite difference method is about as good as the substitute frame of Khan and Sbarounis [8] (Fig. 9e) which has been known to be quite suitable for this type of problem. The finite difference method, however, gives also quite accurately the non-linear distribution of the axial forces in columns due to the horizontal load (Fig. 12), while the substitute frame is unable to provide these results. The substitute frames in which some of the floors are deleted [5] (Fig. 9c,d) are found to give grossly erroneous results (Fig. 11) and should thus never be used.

It is interesting to note, however, that the accuracy of the substitute frame in Fig. 9e in representing the horizontal deflections and joint rotations of a frame with rectangular boundary under uniform lateral load can be almost matched by a much simpler model, replacing the whole frame by a single column (substitute column) which deforms in shear, in addition to bending normal strains. The substitute column is assigned the same moment of inertia of cross-section as the whole horizontal cross-section through the columns of the frame, and the same shear rigidity as the actual frame in relative horizontal displacement of floors without column extensions.

REMARKS ON OTHER APPLICATIONS

An alternate method of analyzing large frames of rectangular boundary under lateral loads and their overall buckling loads is also being studied. In this method the continuum problem is solved by Fourier method, expanding the continuous functions in Fourier series in the vertical direction. This method seems to be particularly convenient when frames of rectangular boundary interact with shear walls of rectangular shape, in which case a solution somewhat similar to that used for folded plates can be developed. In a similar fashion, the framed tube can also be analyzed.

The reduction in the number of unknowns will be of particular advantage in dynamic problems, such as in earthquake analysis.

It should also be noted that in some cases the equilibrium equations cast in form of difference equations admit an exact solution by the methods of finite difference calculus (cf. Ref. 2). But even then the solution by continuous approximation is usually simpler.

CONCLUSIONS

1. The finite difference method applied to the continuous approximation of large regular frameworks gives sufficiently accurate approximate solutions with much fewer unknowns than the exact analysis.

2. A substitute frame in general does not exist but the method presented serves its intended purpose.

3. In the method presented, special attention must be given to boundary disturbances. They may either be removed by modifying the boundary conditions or the grid nodes near the boundary must coincide with the actual joints.

4. The method allows the overall buckling load of a tall and slender frame to be determined and the reduction of the overall lateral stiffness of frame due to axial forces in columns to be evaluated.

5. The continuum approximation may also be used for analytical solutions, e.g. by Fourier series.

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APPENDIX - BASIC NOTATIONS

A = cross section area of a member;
E, E' = Young's modulus and the value EA/L;
c = carry over factor [4] (Eq. 1);
f_x, f_y = prescribed loads in a joint of framework;
h_x, h_y = steps of grid in the finite difference method;
k = EI/L = member stiffness;
I = moment of inertia of cross section of a member;
L = length of member;
m = prescribed moment applied at the joint;
M, N = bending moment and normal force (incremental);
P, P^0 = axial force increment and its initial value;
s, s', s'' = stability functions [4] of P^0 (Eq. 1), and expressions $s(1 + c)$ and $2s' - P^0L/k$, respectively;
T' = shear force in a member in continuum sign convention;
U = potential energy density;
u = horizontal and vertical displacements of joints;
V = shear force in a member;
x, y = cartesian coordinates (parallel to horizontal and vertical members)
Δ_x, Δ_y = difference between adjacent members or between the values of a variable at distances $h_x/2$ and $h_y/2$ from the joint;
ϕ, ψ, ω = joint rotation (or microrotation), rotation of a line connecting adjacent joints, and macrorotation, respectively.

Subscripts:
a, b - for ends a,b of a member
x, y - for directions of x,y;
$,x$ or $,y$ (following a comma) - for partial derivatives (e.g., $v_{,xx} = \partial^2 v/\partial x^2$);
i, j - for joint numbers;
r, s - for numbers of grid nodes when grid does not coincide with frame;
Superscript B stands for values at the frame boundary.

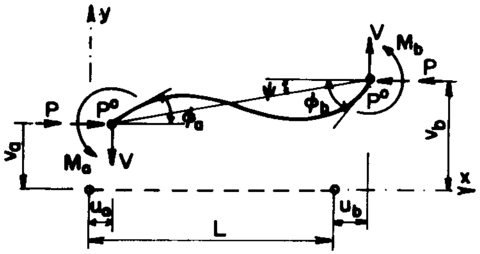


Fig. 1. Positive incremental forces and deformations of a member assumed parallel to the x-axis in its initial stressed state

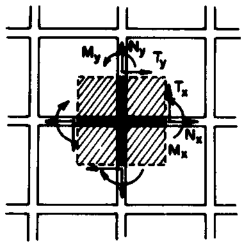


Fig. 2. Internal forces at the member midspans and their intuitive analogy with the stresses acting on an element of micropolar continuum

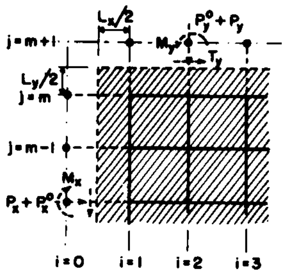


Fig. 3. Imagined extension of the grid beyond its boundary for the formulation of the free boundary conditions

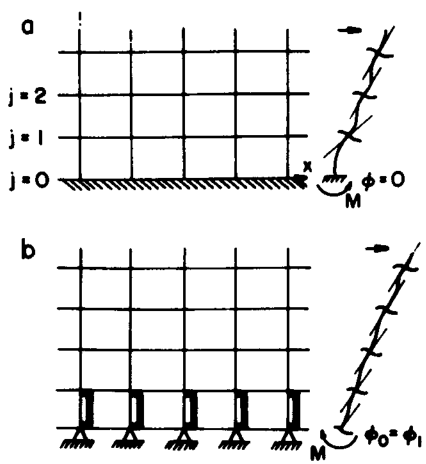
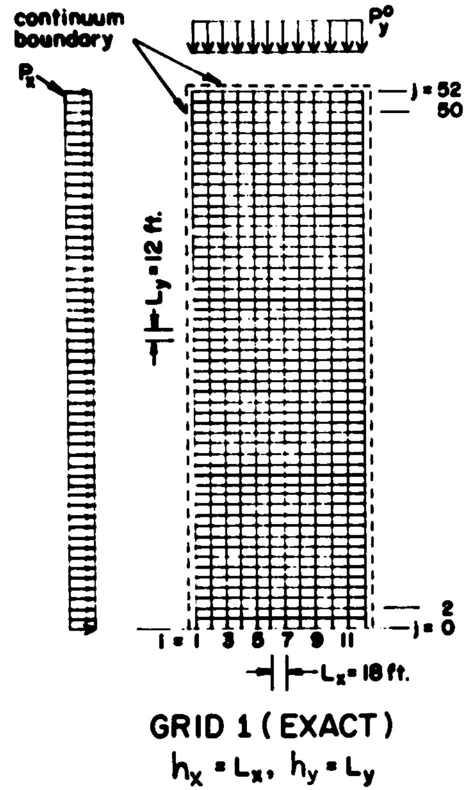


Fig. 4. Boundary condition (a), giving rise to boundary disturbance and modification (b) to remove the disturbance

(a)



(b)

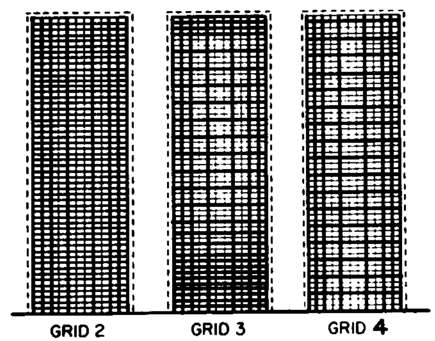


Fig. 5. Frame under initial axial and incremental lateral loads considered in numerical examples and the finite difference grids used for the numerical solution in Figures 6, 7, 8

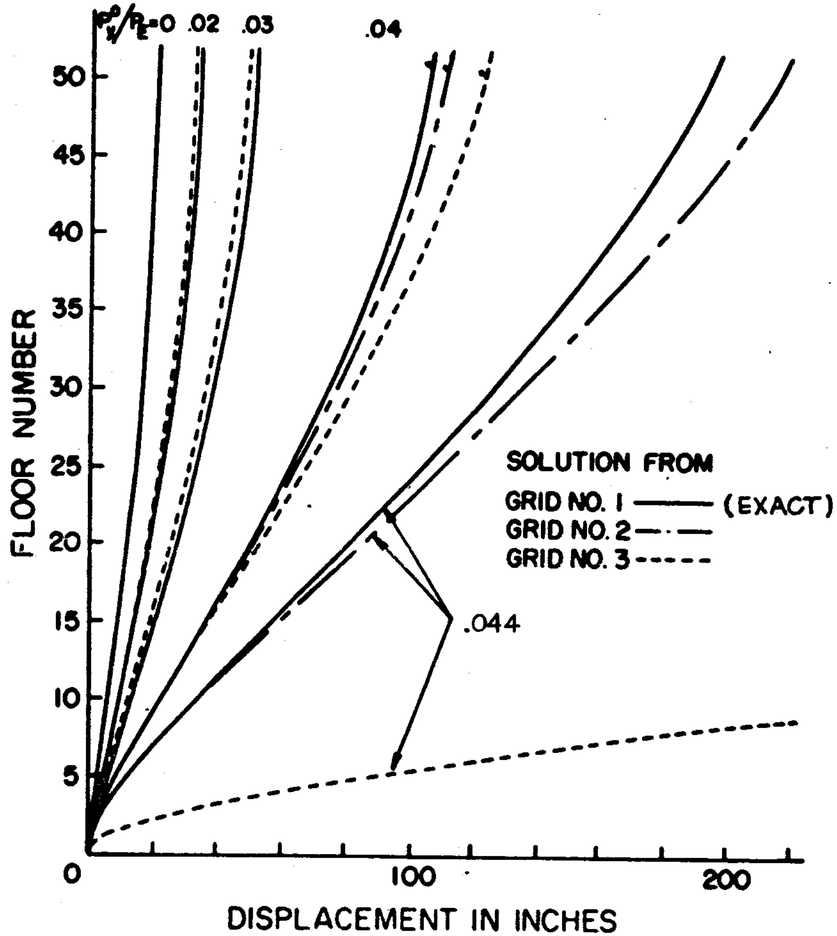


Fig. 6. Horizontal displacement of column line 1 (Fig. 5) under increasing axial load P_y^0

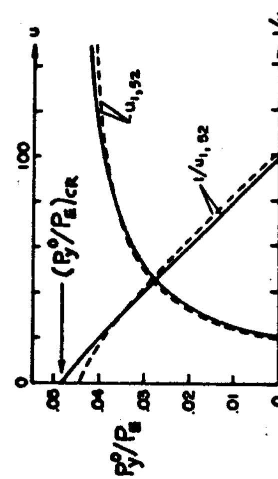


Fig. 8. Horizontal displacement of column line 1 (Fig. 5) at floor 52 for increasing initial load P_y^0 (solid line - exact, dashed line-grid 3)

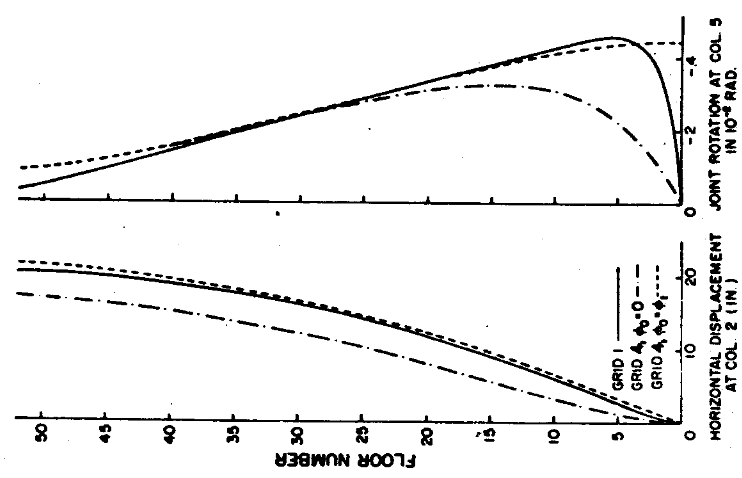


Fig. 7. Comparison of approximate solutions for the frame in Fig. 5 with boundary disturbance present and removed and $P_y^0 = 0$.

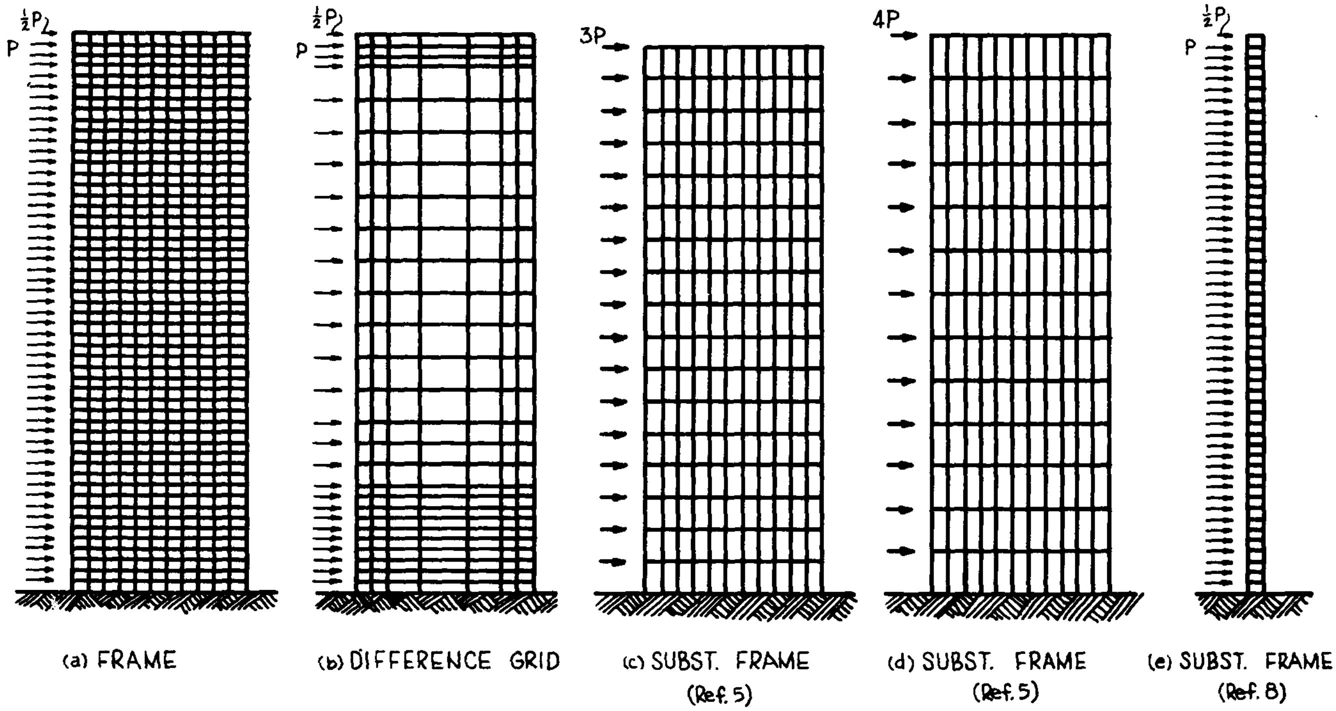


Fig. 9. Finite difference grid and substitute frames considered in the numerical examples in Figures 10, 11 and 12. (Frames (c) and (d) are characterized by $\bar{L}_y = nL_y$, $\bar{I}_y = n^2 I_y$, $\bar{I}_x = n^2 I_x$ where n = 3 in case (c) and n = 4 in case (d).)

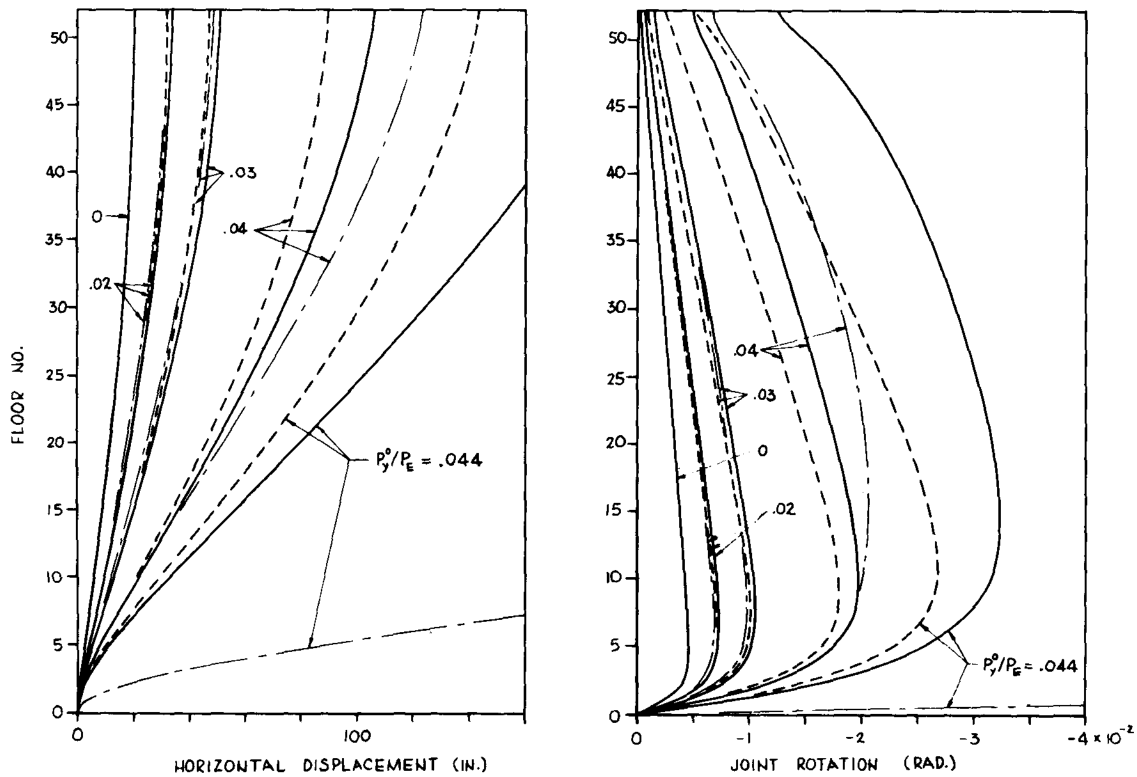


Fig. 10. Comparison of results by finite difference grid from Fig. 9b (dash-dot lines) with the results by substitute frame from Fig. 9e (dashed lines) and the exact solution of the frame in Fig. 9a (solid lines). Where only one line is shown, the results are undistinguishable

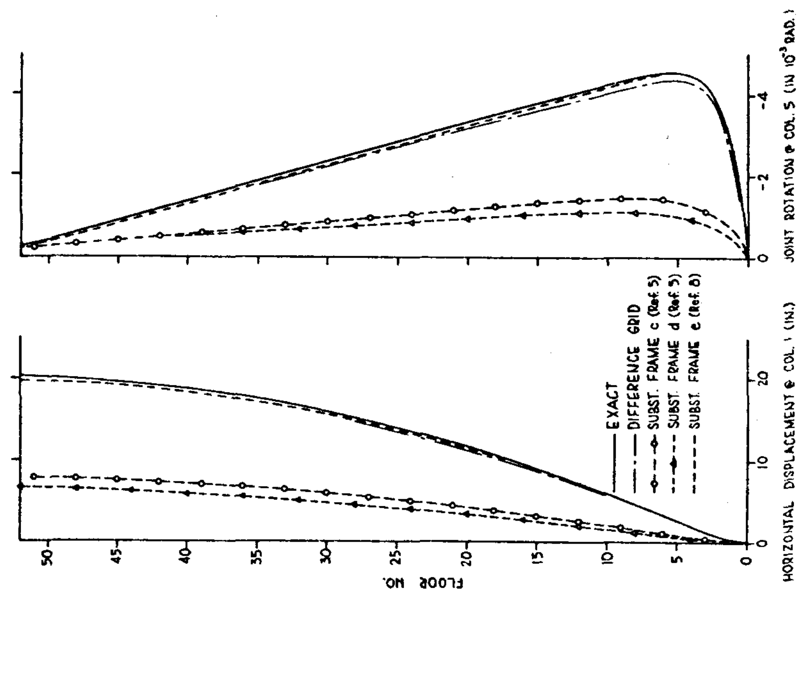


Fig. 11. Comparison of results by various methods (Fig. 9)

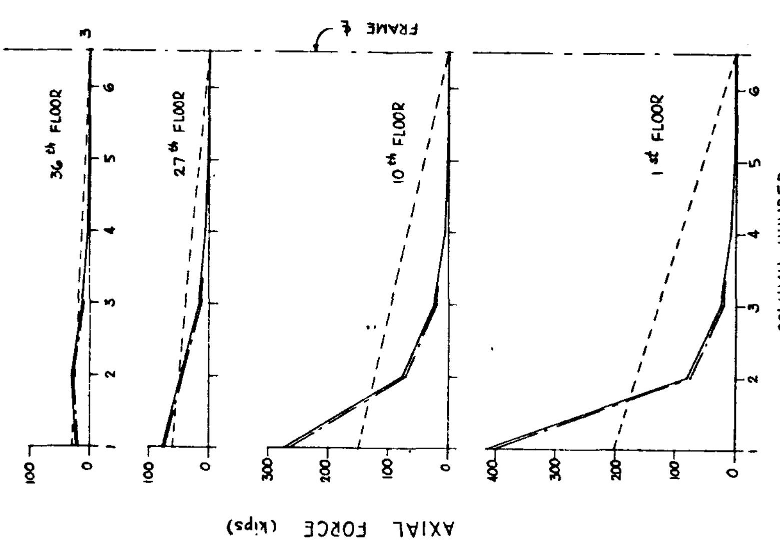


Fig. 12. Comparison of the same cases as in Fig. 10 but for axial column force due to horizontal load