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SINGULAR FINITE ELEMENT FOR DYNAMICS OF ELASTIC SOLIDS WITH CRACKS AND SHARP NOTCHES

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ABSTRACT

The stiffness matrix and the mass matrix of a singular finite element around the crack tip are derived on the basis of limiting near-tip displacement fields expressed as a linear combination of several eigenstates. The stress intensity factors are considered as generalized displacements of the singular element. The homogeneous stress states that can exist near the crack tip are also modeled by the singular element. The stiffness and mass matrices of the adjacent nonsingular elements are transformed as to refer to the generalized displacements of the singular element. An analogous singular element is formulated for the tip of a sharp notch of arbitrary angle.

INTRODUCTION

The convergence of the ordinary finite element method [1] in singular stress problems is known to be very slow [2-5]. Substantial improvement can be achieved by modeling the singularity in terms of a special "singular" finite element which has the exact near-tip field built in [6-12]. A very simple and convenient formulation of a plane singular finite element has been proposed by Walsh [6]. In the present study, Walsh's formulation [6] will be refined by incorporating the homogeneous strain fields which can exist near the crack tip. Also, the adjacent non-singular finite elements will not be lumped with the singular element, as is done by Walsh, but will be treated separately, which is somewhat more expedient for programming. Furthermore, the formulation will be extended to dynamics and to sharp notches of arbitrary angle. Numerical verification will appear in a separate paper by several co-authors.

STIFFNESS MATRIX OF A SINGULAR FINITE ELEMENT FOR CRACK TIP

To describe the stress singularity in plane problems, a singular finite element surrounding the crack tip will be considered (Fig. 1). Its boundary is defined by nodes which are connected by straight lines and are spaced sufficiently closely in terms of angular polar coordinate θ (Fig. 1). As is well known, the stress fields in a sufficiently small neighborhood of the crack tip are fully characterized by the stress inten-

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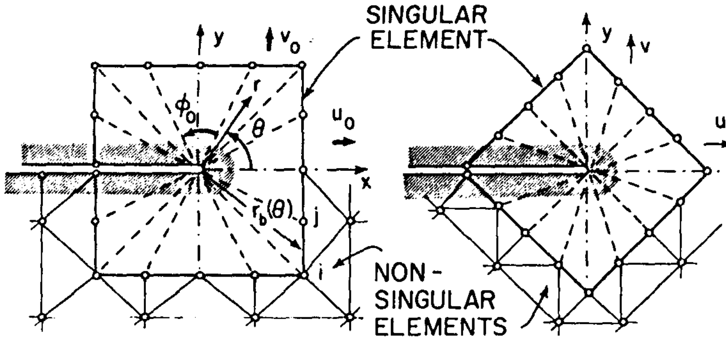

Fig. 1 Two examples of a singular finite element for crack tip in a plane

sity factors K_1 for the opening deformation mode and K_2 for the plane shear mode [13]. A singular finite element which has these stress fields built in has already been developed by Walsh [6]. So far it has been overlooked, however, that a singular stress state need not always arise because a homogeneous state of normal stress parallel to crack may exist near the crack tip. Moreover, if the singular element is not extremely small, it is necessary to include the homogeneous stress state even when singular stress fields are present, or else the numerical error originating from the singular element would be of a higher order of magnitude than that from the non-singular elements. The homogeneous stress state will be characterized by normal strain $\varepsilon_x = \varepsilon_0$ along the crack. Because no homogeneous field of normal stress perpendicular to crack can exist, homogeneous strain $\varepsilon_y = -\nu'\varepsilon_0$ must be imposed simultaneously. Using the expressions for the near-tip fields from [14], stresses σ_x, σ_y, τ_{xy} and displacements u, v (Fig. 1) within the singular element may be considered in the form:

$$\begin{aligned} \sigma_x &= [K_1 s_{x_1}(\theta) + K_2 s_{x_2}(\theta)]r^{-1/2} + E'\varepsilon_0, \\ \sigma_y &= [K_1 s_{y_1}(\theta) + K_2 s_{y_2}(\theta)]r^{-1/2}, \\ \tau_{xy} &= [K_1 s_{xy_1}(\theta) + K_2 s_{xy_2}(\theta)]r^{-1/2}, \end{aligned} \qquad (1)$$

$$\begin{aligned} u &= [K_1 q_{x_1}(\theta) + K_2 q_{x_2}(\theta)]\sqrt{r} + \varepsilon_0 x + u_0 - \phi_0 y, \\ v &= [K_1 q_{y_1}(\theta) + K_2 q_{y_2}(\theta)]\sqrt{r} - \nu'\varepsilon_0 y + v_0 + \phi_0 x \end{aligned} \qquad (2)$$

where, in the case of plane stress, $E' = E$ = Young's modulus, $\nu' = \nu$ = Poisson's ratio, and in the case of plane strain, $E' = E/(1-\nu^2)$, $\nu' = \nu/(1-\nu)$; u_0, v_0, ϕ_0 = rigid body displacements and rotation about crack tip (Fig. 1); r, θ = polar coordinates; x, y = cartesian coordinates (Fig. 1); and

$$\begin{aligned} s_{x_1}(\theta) &= \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)/\sqrt{2}, \\ s_{y_1}(\theta) &= \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)/\sqrt{2}, \\ s_{xy_1}(\theta) &= \left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)/\sqrt{2}, \\ q_{x_1}(\theta) &= \cos\frac{\theta}{2}\left(1 - 2\nu' + \sin^2\frac{\theta}{2}\right)\sqrt{2}(1+\nu')/E', \\ q_{y_1}(\theta) &= \sin\frac{\theta}{2}\left(2 - 2\nu' - \cos^2\frac{\theta}{2}\right)\sqrt{2}(1+\nu')/E', \\ s_{x_2}(\theta) &= \sin\frac{\theta}{2}\left(2 + \cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)/\sqrt{2}, \\ s_{y_2}(\theta) &= \left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2}\right)/\sqrt{2}, \\ s_{xy_2}(\theta) &= \cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right)/\sqrt{2}, \\ q_{x_2}(\theta) &= \sin\frac{\theta}{2}\left(2 - 2\nu' + \cos^2\frac{\theta}{2}\right)(1+\nu')/E'\sqrt{2}, \\ q_{y_2}(\theta) &= \cos\frac{\theta}{2}\left(2\nu' - 1 + \sin^2\frac{\theta}{2}\right)(1+\nu')/E'\sqrt{2}. \end{aligned} \qquad (3)$$

As is seen from (1) and (2), the displacements in the singular element, as well as the stresses and strains, are fully characterized by the column matrix of generalized displacements:

$$\underset{\sim}{u}_0 = [K_1, K_2, \varepsilon_0, u_0, v_0, \phi_0]^T \qquad (4)$$

where superscript T denotes the transpose. It is noteworthy that the stress intensity factors K_1 and K_2 are here adopted as displacement parameters.

Consider now the expression for the virtual work of stresses done within the singular element on the virtual strains $\delta\underset{\sim}{\varepsilon}$ (any kinematically admissible strains), i.e.

$$\delta W = \int_{A_s} (\delta\underset{\sim}{\varepsilon})^T \underset{\sim}{\sigma}\, dA_s, \qquad (5)$$

where A_s = area of the singular element; $\underset{\sim}{\sigma} = [\sigma_x, \sigma_y, \tau_{xy}]^T$ = column matrix formed of stress tensor components and $\underset{\sim}{\varepsilon} = [\varepsilon_x, \varepsilon_y, \gamma_{xy}]^T$ = column matrix formed of strain tensor components. The strains and stresses are related according to the Hooke's law

$$\underset{\sim}{\sigma} = \underset{\sim}{D}\,\underset{\sim}{\varepsilon}, \qquad (6)$$

where $\underset{\sim}{D}$ is the well known square matrix of elastic moduli in plane strain or in plane stress [1].

Using matrix $\underset{\sim}{u}_0$, Eqs. (4) and (1) may be written in the form:

$$\underset{\sim}{\sigma}(\theta,r) = \underset{\sim}{B}(\theta,r)\underset{\sim}{u}_0, \qquad (7)$$

in which

$$\underset{\sim}{B}(\theta,r) = \left[\begin{array}{c|cccc} & E'\varepsilon_0 & 0 & 0 & 0 \\ \{r^{-1/2}\underset{\sim}{S}(\theta)\} & 0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \end{array}\right], \qquad (8)$$

$$S(\theta) = \begin{bmatrix} s_{x_1}(\theta) & s_{x_2}(\theta) \\ s_{y_1}(\theta) & s_{y_2}(\theta) \\ s_{xy_1}(\theta) & s_{xy_2}(\theta) \end{bmatrix} \tag{9}$$

According to (6),

$$\varepsilon(\theta,r) = B'(\theta,r)u_0, \tag{10}$$

in which

$$B'(\theta,r) = \left[\begin{array}{c|cccc} & \varepsilon_0 & 0 & 0 & 0 \\ \{r^{-1/2}D^{-1}S(\theta)\} & -\nu'\varepsilon_0 & 0 & 0 & 0 \\ & 0 & 0 & 0 & 0 \end{array}\right] \tag{11}$$

Thus, expression (5) for the virtual work becomes

$$\delta W = \int_{(A_s)} (\delta u_0)^T B'^T(\theta,r)B(\theta,r)u_0 dA_s = (\delta u_0)^T k_0 u_0, \tag{12}$$

in which

$$k_0 = \int_{(\theta)}\int_{(r)} B'^T(\theta,r)B(\theta,r)r\, dr\, d\theta, \tag{13}$$

where the integration over area A_s of the singular element is expressed in polar coordinates θ and r ($dA_s = r\,d\theta\,dr$). From the structure of the work expression (12) it is seen that k_0 represents the stiffness matrix of the singular element with respect to generalized displacements u_0. The generalized force matrix associated with u_0 is, therefore, expressed as

$$F_0 = k_0\, u_0. \tag{14}$$

Using submatrix S defined by (9), expression (13) for the stiffness matrix may be transformed, after integration over r, to a partitioned form which is more convenient for numerical computation, namely

$$k_0 = \left[\begin{array}{cc|c|ccc} \multicolumn{2}{c|}{\multirow{2}{*}{k_a}} & \multirow{2}{*}{k_b} & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ \hline \multicolumn{2}{c|}{k_b^T} & E'A_s & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{15}$$

where

$$k_a = \int_{-\pi}^{\pi} S^T(\theta)D^{-1}\, S(\theta)\, r_b(\theta)\, d\theta \qquad (2\times 2 \text{ matrix}) \tag{16}$$

$$k_b = \frac{2}{3}\int_{-\pi}^{\pi}[s_{x_1}(\theta) - \nu' s_{y_1}(\theta),\ s_{x_2}(\theta) - \nu' s_{y_2}(\theta)]^T\, r_b^{3/2}(\theta)d\theta \qquad (2\times 1 \text{ matrix}) \tag{17}$$

in which $r_b(\theta)$ is the distance of the general point at the boundary of the singular element from the crack tip. For the evalua-

tion of the integrals (16) and (17), numerical integration over θ is most practical.

The formulation presented above does not guarantee monotonic convergence because the continuity of the displacement field (2) with that in the adjacent elements is assured only in the nodes. However, if desired, functions q_{x_1}, q_{y_1}, q_{x_2}, q_{y_2} may be easily modified as to guarantee full continuity. This is done by retaining their values at the nodal angles as given by (3), and considering a linear variation of these functions with $\cos\theta$ and $\sin\theta$ between the nodal angles.

STIFFNESS MATRICES OF ADJACENT NON-SINGULAR FINITE ELEMENTS

The u- and v-displacements of all nodes of the singular element are not arbitrary but depend on the displacements u_0 as is indicated by relations (2). Hence, these displacements must be eliminated from the system of equilibrium equations.

To this end, consider an adjacent ordinary finite element which contains one or more nodes that lie on the boundary of the singular element. Let the displacements of these nodes be grouped into the column matrix u_b and those of the remaining nodes of this element into the column matrix u_c. Thus, the column matrix of all displacements of this element is partitioned as $u = (u_b,\ u_c)^T$.

Displacements u_c are independent unknowns, but displacements u_b depend upon u_0 according to (2). Thus

$$u_b = C\, u_0 \tag{18}$$

where C is a rectangular matrix. The virtual work of stresses in the element may be expressed as $\delta W' = (\delta u)^T K\, u = (\delta u)^T F$. This may be written in a partitioned form,

$$\delta W' = (\delta u_b^T,\ \delta u_c^T)\left[\begin{array}{c|c} K_{bb} & K_{bc} \\ \hline K_{cb} & K_{cc} \end{array}\right]\left\{\begin{array}{c} u_b \\ \hline u_c \end{array}\right\} = (\delta u_b^T,\ u_c^T)\left\{\begin{array}{c} F_b \\ \hline F_c \end{array}\right\} \tag{19}$$

Substitution of (18) yields

$$\delta W' = (\delta u_0^T\, C^T,\ \delta u_c^T)\left[\begin{array}{c} K_{bb}\, C\, u_0 + K_{bc}\, u_c \\ \hline K_{cb}\, C\, u_0 + K_{cc}\, u_c \end{array}\right] = (\delta u_0^T\, C^T,\ \delta u_c^T)\left\{\begin{array}{c} F_b \\ \hline F_c \end{array}\right\} \tag{20}$$

or

$$\delta W' = (\delta u_0^T,\ \delta u_c^T)K'\left\{\begin{array}{c} u_0 \\ \hline u_c \end{array}\right\} = (\delta u_0^T,\ \delta u_c^T)F' \tag{21}$$

where

$$K' = \left[\begin{array}{c|c} C^T K_{bb}\, C & C^T K_{bc} \\ \hline K_{cb}\, C & K_{cc} \end{array}\right], \quad F' = \left\{\begin{array}{c} C^T F_b \\ \hline F_c \end{array}\right\}. \tag{22}$$

It is clear from the structure of expressions (21) that K' represents the reduced stiffness matrix of the finite element having some of its nodes on the boundary of the singular domain, and $\tilde{F}'$ represents its reduced matrix of applied forces.

MASS MATRICES OF THE SINGULAR ELEMENT AND THE ADJACENT ELEMENTS

The inertia forces are given by the second time derivatives of displacements. These are bounded at every point, including the crack tip. Thus, the effect of the inertia forces is negligible as compared with the effect of stresses and the near-tip fields are the same as in the static case. The displacement field given by (2) may be expressed as:

$$\tilde{v}(r,\theta) = \begin{Bmatrix} u \\ v \end{Bmatrix} = \tilde{C}(r,\theta)\tilde{u}_0 \tag{23}$$

where

$$\tilde{C}(r,\theta) = \begin{bmatrix} q_{x_1}(\theta), & q_{x_2}(\theta), & x, & 1, & 0, & -y \\ q_{y_1}(\theta), & q_{y_2}(\theta), & -\nu' y, & 0, & 1, & x \end{bmatrix} \tag{24}$$

The virtual work of the distributed inertia forces within the singular element is

$$\delta W_{in} = \int_{(A_s)} (\delta \tilde{v})^T \rho \ddot{\tilde{v}} \, dA_s \tag{25}$$

where ρ = mass density. Substitution of (23) yields

$$\delta W_{in} = (\delta \tilde{u}_0)^T \tilde{m}_0 \ddot{\tilde{u}}_0 \text{ , with } \tilde{m}_0 = \int_{(A_s)} \tilde{C}^T(\theta,r)\, \tilde{C}(\theta,r)\rho \, dA_s . \tag{26}$$

$\tilde{m}_0$ is a square matrix of the same size as stiffness matrix $\tilde{k}_0$. According to (26), $\tilde{m}_0$ represents what is generally known as mass matrix [1]. For computation purposes, the mass matrix may be expressed in a similar manner as the stiffness matrix; Eqs. 16-18.

If linear damping is exhibited by the material, the damping matrix of the singular element can be determined in a completely analogous manner.

The mass matrix M' of an adjacent non-singular finite element may be reduced to the displacement vector $\tilde{u}_0$ of the singular element in the same manner as in (19)-(22). In analogy with Eq. (22) (and with the same subscript significance)

$$\tilde{M}' = \left[\begin{array}{c|c} \tilde{C}^T \tilde{M}_{bb} \tilde{C} & \tilde{C}^T \tilde{M}_{bc} \\ \hline \tilde{M}_{bc} \tilde{C} & \tilde{M}_{cc} \end{array}\right] . \tag{27}$$

SINGULAR ELEMENTS FOR SHARP NOTCHES OF ARBITRARY ANGLE

According to [15], the displacement field near the tip of notch of angle 2α (Fig. 2) may be expressed (in polar coordinates r, θ) as:

$$\begin{aligned} u_r &= K_1 r^n f_1(\theta) + K_2 r^m f_2(\theta) + u_0 \cos\theta + v_0 \sin\theta \\ u_\theta &= K_1 r^n g_1(\theta) + K_2 r^m g_2(\theta) + r\,\phi_0 \end{aligned} \tag{28}$$

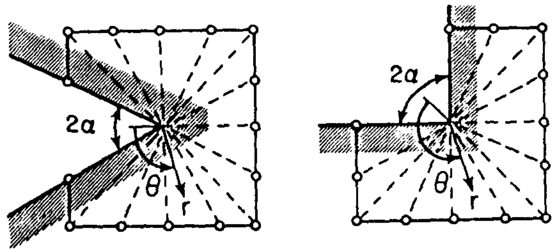

Fig. 2 Two examples of finite elements for sharp notches in a plane

in which

$$f_1(\theta) = \cos(1+n)\theta + a\cos(1-n)\theta, \quad f_2(\theta) = \sin(1+m)\theta + b\sin(1-m)\theta, \tag{29}$$
$$g_1(\theta) = \sin(1+n)\theta + a\gamma\sin(1-n)\theta, \quad g_2(\theta) = \cos(1+m)\theta + b\gamma'\cos(1-m)\theta,$$

where n, m are the smallest roots of the equations $n \sin 2\beta = \sin 2\beta n$ and $m \sin 2\beta = -\sin 2\beta m$, and

$$a = \frac{2n\cos(1+n)\beta}{(\gamma-1)(1+n)\cos(1-n)\beta}, \quad b = \frac{2m\sin(1+m)\beta}{(\gamma'-1)(1+m)\sin(1-m)\beta}, \tag{30}$$
$$\gamma = \frac{s(1-n)+(n+1)}{(1-n)+s(n+1)}, \qquad \gamma' = \frac{s(1-m)+(m+1)}{(1-m)+s(m+1)},$$

with $s = G/(2G+\lambda)$; G, λ = Lamé's elastic constants; $\beta = \pi - \alpha$, α = 1/2 angle of the notch (Fig. 2); K_1, K_2 = parameters analogous to stress intensity factors for cracks. (Near fields are also available for notches with fixed edges, i.e. inclusions [15].) No homogeneous stress field can exist near the tip of a notch. Thus, the near-tip field is completely described by the column matrix $\tilde{u}_0 = (K_1, K_2, u_0, v_0, \phi_0)^T$. Formulation of the finite element of the type shown in Fig. 2 proceeds similarly as for a crack.

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APPENDIX - GENERALIZATION TO CERTAIN THREE-DIMENSIONAL CASES

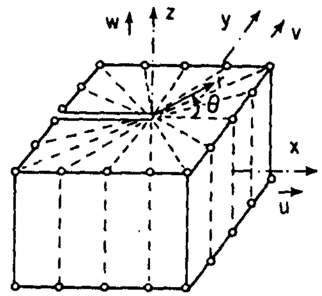

Fig. 3

Consider a point on the crack edge in three dimensions, shown in Fig. 3. The near edge fields in plane (xy) may be considered to have the same form as Eqs. (1) and (2). Displacements w in the z-direction near the crack edge are characterized by stress intensity factor, K_3, for the anti-plane shear mode. Superimposing the plane and the anti-plane deformation modes, the near-edge field is

$$u = [K_1 q_{x_1}(\theta) + K_2 q_{x_2}(\theta) + 0]\sqrt{r} + \varepsilon_{x_0} x + u_0 + \phi_{y_0} z - \phi_{z_0} x \ ,$$

$$v = [K_1 q_{y_1}(\theta) + K_2 q_{y_2}(\theta) + 0]\sqrt{r} - \nu'(\varepsilon_{x_0} + \varepsilon_{z_0})y + v_0 - \phi_{x_0} z + \phi_{z_0} x,$$

$$w = [0 + 0 + K_3 q_{z_3}(\theta)]\sqrt{r} + \varepsilon_{z_0} z + \gamma_{xz_0} x + w_0 + \phi_{x_0} y - \phi_{y_0} x \ , \qquad (31)$$

where $u_0, \ldots, \phi_{z_0}$ = rigid body displacements and rotations;

$\nu' = \nu/(1-\nu)$; ε_{x_0}, ε_{z_0}, γ_{xz_0} = strains characterizing the homogeneous stress fields that can exist near the crack edge; $q_{2_3}(\theta) = \sqrt{2}\sin(\theta/2)$. Expressing strains and stresses from displacements [13], one has $\underset{\sim}{\sigma} = \underset{\sim}{B}(\theta,r)\underset{\sim}{u}_0$, $\underset{\sim}{\varepsilon} = \underset{\sim}{B}'(\theta,r)\underset{\sim}{u}_0$ where

$$\underset{\sim}{u} = [K_1, K_2, K_3, \varepsilon_{x_0}, \varepsilon_{z_0}, \gamma_{xz_0}, u_0, v_0, w_0, \phi_{x_0}, \phi_{y_0}, \phi_{z_o}]^T \qquad (32)$$

$$\underset{\sim}{\sigma} = [\sigma_x, \sigma_y, \sigma_z, \tau_{xy}, \tau_{yz}, \tau_{zx}]^T, \quad \underset{\sim}{\varepsilon} = [\varepsilon_x, \varepsilon_y, \varepsilon_z, \gamma_{xy}, \gamma_{yz}, \gamma_{zx}]^T. \qquad (33)$$

$\underset{\sim}{B}$ and $\underset{\sim}{B}'$ are certain rectangular (6 x 12) matrices similar to (8) and involving $r^{-\frac{1}{2}}$. The derivation of stiffness and mass matrices is similar as before.